Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 40, Special Issue, 2014, Pages 203–214

ON CONSTRUCTION OF A QUADRATIC STURM-LIOUVILLE OPERATOR PENCIL FROM SPECTRAL DATA

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In memory of M. G. Gasymov on his 75th birthday

Abstract. Derivation of fundamental equations of the inverse spectral problem for a quadratic Sturm-Liouville operator pencil is presented. An algorithm for solving the inverse problem is offered.

1. Introduction

In [6] two versions of the inverse spectral problem (the inverse problem from eigenvalues and normalizing numbers and the inverse problem from two spectra) were considered for the following Sturm-Liouville eigenvalue problem with quadratic dependence on the spectral parameter:

$$-y'' + [q(x) + 2\lambda p(x)]y = \lambda^2 y, \quad 0 \le x \le \pi, \tag{1.1}$$

$$y'(0) - hy(0) = 0, \quad y'(\pi) + Hy(\pi) = 0. \tag{1.2}$$

Here $q(x)$, $p(x)$ are real-valued functions, h, H are real numbers, and λ is a spectral parameter.

As is known, at present more completely are studied inverse spectral problems for the Sturm-Liouville and Dirac operators and for their discrete (finite-difference) analogs (see [1, 2, 3, 5, 7, 8, 9, 10, 11, 12, 13, 14, 21]). It turns out that for Eq. (1.1) considered on a finite or infinite interval the inverse spectral problems can also be investigated enough completely since there are the kernels of transformation opeartors for this equation (see [4, 6, 15, 16, 17, 18, 19, 20]). It is remarkable that for Eq. (1.1) the inverse spectral problems in their usual formulations are solvable uniquely: the two functions $q(x)$ and $p(x)$ are determined uniquely from the spectral measure (in particular, from the eigenvalues and normalizing numbers) or from the two spectra.

In the present paper, we display some key points of the inverse spectral problem for (1.1), (1.2), presented in [6] without any proof.

This paper is organized as follows. In Section 2, following [6] we bring out the facts on the problem (1.1), (1.2) needed in the subsequent sections. In Section 3, we derive the so-called fundamental equations of the inverse problem. The

2010 *Mathematics Subject Classification.* 34A55.

Key words and phrases. quadratic pencil, spectral data, inverse problem, fundamental equations.

idea of this derivation belongs to M. G. Gasymov. Afterwards, in Section 4, we indicate an algorithm for constructing boundary value problem (1.1), (1.2) from its spectral data consisting of the eigenvalues and normalizing numbers.

2. Preliminaries

In this section, following [6], we present the facts on the problem (1.1), (1.2), needed in the subsequent sections.

Denote by $W_2^n[0,\pi]$ the Sobolev space consisting of complex-valued functions on $[0,\pi]$ having $n-1$ absolutely continuous derivatives and nth order derivative that is square-integrable on $[0,\pi]$. Note that $W_2^0[0,\pi] = L_2[0,\pi]$.

The following theorem shows that for Eq. (1.1) there exists the so-called *transformation operator* which is crucial in the inverse spectral theory.

Theorem 2.1. *Let $q(x) \in W_2^m[0,\pi]$, $p(x) \in W_2^{m+1}[0,\pi]$ $(m \geq 0)$, and $\varphi(x,\lambda)$ be the solution of Eq. (1.1) satisfying the initial conditions*

$$\varphi(0,\lambda) = 1, \quad \varphi'(0,\lambda) = h. \tag{2.1}$$

Then there exist real-valued functions $A(x,t)$ and $B(x,t)$ having $m+1$ square-integrable derivatives with respect to the both variables such that

$$\varphi(x,\lambda) = \cos[\lambda x - \alpha(x)] + \int_0^x A(x,t)\cos\lambda t dt + \int_0^x B(x,t)\sin\lambda t dt, \tag{2.2}$$

$$\alpha(x) = xp(0) + 2\int_0^x [A(\xi,\xi)\sin\alpha(\xi) - B(\xi,\xi)\cos\alpha(\xi)]d\xi, \tag{2.3}$$

$$q(x) = -p^2(x) + 2\frac{d}{dx}[A(x,x)\cos\alpha(x) + B(x,x)\sin\alpha(x)], \tag{2.4}$$

$$A(0,0) = h, \quad \left.\frac{\partial A(x,t)}{\partial t}\right|_{t=0} = 0, \quad B(x,0) = 0, \tag{2.5}$$

where

$$\alpha(x) = \int_0^x p(t)dt. \tag{2.6}$$

Next, if $m \geq 1$ then

$$\frac{\partial^2 A(x,t)}{\partial x^2} - 2p(x)\frac{\partial B(x,t)}{\partial t} - q(x)A(x,t) = \frac{\partial^2 A(x,t)}{\partial t^2}, \tag{2.7}$$

$$\frac{\partial^2 B(x,t)}{\partial x^2} + 2p(x)\frac{\partial A(x,t)}{\partial t} - q(x)B(x,t) = \frac{\partial^2 B(x,t)}{\partial t^2}. \tag{2.8}$$

Conversely, if given functions $A(x,t)$ and $B(x,t)$ have second order square-integrable partial derivatives satisfying equations (2.7), (2.8) and satisfy conditions (2.3)–(2.6), then the function $\varphi(x,\lambda)$ constructed by formula (2.2) is the solution of Eq. (1.1) subject to initial conditions (2.1).

Let us denote by D the subspace of space $W_2^2[0,\pi]$, consisting of functions $y(x) \in W_2^2[0,\pi]$ satisfying the boundary conditions in (1.2).

Further we will assume that $q(x)$, $p(x)$ are real-valued functions with $q(x) \in L_2[0,\pi]$, $p(x) \in W_2^1[0,\pi]$ and h, H are real numbers such that

$$h\,|y(0)|^2 + H\,|y(\pi)|^2 + \int_0^\pi \{|y'(x)|^2 + q(x)\,|y(x)|^2\}dx > 0 \tag{2.9}$$

for all functions $y(x) \in D$ being not identically zero (the last condition is automatically satisfied if $h \geq 0$, $H \geq 0$, and $q(x) > 0$). Under these conditions boundary value problem (1.1), (1.2) possesses the following spectral properties.

(1) The eigenvalues of boundary value problem (1.1), (1.2) are real, different from zero, and simple. This problem does not have associated functions attached to the eigenfunctions. (Note that without condition (2.9) problem (1.1), (1.2) may have a finite number of nonreal eigenvalues.)

(2) The eigenfunctions $y(x)$ and $z(x)$ of problem (1.1), (1.2) corresponding to the different eigenvalues λ and μ, respectively, satisfy the "orthogonality" relation

$$(\lambda + \mu) \int_0^\pi y(x)\overline{z(x)}dx - 2\int_0^\pi p(x)y(x)\overline{z(x)}dx = 0.$$

(3) The problem (1.1), (1.2) has countably many eigenvalues which can be arranged in the sequence

$$\ldots < \lambda_{-2} < \lambda_{-1} < \lambda_{-0} < 0 < \lambda_{+0} < \lambda_1 < \lambda_2 < \ldots$$

so that for large negative and positive values of n the asymptotic formula

$$\lambda_n = n + c_0 + \frac{c_1}{n} + \frac{c_{1,n}}{n} \tag{2.10}$$

holds, where

$$c_0 = \frac{1}{\pi}\int_0^\pi p(x)dx, \quad \sum_n |c_{1,n}|^2 < \infty,$$

$$c_1 = \frac{1}{\pi}\left(h + H + \frac{1}{2}\int_0^\pi [q(x) + p^2(x)]dx\right). \tag{2.11}$$

(Recall that $\lambda = 0$ is not an eigenvalue of (1.1), (1.2). Note also that the notations $n = -0$ and $n = +0$ are used for the eigenvalues λ_n to have the asymptotic formula just in the form (2.10).)

(4) Obviously, $\varphi_n(x) = \varphi(x, \lambda_n)$ is an eigenfunction of (1.1), (1.2), corresponding to the eigenvalue λ_n. Let us set

$$a_n = \int_0^\pi \varphi_n^2(x)dx - \frac{1}{\lambda_n}\int_0^\pi p(x)\varphi_n^2(x)dx.$$

The numbers a_n we call the *normalizing numbers* of problem (1.1), (1.2).

(5) The normalizing numbers a_n are positive and for large negative and positive values of n the asymptotic formula

$$a_n = \frac{\pi}{2} + \frac{\alpha_1}{n} + \frac{\alpha_{1,n}}{n}$$

holds, where

$$\alpha_1 = -\frac{\pi}{2}p(0), \quad \sum_n |\alpha_{1,n}|^2 < \infty. \tag{2.12}$$

(6) For arbitrary function $f(x)$ in $L_2[0, \pi]$ the "two-fold" expansion formulas

$$\sum_n \frac{1}{\lambda_n a_n}\varphi(x, \lambda_n)\int_0^\pi f(t)\varphi(t, \lambda_n)dt = 0, \tag{2.13}$$

$$\sum_n \frac{1}{2a_n}\varphi(x,\lambda_n)\int_0^\pi f(t)\varphi(t,\lambda_n)dt = f(x) \tag{2.14}$$

hold, where the series converge in the metric of space $L_2[0,\pi]$. Everywhere in the infinite sums n runs all the values $n = \pm 0, \pm 1, \pm 2, \dots$.

(7) The equality

$$\sum_n \frac{1}{\lambda_n a_n} = 0$$

holds, where the infinite sum is understood in the sense of a principial value, i.e. as the limit of sums $\sum_{-N}^{N}$ as $N \to \infty$.

The *inverse spectral problem* consists in recovering the coefficient functions $q(x),\ p(x)$ in Eq. (1.1) and the coefficient numbers $h,\ H$ in boundary conditions (1.2) from the spectral data $\{\lambda_n,\ a_n\}$ of problem (1.1), (1.2).

In [6] the following theorem is stated on solution of the inverse spectral problem.

Theorem 2.2. *In order for real numbers* $\{\lambda_n\}$ *and* $\{a_n\}$ $(n = \pm 0, \pm 1, \pm 2, \dots)$ *with* $a_n > 0$ *and*

$$\dots < \lambda_{-2} < \lambda_{-1} < \lambda_{-0} < 0 < \lambda_{+0} < \lambda_1 < \lambda_2 < \dots \tag{2.15}$$

to be the eigenvalues and the normalizing numbers, respectively, of a boundary value problem of the form (1.1), (1.2) with real-valued functons $q(x) \in L_2[0,\pi],\ p(x) \in W_2^1[0,\pi]$ *and real numbers* $h,\ H$ *it is sufficient that the following conditions are satisfied:*

(i) The equality

$$\sum_n \frac{1}{\lambda_n a_n} = 0 \tag{2.16}$$

holds, where the infinite sum is understood in the sense of a principial value.

(ii) The asymptotic formulas

$$\lambda_n = n + c_0 + \frac{c_1}{n} + \frac{c_{1,n}}{n}, \tag{2.17}$$

$$a_n = \frac{\pi}{2} + \frac{\alpha_1}{n} + \frac{\alpha_{1,n}}{n} \tag{2.18}$$

hold, where $c_0,\ c_1,\ \alpha_1$ *are costants and*

$$\sum_n (c_{1,n}^2 + \alpha_{1,n}^2) < \infty. \tag{2.19}$$

The boundary value problem (1.1), (1.2) is uniquely restored from the spectral data $\{\lambda_n,\ a_n\}$.

In the next section we will derive the so-called fundamental equations of the invese problem, which allow to indicate a procedura for reconstruction of problem (1.1), (1.2) from the spectral data $\{\lambda_n,\ a_n\}$.

3. Fundamental equations of the inverse problem

Assume that for problem (1.1), (1.2) the conditions stated above in Section 2 are satisfied (see (2.9)). Let $\{\lambda_n\}$ be the eigenvalues and $\{a_n\}$ be the corresponding normalizing numbers of problem (1.1), (1.2), where $n = \pm 0, \pm 1, \pm 2, \dots$. In this section we will derive some equations which allow formally solve the inverse spectral problem.

Let us set

$$H(x,t) = \sum_n \frac{1}{2\lambda_n a_n} \varphi(x,\lambda_n) e^{i\lambda_n t}, \tag{3.1}$$

where n in the sum runs all the values $\pm 0, \pm 1, \pm 2, \dots$.

Lemma 3.1. *The equality*

$$H(x,t) = 0 \quad \textit{for} \quad 0 \le t < x \tag{3.2}$$

holds.

Proof. We will handle with the function $H(x,t)$ defined by (3.1) as with a distribution. The formulas (2.13), (2.14) can be written as

$$\sum_n \frac{1}{\lambda_n a_n} \varphi(x,\lambda_n)\varphi(t,\lambda_n) = 0,$$

$$\sum_n \frac{1}{2a_n} \varphi(x,\lambda_n)\varphi(t,\lambda_n) = \delta(x-t),$$

where $\delta(x)$ is the Dirac delta function. Putting here $t = 0$ and taking into account that $\varphi(0,\lambda) = 1$, we conclude by definition (3.1) of $H(x,t)$ that

$$H(x,0) = 0, \qquad \left.\frac{\partial H(x,t)}{\partial t}\right|_{t=0} = i\delta(x).$$

Next, we have

$$\frac{\partial^2 H(x,t)}{\partial x^2} = \sum_n \frac{1}{2\lambda_n a_n} \varphi''(x,\lambda_n) e^{i\lambda_n t}$$

$$= \sum_n \frac{1}{2\lambda_n a_n} \{2\lambda_n p(x)\varphi(x,\lambda_n) + q(x)\varphi(x,\lambda_n) - \lambda_n^2 \varphi(x,\lambda_n)\} e^{i\lambda_n t}$$

$$= -2ip(x)\frac{\partial H(x,t)}{\partial t} + q(x)H(x,t) + \frac{\partial^2 H(x,t)}{\partial t^2}.$$

Therefore $H(x,t)$ is the solution of the following Cauchy problem (initial value problem)

$$\frac{\partial^2 H(x,t)}{\partial t^2} = \frac{\partial^2 H(x,t)}{\partial x^2} + 2ip(x)\frac{\partial H(x,t)}{\partial t} - q(x)H(x,t), \tag{3.3}$$

$$H(x,0) = 0, \qquad \left.\frac{\partial H(x,t)}{\partial t}\right|_{t=0} = i\delta(x). \tag{3.4}$$

Let us reduce the problem (3.3), (3.4) to an integral equation for $H(x,t)$.

For $p(x) \equiv q(x) \equiv 0$ Eq. (3.3) takes the form

$$\frac{\partial^2 H(x,t)}{\partial t^2} = \frac{\partial^2 H(x,t)}{\partial x^2}. \tag{3.5}$$

As is known (D'Alembert's formula) the solution of problem (3.5), (3.4) has the form

$$H_0(x,t) = \frac{i}{2}\int_{x-t}^{x+t} \delta(y)dy. \tag{3.6}$$

Consider now the non-homogeneous equation

$$\frac{\partial^2 H(x,t)}{\partial t^2} - \frac{\partial^2 H(x,t)}{\partial x^2} = g(x,t) \tag{3.7}$$

with a known function $g(x,t)$. Denote by $\widetilde{H}(x,t)$ the solution of Eq. (3.7) satisfying the initial conditions

$$\widetilde{H}(x,0) = 0, \qquad \left.\frac{\partial \widetilde{H}(x,t)}{\partial t}\right|_{t=0} = 0.$$

It is known that

$$\widetilde{H}(x,t) = \frac{1}{2}\iint_{\Delta_{x,t}} g(y,\tau)dyd\tau, \tag{3.8}$$

where $\Delta_{x,t}$ the triangle in the (y,τ) plane with the vertices at the points $(x-t,0)$, (x,t), and $(x+t,0)$.

Then the solution $H(x,t)$ of Eq. (3.7) subject to the initial conditions (3.4) is obtained by the formula

$$\begin{aligned} H(x,t) &= H_0(x,t) + \widetilde{H}(x,t) \\ &= \frac{i}{2}\int_{x-t}^{x+t}\delta(y)dy + \frac{1}{2}\iint_{\Delta_{x,t}} g(y,\tau)dyd\tau. \end{aligned} \tag{3.9}$$

Using the formula (3.9) it is not difficult to get an integral equation for the solution of problem (3.3), (3.4). Indeed, regarding the function $2ip(x)\frac{\partial H(x,t)}{\partial t} - q(x)H(x,t)$ in Eq. (3.3) as a known function and applying the formula (3.9) we obtain the following integro-differential equation which is equivalent to problem (3.3), (3.4):

$$\begin{aligned} H(x,t) &= \frac{i}{2}\int_{x-t}^{x+t}\delta(y)dy \\ &+\frac{1}{2}\int_0^t d\tau\int_{x-(t-\tau)}^{x+(t-\tau)}\left\{2ip(y)\frac{\partial H(y,\tau)}{\partial\tau} - q(y)H(y,\tau)\right\}dy. \end{aligned} \tag{3.10}$$

Further, taking into account that

$$\begin{aligned} &\frac{\partial}{\partial\tau}\int_{x-(t-\tau)}^{x+(t-\tau)} ip(y)H(y,\tau)dy = -ip(x+t-\tau)H(x+t-\tau,\tau) \\ &-ip(x-t+\tau)H(x-t+\tau,\tau) + \int_{x-(t-\tau)}^{x+(t-\tau)} ip(y)\frac{\partial H(y,\tau)}{\partial\tau}dy, \end{aligned}$$

we have

$$\begin{aligned} &\int_0^t d\tau\int_{x-(t-\tau)}^{x+(t-\tau)} ip(y)\frac{\partial H(y,\tau)}{\partial\tau}dy = i\int_0^t p(x+t-\tau)H(x+t-\tau,\tau)d\tau \\ &+i\int_0^t p(x-t+\tau)H(x-t+\tau,\tau)d\tau + \left.\int_{x-(t-\tau)}^{x+(t-\tau)} ip(y)H(y,\tau)dy\right|_{\tau=0}^{\tau=t}. \end{aligned}$$

Besides

$$\int_{x-(t-\tau)}^{x+(t-\tau)} ip(y)H(y,\tau)dy\Bigg|_{\tau=0}^{\tau=t} = -\int_{x-t}^{x+t} ip(y)H(y,0)dy = 0$$

by (3.4). Substituting these in the right-hand side of (3.10), we get

$$H(x,t) = \frac{i}{2}\int_{x-t}^{x+t}\delta(y)dy + i\int_0^t p(x+t-\tau)H(x+t-\tau,\tau)d\tau$$

$$+i\int_0^t p(x-t+\tau)H(x-t+\tau,\tau)d\tau - \frac{1}{2}\int_0^t d\tau\int_{x-(t-\tau)}^{x+(t-\tau)} q(y)H(y,\tau)dy. \quad (3.11)$$

If $t < x$, then $x - t > 0$ and hence

$$\int_{x-t}^{x+t}\delta(y)dy = 0.$$

Consequently, we get from (3.11) that

$$H(x,t) = i\int_0^t p(x+t-\tau)H(x+t-\tau,\tau)d\tau + i\int_0^t p(x-t+\tau)H(x-t+\tau,\tau)d\tau$$

$$-\frac{1}{2}\int_0^t d\tau\int_{x-(t-\tau)}^{x+(t-\tau)} q(y)H(y,\tau)dy \quad \text{for} \quad 0 \le t < x.$$

The last equation is a Volterra type linear homogeneous integral equation for fixed x and, therefore, it has only the trivial solution, i.e. $H(x,t) = 0$ for $0 \le t < x$. □

Theorem 3.1. *The kernels $A(x,t)$ and $B(x,t)$ involved in the representation (2.2) of $\varphi(x,\lambda)$ satisfy the following system of linear integral equations:*

$$F_{11}(x,t)\cos\alpha(x) + F_{12}(x,t)\sin\alpha(x) + A(x,t)$$

$$+\int_0^x A(x,\xi)F_{11}(\xi,t)d\xi + \int_0^x B(x,\xi)F_{12}(\xi,t)d\xi = 0, \quad 0 \le t < x, \quad (3.12)$$

$$F_{21}(x,t)\cos\alpha(x) + F_{22}(x,t)\sin\alpha(x) + B(x,t)$$

$$+\int_0^x A(x,\xi)F_{21}(\xi,t)d\xi + \int_0^x B(x,\xi)F_{22}(\xi,t)d\xi = 0, \quad 0 \le t < x, \quad (3.13)$$

where

$$F_{11}(x,t) = \frac{1}{\pi}\cos c_0x\cos c_0t$$

$$+\sum_n\left\{\frac{1}{2a_n}\cos\lambda_nx\cos\lambda_nt - \frac{1}{\pi}\cos(n+c_0)x\cos(n+c_0)t\right\}, \quad (3.14)$$

$$F_{12}(x,t) = \frac{1}{\pi}\sin c_0x\cos c_0t$$

$$+\sum_n\left\{\frac{1}{2a_n}\sin\lambda_nx\cos\lambda_nt - \frac{1}{\pi}\sin(n+c_0)x\cos(n+c_0)t\right\}, \quad (3.15)$$

$$F_{21}(x,t) = \frac{1}{\pi}\cos c_0x\sin c_0t$$

$$+\sum_n\left\{\frac{1}{2a_n}\cos\lambda_nx\sin\lambda_nt - \frac{1}{\pi}\cos(n+c_0)x\sin(n+c_0)t\right\}, \quad (3.16)$$

$$F_{21}(x,t) = \frac{1}{\pi}\sin c_0 x \sin c_0 t$$
$$+\sum_n \left\{ \frac{1}{2a_n}\sin\lambda_n x \sin\lambda_n t - \frac{1}{\pi}\sin(n+c_0)x\sin(n+c_0)t \right\} \qquad (3.17)$$

in which in infinite sums n runs all the values $\pm 0, \pm 1, \pm 2, \ldots$ and we take convention that $(\pm 0)a = 0$ for any number a.

Proof. To derive integral equations for the functions $A(x,t)$ and $B(x,t)$ we use the equality

$$\frac{\partial H(x,t)}{\partial t} = 0 \quad \text{for} \quad 0 \le t < x,$$

which follows from (3.2). From this equality we get, by (3.1),

$$\sum_n \frac{1}{2a_n}\varphi(x,\lambda_n)e^{i\lambda_n t} = 0 \quad \text{for} \quad 0 \le t < x,$$

which is understood in the sense of distributions (because the series in this equality does not converge in ordinary sense). Substituting here for $\varphi(x,\lambda_n)$ the expression

$$\varphi(x,\lambda_n) = \cos\lambda_n x \cos\alpha(x) + \sin\lambda_n x \sin\alpha(x)$$
$$+\int_0^x A(x,t)\cos\lambda_n t dt + \int_0^x B(x,t)\sin\lambda_n t dt$$

obtained from (2.2) and equating then to zero the real and imaginary parts of obtained equation, we get

$$\sum_n \frac{1}{2a_n}\{[\cos\lambda_n x\cos\alpha(x) + \sin\lambda_n x \sin\alpha(x)]\cos\lambda_n t$$
$$+\int_0^x A(x,\xi)\cos\lambda_n\xi\cos\lambda_n t d\xi + \int_0^x B(x,\xi)\sin\lambda_n\xi\cos\lambda_n t d\xi\} = 0,$$
$$\sum_n \frac{1}{2a_n}\{[\cos\lambda_n x\cos\alpha(x) + \sin\lambda_n x \sin\alpha(x)]\sin\lambda_n t$$
$$+\int_0^x A(x,\xi)\cos\lambda_n\xi\sin\lambda_n t d\xi + \int_0^x B(x,\xi)\sin\lambda_n\xi\sin\lambda_n t d\xi\} = 0,$$

for $0 \le t < x$. To have in the last equations series convergent in ordinary sense we rewrite these equations in the form

$$\sum_n \left\{ \frac{1}{2a_n}\cos\lambda_n x\cos\lambda_n t - \frac{1}{\pi}\cos(n+c_0)x\cos(n+c_0)t \right\}\cos\alpha(x)$$
$$+\sum_n \left\{ \frac{1}{2a_n}\sin\lambda_n x\cos\lambda_n t - \frac{1}{\pi}\sin(n+c_0)x\cos(n+c_0)t \right\}\sin\alpha(x)$$
$$+\int_0^x A(x,\xi)\left\{ \sum_n \left[\frac{1}{2a_n}\cos\lambda_n\xi\cos\lambda_n t - \frac{1}{\pi}\cos(n+c_0)\xi\cos(n+c_0)t \right] \right\} d\xi$$
$$+\int_0^x B(x,\xi)\left\{ \sum_n \left[\frac{1}{2a_n}\sin\lambda_n\xi\cos\lambda_n t - \frac{1}{\pi}\sin(n+c_0)\xi\cos(n+c_0)t \right] \right\} d\xi$$

$$+\left\{\sum_n \frac{1}{\pi}\cos(n+c_0)x\cos(n+c_0)t\right\}\cos\alpha(x)$$
$$+\left\{\sum_n \frac{1}{\pi}\sin(n+c_0)x\cos(n+c_0)t\right\}\cos\alpha(x)$$
$$+\int_0^x A(x,\xi)\left\{\sum_n \frac{1}{\pi}\cos(n+c_0)\xi\cos(n+c_0)t\right\}d\xi$$
$$+\int_0^x B(x,\xi)\left\{\sum_n \frac{1}{\pi}\sin(n+c_0)\xi\cos(n+c_0)t\right\}d\xi, \tag{3.18}$$

$$\sum_n\left\{\frac{1}{2a_n}\cos\lambda_n x\sin\lambda_n t-\frac{1}{\pi}\cos(n+c_0)x\sin(n+c_0)t\right\}\cos\alpha(x)$$
$$+\sum_n\left\{\frac{1}{2a_n}\sin\lambda_n x\sin\lambda_n t-\frac{1}{\pi}\sin(n+c_0)x\sin(n+c_0)t\right\}\sin\alpha(x)$$
$$+\int_0^x A(x,\xi)\left\{\sum_n\left[\frac{1}{2a_n}\cos\lambda_n\xi\sin\lambda_n t-\frac{1}{\pi}\cos(n+c_0)\xi\sin(n+c_0)t\right]\right\}d\xi$$
$$+\int_0^x B(x,\xi)\left\{\sum_n\left[\frac{1}{2a_n}\sin\lambda_n\xi\sin\lambda_n t-\frac{1}{\pi}\sin(n+c_0)\xi\sin(n+c_0)t\right]\right\}d\xi$$
$$+\left\{\sum_n \frac{1}{\pi}\cos(n+c_0)x\sin(n+c_0)t\right\}\cos\alpha(x)$$
$$+\left\{\sum_n \frac{1}{\pi}\sin(n+c_0)x\sin(n+c_0)t\right\}\sin\alpha(x)$$
$$+\int_0^x A(x,\xi)\left\{\sum_n \frac{1}{\pi}\cos(n+c_0)\xi\sin(n+c_0)t\right\}d\xi$$
$$+\int_0^x B(x,\xi)\left\{\sum_n \frac{1}{\pi}\sin(n+c_0)\xi\sin(n+c_0)t\right\}d\xi, \tag{3.19}$$

for $0\le t<x$. Next, since

$$\sum_n \cos nx\sin nt=0,$$

$$-\frac{1}{\pi}+\sum_n\frac{1}{\pi}\cos nx\cos nt=\delta(x-t),\quad \sum_n\frac{1}{\pi}\sin nx\sin nt=\delta(x-t),$$

it is easy to show that

$$\sum_n\frac{1}{\pi}\cos(n+c_0)x\cos(n+c_0)t=\frac{1}{\pi}\cos c_0x\cos c_0t+\delta(x-t)\cos c_0(x-t), \tag{3.20}$$

$$\sum_n\frac{1}{\pi}\cos(n+c_0)x\sin(n+c_0)t=\frac{1}{\pi}\cos c_0x\sin c_0t+\delta(x-t)\sin c_0(x-t), \tag{3.21}$$

$$\sum_n\frac{1}{\pi}\sin(n+c_0)x\sin(n+c_0)t=\frac{1}{\pi}\sin c_0x\sin c_0t+\delta(x-t)\cos c_0(x-t)). \tag{3.22}$$

From (3.18) and (3.19), taking into account equations (3.20)–(3.22) and notations (3.14)–(3.17) we get the statement of the theorem. □

4. Algorithm for solving the inverse problem

The equations (3.12), (3.13) are called the *fundamental equations* of the inverse problem. They allow to solve the inverse problem and prove Theorem 2.2 as follows.

Let a collection of numbers $\{\lambda_n,\ a_n\}$ be given that satisfies the conditions of Theorem 2.2. Using this collection we construct the functions $F_{jk}(x,t)$ $(j,k=1,2)$ by (3.14)–(3.17) and consider for each fixed x the system of Fredholm linear integral equations (3.12), (3.13) with respect to unknown functions $A(x,t),\ B(x,t)$ assuming $\alpha(x)$ in these equations an arbitrarily given function. It turns out that these equations are uniquely solvable and dependence of its solution on $\alpha(x)$ can be expressed explicitly.

Theorem 4.1. *For any continuous function $\alpha(x)$ the system of integral equations (3.12), (3.13) has a unique solution $A(x,t),\ B(x,t)$ and dependence of this solution on the function $\alpha(x)$ is expressed by the formulas*

$$A(x,t) = A_0(x,t)\cos\alpha(x) + A_1(x,t)\sin\alpha(x), \tag{4.1}$$

$$B(x,t) = B_0(x,t)\cos\alpha(x) + B_1(x,t)\sin\alpha(x), \tag{4.2}$$

where $A_0(x,t),\ B_0(x,t)$ form the solution of system (3.12), (3.13) with $\alpha(x)\equiv 0$,

$$\begin{gathered} F_{11}(x,t) + A_0(x,t) \\ + \int_0^x A_0(x,\xi)F_{11}(\xi,t)d\xi + \int_0^x B_0(x,\xi)F_{12}(\xi,t)d\xi = 0, \quad 0 \le t < x, \end{gathered} \tag{4.3}$$

$$\begin{gathered} F_{21}(x,t) + B_0(x,t) \\ + \int_0^x A_0(x,\xi)F_{21}(\xi,t)d\xi + \int_0^x B_0(x,\xi)F_{22}(\xi,t)d\xi = 0, \quad 0 \le t < x, \end{gathered} \tag{4.4}$$

and $A_1(x,t),\ B_1(x,t)$ form the solution of system (3.12), (3.13) with $\alpha(x)\equiv \pi/2$,

$$\begin{gathered} F_{12}(x,t) + A_1(x,t) \\ + \int_0^x A_1(x,\xi)F_{11}(\xi,t)d\xi + \int_0^x B_1(x,\xi)F_{12}(\xi,t)d\xi = 0, \quad 0 \le t < x, \end{gathered} \tag{4.5}$$

$$\begin{gathered} F_{22}(x,t) + B_1(x,t) \\ + \int_0^x A_1(x,\xi)F_{21}(\xi,t)d\xi + \int_0^x B_1(x,\xi)F_{22}(\xi,t)d\xi = 0, \quad 0 \le t < x. \end{gathered} \tag{4.6}$$

We need to get an equation for $\alpha(x)$. In the part of direct spectral problem we have relation (2.3). Substituting (4.1), (4.2) in this relation we get for $\alpha(x)$ the nonlinear Volterra integral equation

$$\alpha(x) = xp(0) + \int_0^x \Phi(\xi,\alpha(\xi))d\xi, \tag{4.7}$$

where

$$\Phi(\xi,z) = 2A_1(\xi,\xi)\sin^2 z - 2B_0(\xi,\xi)\cos^2 z + [A_0(\xi,\xi) - B_1(\xi,\xi)]\sin 2z. \tag{4.8}$$

Thus we get the following algoritm for solution of the inverse problem.

Given a collection of numbers $\{\lambda_n,\ a_n\}$ satisfying the conditions of Theorem 2.2, we construct the functions $F_{jk}(x,t)$ $(j,k=1,2)$ by (3.14)–(3.17) and consider the two sistems of equations (4.3), (4.4) and (4.5), (4.6) with respect to $A_0(x,t)$, $B_0(x,t)$ and $A_1(x,t)$, $B_1(x,t)$, respectively. Solving these systems we find $A_0(x,t)$, $B_0(x,t)$ and $A_1(x,t)$, $B_1(x,t)$. Then we form the function $\Phi(\xi,z)$ by (4.8) and consider equation (4.7) for $\alpha(x)$ where the number $p(0)$ is taken from $p(0)=-2\alpha_1/\pi$ according to (2.12) with α_1 given in (2.18). Solving this equation we find $\alpha(x)$ and then $p(x)$ by $p(x)=\alpha'(x)$ according to (2.6). Next, define $A(x,t)$, $B(x,t)$ by (4.1), (4.2) and then $q(x)$ by (2.4). The number h is defined by $h=A(0,0)$ according to (2.5) and the number H is defined from (2.11) with c_1 given in (2.17).

These reasonings prove, in particular, the uniqueness of solution of the inverse problem: the coefficient functions $p(x)$, $q(x)$ of Eq. (1.1) and the numbers h, H in boundary conditions (1.2) are determined uniquely from the spectral data $\{\lambda_n,\ a_n\}$ of the boundary value problem (1.1), (1.2). A complete proof of Theorem 2.2 will be presented by the author elsewhere.

References

[1] F. V. Atkinson, *Discrete and Continuous Boundary Problems*, Academic Press, New York, 1964.

[2] Yu. M. Berezanskii, *Expansions in Eigenfunctions of Self-Adjoint Operators*, Transl. Math. Monographs, vol. 17, Amer. Math. Soc., Providence, RI, 1968.

[3] M. G. Gasymov, The inverse scattering problem for a system of Dirac equations of order $2n$, *Trudy Moskov. Mat. Obsc.* **19** (1968), 41–119 (Russian); English transl., *Trans. Moscow Math. Soc.* **19** (1968).

[4] M. G. Gasymov, On the spectral theory of differential operators polynomially depending on a parameter, *Uspekhi Mat. Nauk* **37**(4) (1982), p. 99 (Russian).

[5] M. G. Gasymov, The inverse problem on the half-line from the spectral function, Chapter 12 in B. M. Levitan and I. S. Sargsjan, *Sturm-Liouville and Dirac Operators*, Kluwer, Dordrecht, 1991.

[6] M. G. Gasymov and G. Sh. Guseinov, Determination of a diffusion operator from spectral data, *Dokl. Akad. Nauk Azerbaijan SSR* **37**(2) (1981), 19–22 (Russian).

[7] M. G. Gasymov and G. Sh. Guseinov, Some uniqueness theorems in inverse problems of spectral analysis for Sturm-Liouville opeartors in the case of the Weyl limit circle, *Differentsial'nye Uravneniya* **25**(4) (1989), 588–599 (Russian); English transl., *Differential Equations* **25**(4) (1989), 394–402.

[8] M. G. Gasymov and G. Sh. Guseinov, On inverse problems of spectral analysis for infinite Jacobi matrices in the limit-circle case, *Dokl. Akad. Nauk SSSR* **309** (1989), 1293–1296 (Russian); English transl. *Soviet Math. Dokl.* **40** (1990), 627–630.

[9] M. G. Gasymov and B. M. Levitan, Determination of a differential equation by two of its spectra, *Uspekhi Mat. Nauk* **19**(2) (1964), 3–63 (Russian); English transl., *Russ. Math. Surv.* **19** (1964), 1–63.

[10] M. G. Gasymov and B. M. Levitan, The inverse problem for the Dirac system, *Dokl. Akad. Nauk SSSR* **167** (1966), 967–970 (Russian).

[11] I. M. Gelfand and B. M. Levitan, On the determination of a differential equation from its spectral function, *Izv. Akad. Nauk, Ser. Mat.* **15** (1951), 309–360 (Russian); English transl., *Amer. Math. Soc. Transl.* (2) **1** (1955) 253–304.

[12] G. Sh. Guseinov, The determination of an infinite Jacobi matrix from the scattering data, *Dokl. Akad. Nauk SSSR* **227** (1976), 1289–1292 (Russian); English transl., *Soviet Math. Dokl.* **17** (1976), 596–600.

[13] G. Sh. Guseinov, The inverse problem of scattering theory for a second-order difference equation on the whole axis, *Dokl. Akad. Nauk SSSR* **231** (1976), 1045–1048 (Russian); English transl., *Soviet Math. Dokl.* **17** (1976), 1684–1688.

[14] G. Sh. Guseinov, The scattering problem for an infinite Jacobi matrix, *Izv. Akad. Nauk Armyan. SSR, Ser. Mat.* **12** (1977), 365–379 (Russian).

[15] G. Sh. Guseinov, A quadratic bundle of Sturm-liouville operators with periodic coefficients, *Vestnik. Moskov. Univ. Ser. I Mat. Mekh.* no. 3 (1984), 14–21 (Russian); English transl., *Moscow Univ. Math. Bull.* 39(3) (1984), 17–25.

[16] G. Sh. Guseinov, On the spectral analysis of a quadratic pencil of Sturm-Liouville operators, *Dokl. Akad. Nauk SSSR* **285** (1985), 1292–1296 (Russian); English transl., *Soviet Math. Dokl.* **32** (1985), 859–862.

[17] G. Sh. Guseinov, The spectrum and eigenfunction expansions of a quadratic operator pencil of Sturm-Liouville operators with periodic coefficients, in *Spectral Theory of Operators and Its Applications*, no. 6 (1985), 56–97, “Elm”, Baku (Russian).

[18] G. Sh. Guseinov, Inverse spectral problems for a quadratic pencil of Sturm-Liouville operators on a finite interval, in *Spectral Theory of Operators and Its Applications*, no. 7 (1986), 51–101, “Elm”, Baku (Russian).

[19] F. G. Maksudov and G. Sh. Guseinov, On the solution of the inverse scattering problem for a quadratic pencil of one-dimensional Schrödinger operators on the whole axis, *Dokl. Akad. Nauk SSSR* **289** (1986), 42–46 (Russian); English transl., *Soviet Math. Dokl.* **34** (1987), 34–38

[20] F. G. Maksudov and G. Sh. Guseinov, An inverse scattering problem for a quadratic pencil of Sturm-Liouville operators on the full line, in *Spectral Theory of Operators and Its Applications*, no. 9 (1989), 176–211, “Elm”, Baku (Russian).

[21] V. A. Marchenko, *Sturm-Liouville Operators and Applications,* Birkhäuser, Basel, 1986.

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Received: June 23, 2014; Accepted: July 14, 2014