Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 40, Special Issue, 2014, Pages 215–225

AN INVERSE SCATTERING PROBLEM FOR A SYSTEM OF DIRAC EQUATIONS WITH DISCONTINUITY CONDITIONS

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In memory of M. G. Gasymov on his 75th birthday

Abstract. We solve an inverse scattering problem for the Dirac system with discontinuity conditions at a point.

1. Introduction

Let's consider a system of Dirac equations

$$By' + \Omega(x)\,y = \lambda y, \quad 0 < x < \infty, \tag{1}$$

with discontinuity conditions at some point $a \in (0, \infty)$

$$y(a-0) = My(a+0) \tag{2}$$

and with the boundary condition

$$y_1(0) = 0. \tag{3}$$

Here

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ M = \begin{pmatrix} \alpha & 0 \\ \beta & \alpha^{-1} \end{pmatrix}, \Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix},\ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{4}$$

α, β are real numbers $\alpha \neq 0$, $p(x)$, $q(x)$ are real-valued functions, satisfying the condition

$$\int_0^\infty \|\Omega(x)\|\,dx < \infty, \tag{5}$$

where $\|\cdot\|$ is the operator norm in the Euclidean space $\mathbb{C}^2$.

There exists the solution of problem (1)-(3) (see formula (16)) $u(x,\lambda)$ such that it holds the Parseval equality

$$\frac{1}{\pi}\int_{-\infty}^{\infty} u(x,\lambda)\,u^*(t,\lambda)\,d\lambda = \delta(x-t)\,E_2,$$

2010 *Mathematics Subject Classification.* 34L25, 34L40, 47A40.
Key words and phrases. inverse scattering problems, system of Dirac equations, discontinuity conditions, scattering function, Jost solution.

where $E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, δ is the Dirac delta-function, and as $x \to +\infty$

$$u(x, \lambda) = \left\{ \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{i\lambda x} - \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-i\lambda x} S(x) \right\} [1 + o(1)],$$

the function $S(\lambda)$ is called a scattering function of boundary value problem (1)-(3).

Obviously, for determining asymptotic behavior of the normed generalized eigen function $u(x, \lambda)$ it suffices to know the scattering function . Therefore, the inverse problem of the scattering theory for a boundary value problem is formulated as follows. Knowing the scattering function $S(x)$ show the way for determining the potential $\Omega(x)$ and find necessary and sufficient conditions for the pregiven function be the scattering function of the problem as (1)-(3). In the present paper this problem is completely solved.

When there are no discontinuity conditions i.e. when $M = E_2$, the inverse scattering problems for the system of Dirac equations of mass, and also for the system of order $2n$ Dirac equations with general self-adjoint boundary conditions were solved in the papers [1], [4]. Note also the papers [2], [3] and others where the inverse problems for the system of Dirac equations are considered in other statements.

2. On the Jost solution

We call the matrix function $E(x, \lambda)$ satisfying equation (1), condition (2) and the condition at infinity $\lim\limits_{x \to +\infty} E(x, \lambda) e^{\lambda B x} = E_2$ the Jost solution. It is easy to show that if $\Omega(x) \equiv 0$, then the Jost solution is the function

$$E_0(x, \lambda) = \begin{cases} e^{-\lambda B x}, & x > a, \\ M^- e^{-\lambda B x} + M^+ e^{-\lambda(2a - x)B}, & 0 < x < a, \end{cases}$$

where

$$M^{\pm} = \frac{1}{2}(M \pm BMB) = \begin{pmatrix} \alpha^{\mp} \pm B \\ \beta \mp \alpha^{\mp} \end{pmatrix}, \ \alpha^{\pm} = \frac{1}{2}\left(\alpha \pm \alpha^{-1}\right).$$

Theorem 1. *Under condition (5) equation (1) with discontinuity condition (2) for all real λ has the Jost solution $E(x, \lambda)$ representable in the form*

$$E(x, \lambda) = E_0(x, \lambda) + \int_x^{+\infty} K(x, t) e^{-\lambda B t} dt, \tag{6}$$

and the kernel $K(x, t)$ satisfies the inequality

$$\int_x^{+\infty} \|K(x, t)\| dt \le e^{2C\sigma(x)} - 1, \tag{7}$$

where

$$C = \max\left(1, \left\|M^-\right\|, \left\|M^+\right\|\right), \quad \sigma(x) = \int_x^{+\infty} \|\Omega(s)\| ds.$$

Furthermore, the following relations are fulfilled

$$\lim_{t\to+0}\int\limits_a^{+\infty}\|BK(x,x+t)-K(x,x+t)B-\Omega(x)\|\,dx=0,$$

$$\lim_{t\to+0}\int\limits_0^{a}\left\|BK(x,x+t)-K(x,x+t)B-\Omega(x)M^-\right\|dx=0, \tag{8}$$

$$\lim_{\delta\to+0}\int\limits_a^{a}\left\|\left[B,K(x,t)\right]\Big|_{t=2a-x-\delta}^{2a-x+\delta}-\Omega(x)M^+\right\|dx=0.$$

Proof. It is easy to show that the Jost solution (if it exists) satisfies the integral equation

$$E(x,\lambda)=E_0(x,\lambda)-\int\limits_x^{+\infty}E_0(x,\lambda)E_0^{-1}(t,\lambda)B\Omega(t)E(t,\lambda)\,dt.$$

Substituting here instead of $E(x,\lambda)$ its representation of the form (6), we get

$$\int\limits_x^{+\infty}K(x,t)e^{-\lambda Bt}dt=-\int\limits_x^{+\infty}E_0(x,\lambda)E_0^{-1}(t,\lambda)B\Omega(t)E_0(t,\lambda)\,dt-$$

$$\int\limits_x^{+\infty}E_0(x,\lambda)E_0^{-1}(t,\lambda)B\Omega(t)\int\limits_t^{+\infty}K(x,\lambda)e^{-\lambda Bs}dsdt\quad(\lambda\in\mathbb{R}). \tag{9}$$

Suppose $x>0$. Then relation (9) takes the form:

$$\int\limits_x^{+\infty}K(x,t)e^{-\lambda Bt}dt=-\int\limits_x^{+\infty}e^{-\lambda B(x-t)}B\Omega(t)e^{-\lambda Bt}dt-$$

$$\int\limits_x^{+\infty}e^{-\lambda B(x-t)}B\Omega(t)\int\limits_t^{+\infty}K(t,s)e^{-\lambda Bs}sdsdt.$$

Hence it is easy to get the following relations for the functions $K^{\pm}(x,t)=\frac{1}{2}(K(x,t)\pm BK(x,t)B)$:

$$\int\limits_x^{+\infty}K^+(x,t)e^{-\lambda Bt}dt=\frac{1}{2}\int\limits_x^{+\infty}\Omega\left(\frac{x+t}{2}\right)Be^{-\lambda Bt}dt+$$

$$\int\limits_x^{+\infty}e^{-\lambda B(x-t)}\Omega(t)\int\limits_t^{+\infty}K^-(t,s)e^{-\lambda Bs}dsdt,$$

$$\int\limits_x^{+\infty}K^-(x,t)e^{-\lambda Bt}dt=-\int\limits_x^{+\infty}e^{-\lambda B(x-t)}B\Omega(t)\int\limits_x^{+\infty}K^+(t,s)e^{-\lambda Bs}dsdt.$$

Hence, using

$$BK^{\pm}(x,t) = \mp K^{\pm}(x,t)\,B, \quad e^{-\lambda Bx}K^{\pm}(x,t) = K^{\pm}(x,t)\,e^{\pm\lambda Bx},$$

we finally get integral equations for the matrix-functions $K^{\pm}(x,t)$ for $x > a$:

$$\begin{cases} K^{+}(x,t) = \frac{1}{2}\Omega\left(\frac{x+t}{2}\right)B + \int\limits_{x}^{\frac{x+t}{2}} \Omega(\xi)\,BK^{-}(\xi, t+x-\xi)\,d\xi, \\ K^{-}(x,t) = \int\limits_{x}^{+\infty} \Omega(\xi)\,BK^{+}(\xi, t-x+\xi)\,d\xi. \end{cases} \tag{10}$$

Now consider the case $0 < x < a$. In this case we have

$$E_0(x,\lambda) = M^{-}e^{-\lambda Bx} + M^{+}e^{-\lambda(2a-x)B},$$

$$E_0(x,\lambda)\,E_0^{-1}(t,\lambda) = \begin{cases} e^{-\lambda B(x-t)}, & for \quad a<x<t \quad or \quad x<t<a, \\ M^{-}e^{-\lambda B(x-t)} + M^{+}e^{-\lambda B(2a-x-t)}, & for \quad x<a<t. \end{cases}$$

Therefore, proceeding from relation (9), similar to the case mentioned above for the matrix-function $K^{\pm}(x,t)$ we get the integral equations:

$$\begin{cases} K^{+}(x,t) = -\frac{1}{2}B\Omega\left(\frac{x+t}{2}\right)M^{-} - \int\limits_{x}^{\frac{x+t}{2}} B\Omega(s)\,K^{-}(s,t+x-s)\,ds, \\ K^{-}(x,t) = -\frac{1}{2}B\Omega\left(\frac{x+2a-t}{2}\right)M^{+} - \frac{1}{2}M^{+}B\Omega\left(\frac{t-x+2a}{2}\right) - \\ \int\limits_{x}^{a} B\Omega(s)\,K^{+}(s,t-x+s)\,ds - \int\limits_{a}^{+\infty} M^{-}B\Omega(s)\,K^{+}(s,t-x+s)\,ds - \\ \int\limits_{a}^{\frac{t+2a-x}{2}} M^{+}B\Omega(s)\,K^{-}(s,t+2a-x-s)\,ds, \quad 0<x<a, \quad x<t<2a-x, \end{cases}$$

$$\begin{cases} K^{+}(x,t) = -\frac{1}{2}M^{-}B\Omega\left(\frac{x+t}{2}\right) - \int\limits_{a}^{\frac{t+x}{2}} M^{-}B\Omega(s)\,K^{-}(s,t+x-s)\,ds - \\ \int\limits_{x}^{a} B\Omega(s)\,K^{-}(x,t+x-s)\,ds - \int\limits_{a}^{+\infty} M^{+}B\Omega(s)\,K^{+}(s,t-2a+x+s)\,ds, \\ K^{-}(x,t) = -\frac{1}{2}M^{+}B\Omega\left(\frac{t-x+2a}{2}\right) - \int\limits_{x}^{a} B\Omega(s)\,K^{+}(s,t-x+s)\,ds - \\ \int\limits_{a}^{+\infty} M^{-}B\Omega(s)\,K^{+}(s,t-x+s)\,ds - \int\limits_{a}^{\frac{t+2a-x}{2}} M^{+}B\Omega(s)\,K^{-}(s,t+2a-x-s)\,ds, \\ 0<x<a, \quad 2a-x<t<\infty. \end{cases} \tag{11}$$

For proving the existence of the solution $E(x,\lambda)$ it suffices to show that the systems of equations (10), (11) have the solutions $K^{\pm}(x,t)$ satisfying the inequalities

$$\int\limits_{x}^{+\infty} \left\|K^{\pm}(x,t)\right\| dt \le \frac{1}{2}\left\{e^{2C\sigma(x)} - 1\right\}, \tag{12}$$

and hence for the kernel $K(x,t) = K^{+}(x,t) + K^{-}(x,t)$ estimation (7) will follow.

Assume

$$K_0^{+}(x,t) = \begin{cases} \frac{1}{2}\Omega\left(\frac{x+t}{2}\right)B, & x>a, \quad t>s, \\ \frac{1}{2}\Omega\left(\frac{x+t}{2}\right)BM^{-}, & 0<x<a, \quad x<t<2a-x, \\ \frac{1}{2}M^{-}\Omega\left(\frac{x+t}{2}\right)B, & 0<x<a, \quad t>2a-x, \end{cases}$$

$$K_0^-(x,t) =$$

$$\begin{cases} 0, \quad x>a, \quad t>s, \\ \frac{1}{2}\Omega\left(\frac{x+2a-t}{2}\right)BM^+ + \frac{1}{2}M^+\Omega\left(\frac{t+2a-x}{2}\right)B, \quad 0<x<a, \quad x<t<2a-x, \\ \frac{1}{2}M^+\Omega\left(\frac{t+2a-x}{2}\right)B, \quad 0<x<a, \quad t>2a-x, \end{cases}$$

$$K_n^+(x,t) =$$

$$\begin{cases} \int\limits_x^{\frac{x+t}{2}} \Omega(s)BK_{n-1}^-(s,t+x-s)\,ds, x>a, t>x \text{ or } 0<x<a, x<t<2a-x, \\ \int\limits_a^{\frac{t+x}{2}} M^-\Omega(s)BK_{n-1}^-(s,t+x-s)\,ds + \int\limits_x^{a}\Omega(s)BK_{n-1}^-(s,t+x-s)\,ds+ \\ + \int\limits_a^{+\infty} M^+\Omega(s)BK_{n-1}^+(s,t-2a+x+s)\,ds, \quad 0<x<a, \quad t>2a-x, \end{cases}$$

$$K_n^-(x,t) =$$

$$\begin{cases} \int\limits_x^{+\infty} \Omega(s)BK_{n-1}^+(s,t-x+s)\,ds, \quad x>a, \quad t>x, \\ \int\limits_x^{a}\Omega(s)BK_{n-1}^+(s,t-x+s)\,ds + \int\limits_a^{+\infty} M^-\Omega(s)BK_{n-1}^+(s,t-x+s)\,ds+ \\ + \int\limits_a^{\frac{t+2a-x}{2}} M^+\Omega(s)BK_{n-1}^-(s,t+2a-x-s)\,ds, \ 0<x<a, \ x<t<2a-x, \\ \int\limits_x^{a}\Omega(s)BK_{n-1}^+(s,t-x+s)\,ds + \int\limits_a^{+\infty} M^-\Omega(s)BK_{n-1}^+(s,t-x+s)\,ds+ \\ + \int\limits_a^{\frac{t+2a-x}{2}} M^+\Omega(s)BK_{n-1}^-(s,t+2a-x-s)\,ds, \ 0<x<a, \ t>2a-x, \end{cases}$$

From the definition $K_n^\pm(x,t)\,(n=0,1,2,...)$ it follows

$$\int\limits_x^{+\infty} \left\|K_0^\pm(x,t)\right\|dt \le C\int\limits_x^{+\infty}\|\Omega(s)\|\,ds = C\sigma(x),$$

$$\int\limits_x^{+\infty} \left\|K_n^\pm(x,t)\right\|dt \le C\int\limits_x^{+\infty}\|\Omega(s)\|\left(\int\limits_s^{\infty}\left\|K_{n-1}^+(s,\xi)\right\|d\xi + \int\limits_s^{\infty}\left\|K_{n-1}^-(s,\xi)\right\|d\xi\right)d\xi.$$

Applying the mathematical induction method, we have

$$\int\limits_x^{+\infty} \left\|K_n^\pm(x,t)\right\|dt \le \frac{2^nC^{n+1}\{\sigma(x)\}^{n+1}}{(n+1)!}.$$

Hence it follows that the matrix series $\sum\limits_{n=0}^{\infty} K_n^\pm(x,\cdot) = K^\pm(x,\cdot)$ converge uniformly with respect to $x\in(0,\infty)$ in the space L_1 and estimations (12) are fulfilled.

Now prove relations (8). Write the first equation from the system (10) in the form

$$-2BK^{+}(x,t)=\Omega\left(\frac{x+t}{2}\right)+2\int\limits_{x}^{\frac{x+t}{2}}\Omega(s)\,K^{-}(s,t+x-s)\,ds$$

or having made a substitution $t\to x+t$, in the form

$$-BK(x,x+t)+K(x,x+t)\,B=\Omega\left(x+\frac{t}{2}\right)+2\int\limits_{x}^{x+\frac{t}{2}}\Omega(s)\,K^{-}(s,2x+t-s)\,ds.$$

Hence

$$\int\limits_{x}^{+\infty}\left\|K(x,x+t)\,B-BK(x,x+t)-\Omega(x)\right\|dx\le\int\limits_{0}^{\infty}\left\|\Omega\left(x+\frac{t}{2}\right)-\Omega(x)\right\|dx$$

$$+2\int\limits_{0}^{+\infty}\left\|\int\limits_{x}^{x+\frac{t}{2}}\Omega(s)\,K^{-}(s,2x+t-s)\,ds\right\|dx=I_1(t)+I_2(t).$$

In order to set up the first relation from (8), it suffices to show that

$$\lim_{t\to+0}I_2(t)=0. \tag{13}$$

Changing the integration order and making change of variables $\xi=2x+t-s$, we get

$$I_2(t)\le2\int\limits_{0}^{+\infty}\left(\int\limits_{x}^{x+\frac{t}{2}}\|\Omega(s)\|\left\|K^{-}(s,2x+t-s)\right\|ds\right)dx=$$

$$\int\limits_{0}^{\frac{t}{2}}\|\Omega(s)\|\left(\int\limits_{t-s}^{t+s}\left\|K^{-}(s,\xi)\right\|d\xi\right)ds+\int\limits_{\frac{t}{2}}^{\infty}\|\Omega(s)\|\left(\int\limits_{s}^{s+t}\left\|K^{-}(s,\xi)\right\|d\xi\right)ds. \tag{14}$$

According to estimations (12), we have:

$$\int\limits_{0}^{\frac{t}{2}}\|\Omega(s)\|\int\limits_{t-s}^{t+s}\left\|K^{-}(s,\xi)\right\|d\xi ds\le$$

$$\int\limits_{0}^{\frac{t}{2}}\|\Omega(s)\|\int\limits_{s}^{\infty}\left\|K^{-}(s,\xi)\right\|d\xi ds\le\frac{1}{2}\left\{e^{2C\sigma(0)}-1\right\}\int\limits_{0}^{\frac{t}{2}}\|\Omega(s)\|\,ds,$$

thus, the first summand from the right hand side of (14) tends to zero as $t\to+0$.

In the sequel, since $\varphi_t(s) = \|\Omega(s)\| \int\limits_s^{s+t} \|K^-(s,\xi)\| d\xi \to 0$ as $t \to +0$ and according to (12)

$$\varphi_t(s) \leq \|\Omega(s)\| \int\limits_s^{\infty} \left\|K^-(s,\xi)\right\| d\xi \leq C_1 \|\Omega(s)\|,$$

then by the theorem on limit passage under the sign of integral, we have

$$\lim_{t\to+0} \int\limits_{\frac{t}{2}}^{\infty} \varphi_t(s)\, ds = 0.$$

Thus, (13) is valid, and so the first equality from (8) is valid as well. Proceeding from equations (10), (11), the remaining relations from (8) are proved in the same way. The theorem is proved.

Corollary 1. *The vector-function* $e(x,\lambda)$ *and* $\overline{e(x,\lambda)}$*, where*

$$e(x,\lambda) = e_0(x,\lambda) + \int\limits_x^{+\infty} K(x,t) \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{i\lambda t} dt, \tag{15}$$

$$e_0(x,\lambda) = \begin{cases} \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{i\lambda x}, x > a, \\ M^- e^{i\lambda x} \begin{pmatrix} 1 \\ -i \end{pmatrix} + M^+ e^{i\lambda(2a-x)} \begin{pmatrix} 1 \\ -i \end{pmatrix}, \ 0 < x < a, \end{cases}$$

are a fundamental system of solutions of problem (1), (2).

3. Basic equations of the inverse problem

Let's consider the boundary value problem (1)-(3).

Denote by $\varphi(x,\lambda)$ the solution of equation (1) with discontinuity conditions (2) and initial conditions

$$\varphi_1(0,\lambda) = 0, \quad \varphi_2(0,\lambda) = 1.$$

Lemma 1. For all $\lambda \in \mathbb{R}$ the following identity is valid

$$u(x,\lambda) = \frac{2i\varphi(x,\lambda)}{e_1(0,\lambda)} = \overline{e(x,\lambda)} - S(\lambda)\, e(x,\lambda), \tag{16}$$

where $S(\lambda) = \frac{\overline{e_1(x,\lambda)}}{e_1(x,\lambda)}$ is the scattering function of problem (1)-(3) continuous on the whole axis, and $S(\lambda) - S_0(\lambda) \in L_2(-\infty,+\infty)$. Here

$$S_0(\lambda) = \frac{\alpha^+ - i\beta + (\alpha^- + i\beta)\, e^{-2ia\lambda}}{\alpha^+ + i\beta + (\alpha^- - i\beta)\, e^{2ia\lambda}} \tag{17}$$

is the scattering function of problem (1)-(3), when $\Omega(x) \equiv 0$.

Theorem 2. *The kernel of the representation (6) satisfies the functional-integral equation (the basic equation of the inverse problem)*

$$F_1(x,y) + K(x,y) - K(x,2a-y) \left\{ \frac{\alpha^+\alpha^- + \beta^2}{\alpha^{+2} + \beta^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \right.$$

$$\frac{2\beta\alpha^{-1}}{\alpha^{+2}+\beta}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+\int\limits_{x}^{+\infty} K\left(x,t\right)F_s\left(t+y\right)dt=0, \tag{18}$$

$$F_s\left(x\right)=\operatorname{Re}\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left[S_0\left(\lambda\right)-S\left(\lambda\right)\right]\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}e^{i\lambda x}d\lambda, \tag{19}$$

$$F_1\left(x,y\right)=\begin{cases} F_s\left(x+y\right), & x>a, \\ M^{-}F_s\left(x+y\right)+M^{+}F_s\left(2a-x+y\right), 0<x<a. \end{cases}$$

Proof. Write equality (16) in the case $\Omega\left(x\right)\equiv 0$ $(\lambda\in\mathbb{R})$:

$$\frac{2i\varphi_0\left(x,\lambda\right)}{e_{10}\left(x,\lambda\right)}=\overline{e_0\left(x,\lambda\right)}-S_0\left(\lambda\right)e_0\left(x,\lambda\right). \tag{16_0}$$

From relations (16), (16_0), (13) we have

$$\frac{2i\varphi\left(x,\lambda\right)}{e_i\left(0,\lambda\right)}-\frac{2i\varphi_0\left(x,\lambda\right)}{e_{10}\left(0,\lambda\right)}=\int\limits_{x}^{+\infty}K\left(x,t\right)\begin{pmatrix} 1 \\ i \end{pmatrix}e^{-i\lambda t}dt+$$

$$\left[S_0\left(\lambda\right)-S\left(\lambda\right)\right]e_0\left(x,\lambda\right)+\int\limits_{x}^{+\infty}K\left(x,t\right)\left[S_0\left(\lambda\right)-S\left(\lambda\right)\right]\begin{pmatrix} 1 \\ -i \end{pmatrix}e^{i\lambda t}dt-$$

$$S_0\left(\lambda\right)\int\limits_{x}^{+\infty}K\left(x,t\right)\begin{pmatrix} 1 \\ -i \end{pmatrix}e^{i\lambda t}dt.$$

Multiply the both hand sides of equality at first by $\frac{1}{2\pi}\left(1,-i\right)e^{i\lambda y}$, where $y>x$, and integrate with respect to λ within $-\infty$ and $+\infty$:

$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left\{\frac{2i\varphi\left(x,\lambda\right)}{e_1\left(0,\lambda\right)}-\frac{2i\varphi_0\left(x,\lambda\right)}{e_{10}\left(0,\lambda\right)}\right\}\left(1,-i\right)e^{i\lambda y}d\varphi=$$

$$K\left(x,y\right)\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}+\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left[S_0\left(\lambda\right)-S\left(\lambda\right)\right]e_0\left(x,\lambda\right)\left(1,-i\right)e^{i\lambda y}d\lambda+$$

$$\int\limits_{x}^{+\infty}K\left(x,t\right)\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left[S_0\left(\lambda\right)-S\left(\lambda\right)\right]\begin{pmatrix} 1 \\ -i \end{pmatrix}\left(1,-i\right)e^{i\lambda(t+y)}d\lambda dt-$$

$$\int\limits_{x}^{-\infty}K\left(x,t\right)\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}S_0\left(\lambda\right)\begin{pmatrix} 1 \\ -i \end{pmatrix}\left(1,-i\right)e^{i\lambda(t+y)}d\lambda dt. \tag{20}$$

On the other hand, according to formula (17) we have

$$S_0\left(\lambda\right)=$$

$$\frac{\alpha^{+}-i\beta+\left(\alpha^{-}+i\beta\right)e^{-2ia\lambda}}{\alpha^{+}+i\beta}\left[1-\frac{\alpha^{-}-i\beta}{\alpha^{+}+i\beta}e^{2ia\lambda}+\left(\frac{\alpha^{-}-i\beta}{\alpha^{+}+i\beta}\right)^{2}e^{4ia\lambda}+...\right].$$

Consequently,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_0(\lambda)\, e^{i\lambda(t+y)} dy = \left[\frac{\alpha^+ - i\beta}{\alpha^+ + i\beta} - \frac{\alpha^{-2}+\beta^2}{(\alpha^+ + i\beta)^2}\right]\delta(t+y) +$$

$$\frac{\alpha^- + i\beta}{\alpha^+ + i\beta}\delta(t+y-2a) - \frac{(\alpha^+ - i\beta)(\alpha^- + i\beta)}{(\alpha^+ + i\beta)^2}\delta(t+y+2a) + ...,$$

and since for $y > x$ $K(x,-y) = 0$, $K(x,-2la-y) = 0$, $l = 1,2,...$, then the last summand in the right-hand side of equation (20) takes the form

$$\frac{\alpha^- + i\beta}{\alpha^+ + i\beta} K(x, 2a-y) \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$

In what follows, the subintegrand matrix-function in the left-hand side of equality (20) is regular, bounded in the closed upper half-plane for $y > x$. Then, applying the Jordan lemma, we get that this integral equals zero. Taking into account the above stated ones, and also real-valuededness of the elements of the matrix-function, from relation (20) we get the basic equation (18) of the inverse problem. The theorem is proved.

4. Solvability of the basic equation. The uniqueness theorem of the solution of the inverse problem

Theorem 3. *For any fixed $x \geq 0$ the basic equation (18) has the matrix solution $K(x,\cdot)$ with the elements from $L_2(x,\infty)$.*

Proof. Note that for any fixed $x \geq 0$ the operation

$$M_x(f) =$$

$$\begin{cases} f(y)\,E_2, \quad x > a, \\ f(y)\,E_2 - f(2a-y)\left[\frac{\alpha^+\alpha^- + \beta^2}{\alpha^{+2}+\beta^2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{2\beta\alpha^{-1}}{\alpha^{+2}+\beta^2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right], 0 < x < a, \end{cases}$$

is invertible in the space $L_2(x,\infty;\mathbb{C}^2)$. Therefore the basic equation (18) is equivalent to the following equation with a completely continuous operator:

$$K(x,y) + (M_x)^{-1} F_1(x,y) + (M_x)^{-1}\mathbf{F}K(x,\cdot)(y) = 0,\ y > x.$$

Consequently, in order to prove the theorem, it suffices to show that the homogeneous equation

$$f_x(y) - f_x(2a-y)\left[\frac{\alpha^+\alpha^- + \beta^2}{\alpha^{+2}+\beta^2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{2\beta\alpha^{-1}}{\alpha^{+2}+\beta^2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

$$+\int_{x}^{+\infty} f_x(t)\, F_s(t+y)\, dt = 0, \quad y > x, \tag{21}$$

has only a trivial solution $f_x(\cdot) \in L_2(x,\infty;\mathbb{C}^2)$. Multiply scalarly the equation (21) by $f_x(y)$ and integrate with respect to in the interval (x,∞). As a result we get

$$\int_x^{+\infty}(f_x(y),f_x(y))\,dy+\int_x^{+\infty}(f_x(y),f_x(2a-y))\,dy\left[\frac{\alpha^+\alpha^-+\beta^2}{\alpha^{+2}+\beta^2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right.$$

$$\left.+\frac{2\beta\alpha^{-1}}{\alpha^{+2}+\beta^2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\right]+\int_x^{+\infty}(f_x(y),f_x(t)F_s(t+y))\,dt=0. \qquad (22)$$

In this equation, instead of $F_s(t+y)$ substitute its expression from formula (19) and take into account the relation

$$\int_x^{+\infty}(f_x(y),f_x(2a-y))\,dy\left[\frac{\alpha^+\alpha^-+\beta^2}{\alpha^{+2}+\beta^2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\frac{2\beta\alpha^{-1}}{\alpha^{+2}+\beta^2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\right]=$$

$$\operatorname{Re}\frac{1}{2\pi}\int_{-\infty}^{\infty}S_0(\lambda)\,\Phi(\lambda)\,\widetilde{\Phi}(\lambda)\,d\lambda,$$

where

$$\Phi(\lambda)=\int_x^{\infty}f_x(t)\begin{pmatrix}1\\-i\end{pmatrix}e^{i\lambda t}dt,$$

and the Parseval formula

$$\int_x^{+\infty}(f_x(y),f_x(y))\,dy=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\lambda)\,\Phi^*(\lambda)\,d\lambda.$$

Then equality (22) will take the form

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\left\{\Phi(\lambda)-\overline{\Phi}(\lambda)\,S(\lambda)\right\}\left\{\Phi^*(\lambda)-\widetilde{\Phi}(\lambda)\,\overline{S(\lambda)}\right\}d\lambda=0.$$

Hence we have

$$\Phi(\lambda)=\overline{\Phi}(\lambda)\,S(\lambda).$$

By definition

$$S(\lambda)=\frac{\overline{e_1(0,\lambda)}}{e_1(0,\lambda)}.$$

Then the vector-function $Z(\lambda)=\Phi(\lambda)\,e_1(0,\lambda)$ is regular in the half-plane $\operatorname{Im}\lambda>0$ continuous up to the real axis and satisfies the condition

$$\overline{Z(\lambda)}=Z(\lambda),\quad -\infty<\lambda<\infty.$$

Consequently, $Z(\lambda)$ is an entire vector-function. From the definition of the vector-function $Z(\lambda)$ it follows that it converges to zero as $\lambda\to\infty$. Therefore, by the Liouville theorem $Z(\lambda)\equiv 0$, $\Phi(\lambda)\equiv 0$ and so $f_x(y)=0$. The theorem is proved.

Corollary 2. *The potential $\Omega(x)$ is uniquely determined from the given scattering function $S(\lambda)$.*

The solution of the inverse scattering problem in the class (5) gives

Theorem 4. *For the function $S(\lambda)$, $-\infty < \lambda < \infty$ to be a scattering function of the problem of the form (1)-(4) with real $p(x)$ and $q(x)$, satisfying inequality (5) it is necessary and sufficient that the following conditions to be fulfilled:*

1^0. *the function $S(\lambda)$ is continuous on the whole axis, $\overline{S(\lambda)} = S^{-1}(\lambda)$, each element of the matrix function $F_s(x)$ belongs to $L_2(-\infty, \infty)$ and*

$$\int_0^\infty \|F_s(x)\|\, dx < \infty;$$

2^0. *homogeneous equation*

$$f(y) - f(2a-y)\left[\frac{\alpha^+\alpha^- + \beta^2}{\alpha^{+2}+\beta^2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{2\beta\alpha^{-1}}{\alpha^{+2}+\beta^2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] +$$

$$\int_0^\infty f(t)\, F_s(t+y)\, dt = 0$$

has only a zero vector-solution with the components from $L_2(0, \infty)$;

3^0. *the homogeneous equation*

$$-f(y) + f(2a-y)\left[\frac{\alpha^+\alpha^- + \beta^2}{\alpha^{+2}+\beta^2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{2\beta\alpha^{-1}}{\alpha^{+2}+\beta^2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] +$$

$$\int_{-\infty}^0 f(t)\, F_s(t+y)\, dt = 0$$

has only a zero solution with the components from $L_2(-\infty, 0)$.

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Received: July 28, 2014; Accepted: September 05, 2014