Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 40, Special Issue, 2014, Pages 226–232

INVERSE SCATTERING PROBLEM FOR DIRAC-TYPE SYSTEMS WITH A SELF-ADJOINT POTENTIAL

MANSUR I. ISMAILOV

In memory of M. G. Gasymov on his 75th birthday

Abstract. Nonstationary approach to scattering problem for Dirac-type systems with a self-adjoint potential on the whole axis is considered. The uniqueness of the solution of the inverse scattering problem is proved and some of its possibly applications are pointed.

1. Introduction

Let us consider the system of first-order ordinary differential equations of the form

$$-i\sigma\frac{d}{dx}\psi + Q(x)\psi = \lambda\psi,\ -\infty < x < +\infty, \tag{1.1}$$

where

$$\sigma = \begin{bmatrix} I_{n_1} & 0 \\ 0 & -I_{n_2} \end{bmatrix}, Q(x) = \begin{bmatrix} 0 & q_{12}(x) \\ q_{21}(x) & 0 \end{bmatrix},$$

λ is a parameter, I_p is the $p \times p$ identity matrix, q_{12}, q_{21} are $n_1\times\ n_2$ and $n_2\times$ n_1 complex matrix functions, respectively. When $q_{21}^* = q_{12}$, the system (1.1) is self-adjoint.

Dirac-type systems (also called Zakharov-Shabat (ZS) systems) are of great interest as auxiliary systems for the integrable nonlinear equations. The name ZS is used because of the fact that the system (1.1) is an auxiliary linear system for many important nonlinear integrable wave equations. For example, in the case of $q_{21}^* = q_{12}$, the equation (1.1) arises in [12] as auxiliary equation of some nonlinear evolution system of equations with 1+1 dimensions.

Many publications are devoted to the inverse scattering problems (ISP's) for the Dirac-type systems and their matrix generalizations, mostly for $n_1 = n_2$, has been intensively studied in [1, 2, 4, 5, 6, 10, 11, 13] and the references therein. See also [3, 9, 15] for the case $n_1 \neq n_2$. In general, it is observed from [16] that such type of ISP's are reduced to Riemann-Hilbert problem.

In the present paper we consider the system (1.1) with self-adjoint potential $q_{21}^* = q_{12}$. An important paper [7] studies the case of a non-stationary potential on the plane and it gives an idea of nonstationary approach to the ISP on the

---

2010 *Mathematics Subject Classification.* Primary 35R30 ; Secondary 35L50, 35P25, 37K15.
Key words and phrases. inverse scattering problem, Dirac-type systems, self-adjoint potential.

whole axis for the equation (1.1). In the case of regular Riemann-Hilbert problem, the nonstationary approach to ISP for the system (1.1) is studied in [9] without assuming that the potential is self-adjoint. Different approaches to ISP in whole axis for the equation (1.1) with self-adjoint potential are studied in [3, 15].

In what follows we shall assume that q_{ij} is complex valued measurable matrix functions satisfying the estimate

$$\|q_{ij}(x)\| \leq ce^{-\varepsilon|x|},\ c \text{ is constant},\ \varepsilon > 0. \tag{1.2}$$

The paper organized as follows: In Chapter 2, we recall the definition of the scattering matrix and conditionally inverse scattering problem's results for the system (1.1) in the non-selfadjoint potential case [9]. In Chapter 3, the inverse scattering problem for the system (1.1) is studied in the self-adjoint potential case.

2. ISP in the case of nonself-adjoint potential

Let $\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$ where ψ_1 represents the first n_1 components and ψ_2 the remaining n_2 components of of ψ.

Introduce the following boundary conditions at infinity:

$$\lim_{x\to-\infty} \psi_1(x,\lambda)e^{-i\lambda x} = a_1, \tag{2.1-a}$$

$$\lim_{x\to-\infty} \psi_2(x,\lambda)e^{i\lambda x} = a_2, \tag{2.1-b}$$

$$\lim_{x\to+\infty} \psi_1(x,\lambda)e^{-i\lambda x} = b_1, \tag{2.1-c}$$

$$\lim_{x\to+\infty} \psi_2(x,\lambda)e^{i\lambda x} = b_2. \tag{2.1-d}$$

Lemma 2.1. *(Theorem 1, [9]) Let λ be a real number and the coefficients of system* (1.1) *satisfy condition* (1.2). *Then the following statements hold:*

i) There exists a unique solution in the class of bounded function of problems (1.1), $(2.1-a)$, $(2.1-b)$ *and* (1.1), $(2.1-c)$, $(2.1-d)$.

ii) For any bounded solution $\psi(x,\lambda)$ of the system (1.1) *there exist limits* (2.1).

The following theorem allows us to determine the scattering matrix for the system (1.1) on the whole axis.

Theorem 2.1. *(Theorem 2, [9]) Let λ be a real number and the coefficients of system* (1.1) *satisfy the condition* (1.2). *Then the solution of problem* (1.1), $(2.1-a)$, $(2.1-b)$ *can be represented in the form*

$$\begin{cases} \psi_1(x,\lambda) = a_1e^{i\lambda x} + \int_{-\infty}^{x} A_{11}(x,t)\,a_1e^{i\lambda t}dt + \int_{-\infty}^{x} A_{12}(x,t)\,a_2e^{-i\lambda t}dt, \\ \psi_2(x,\lambda) = a_2e^{-i\lambda x} + \int_{-\infty}^{x} A_{21}(x,t)\,a_1e^{i\lambda t}dt + \int_{-\infty}^{x} A_{22}(x,t)\,a_2e^{-i\lambda t}dt, \end{cases} \tag{2.2}$$

and solution of problem (1.1), $(2.1-c)$, $(2.1-d)$ *can be represented in the form*

$$\begin{cases} \psi_1(x,\lambda) = b_1 e^{i\lambda x} + \int_x^{+\infty} B_{11}(x,t)\, b_1 e^{i\lambda t} dt + \int_x^{+\infty} B_{12}(x,t)\, b_2 e^{-i\lambda t} dt, \\ \psi_2(x,\lambda) = b_2 e^{-i\lambda x} + \int_x^{+\infty} B_{21}(x,t)\, b_1 e^{i\lambda t} dt + \int_x^{+\infty} B_{22}(x,t)\, b_2 e^{-i\lambda t} dt, \end{cases} \tag{2.3}$$

where the kernels satisfy the estimation

$$\|A_{kj}(x,t)\|, \|B_{kj}(x,t)\| \leq C_1 e^{-\varepsilon \frac{|x+t|}{2}}, \ k,j = 1,2. \tag{2.4}$$

Now from (2.2) and (2.3) we have

$$(I_n + A_-(x,\lambda)) e^{i\lambda\sigma x} \begin{bmatrix} a_1 \\ b_2 \end{bmatrix} = (I_n + B_+(x,\lambda)) e^{i\lambda\sigma x} \begin{bmatrix} b_1 \\ a_2 \end{bmatrix}, \tag{2.5}$$

where

$$A_-(x,\lambda) = \int_{-\infty}^{0} \begin{bmatrix} A_{11}(x,x+t) & B_{12}(x,x-t) \\ A_{21}(x,x+t) & B_{22}(x,x-t) \end{bmatrix} e^{i\lambda t} dt,$$

$$B_+(x,\lambda) = \int_{0}^{+\infty} \begin{bmatrix} B_{11}(x,x+t) & A_{12}(x,x-t) \\ B_{21}(x,x+t) & A_{22}(x,x-t) \end{bmatrix} e^{i\lambda t} dt,$$

$$e^{i\lambda\sigma x} = \begin{bmatrix} e^{i\lambda x} I_{n_1} & 0 \\ 0 & -e^{-i\lambda x} I_{n_2} \end{bmatrix}$$

are denoted.

It is easy to see that matrix functions $A_-(x,\lambda)$ and $B_+(x,\lambda)$ admit analytical extension to lower $(\mathrm{Im}\lambda \leq 0)$ and upper $(\mathrm{Im}\lambda \geq 0)$ half-plane, respectively. If we suppose that matrix functions $A_-(x,\lambda)$ and $B_+(x,\lambda)$ nowhere degenerate in their domains of analyticity, i.e.

$$\begin{cases} \det(I_n + A_-(x,\lambda)) \neq 0, \ \mathrm{Im}\lambda \leq 0, \\ \det(I_n + B_+(x,\lambda)) \neq 0, \ \mathrm{Im}\lambda \geq 0, \end{cases} \tag{2.6}$$

then from (2.5) we obtain

$$e^{i\lambda\sigma x} S(\lambda) e^{-i\lambda\sigma x} = (I_n + B_+(x,\lambda))^{-1} (I_n + A_-(x,\lambda)).$$

It is easy to see that

$$S(\lambda) : \begin{bmatrix} a_1 \\ b_2 \end{bmatrix} \to \begin{bmatrix} b_1 \\ a_2 \end{bmatrix}.$$

According to nonstationary potential's case [7], the matrix $S(\lambda), \lambda \in \mathbb{R}$, is called the scattering matrix for the system (1.1) on the whole axis.

Thus we obtain regular Riemann-Hilbert problem with the normalization to unit matrix at infinity.

Theorem 2.2. *(Theorem 3, [9]) Let $S(\lambda)$ be a scattering matrix for the system of equation* (1.1) *with the coefficients satisfying condition* (1.2). *Then coefficients of the system* (1.1) *on the whole axis are uniquely determined from $S(\lambda)$ under the condition* (2.6).

3. ISP in the case of self-adjoint potential

Consider system (1.1) on the whole axis under self-adjoint potential ($q_{21}^* = q_{12}$). In this case the operator L defined in the space $L_2(\mathbb{R}, \mathbb{C}^3)$ by the differential expression $l(\psi) \equiv -i\sigma\frac{d}{dx}\psi + Q(x)\psi$ is self-adjoint. We will show that the conditions (2.6), which are sufficient for uniqueness for ISP in general case, is satisfied automatically for the self-adjoint case of problem.

Lemma 3.1. *Let the coefficients of system* (1.1) *with self-adjoint potential satisfy condition* (1.2). *Then the problems* (1.1), $(2.1-a)$, $(2.1-d)$ *(*$Im\lambda \leq 0$*) and* (1.1), $(2.1-b)$, $(2.1-c)$ *(*$Im\lambda \geq 0$*) have the unique solutions in the class of bounded functions.*

Proof. The problem (1.1), $(2.1-b)$, $(2.1-c)$ is equivalent to system of integral equation

$$\begin{cases} \psi_1(x,\lambda) = b_1 e^{i\lambda x} + i\int_x^{+\infty} e^{i\lambda(x-s)} q_{12}(s)\psi_2(s)ds, \\ \\ \psi_2(x,\lambda) = a_2 e^{-i\lambda x} + i\int_{-\infty}^{x} e^{-i\lambda(x-s)} q_{21}(s)\psi_1(s)ds. \end{cases} \tag{3.1}$$

First, let λ be a complex number with $\mathrm{Im}\lambda > 0$. It is easy to verify that the bounded solution of the system (1.1) must be satisfied the system (3.1) with $b_1 = a_2 = 0$. Therefore $|\psi_k(x,\lambda)| \leq Ce^{-\varepsilon|x|}$, $k = 1, 2$. It means that $\psi(x,\lambda) \in L_2(\mathbb{R}, \mathbb{C}^3)$ and, consequently, $\psi(x,\lambda)$ is eigenfunction of the self-adjoint operator L and non-real number λ is its eigenvalue. In this way $\psi(x,\lambda) = 0$.

Now, let λ be a real number and $\psi(x,\lambda)$ be solution of $-i\sigma\frac{d}{dx}\psi + Q(x)\psi = \lambda\psi$, where $Q = \begin{bmatrix} 0 & q_{21}^* \\ q_{21} & 0 \end{bmatrix}$ with homogeneous boundary condition $\lim\limits_{x\to+\infty} \psi_1(x,\lambda)e^{-i\lambda x} = 0$, $\lim\limits_{x\to-\infty} \psi_2(x,\lambda)e^{i\lambda x} = 0$. It is easy to show that

$$\begin{aligned} & \psi^*(x,\lambda)\,\sigma\psi(x,\lambda) \\ = \; & \psi_1^*(x,\lambda)\,\psi_1(x,\lambda) - \psi_2^*(x,\lambda)\,\psi_2(x,\lambda) \end{aligned}$$

is not dependent by x. It is clear that

$$\begin{aligned} \lim_{x\to-\infty} \psi_1^*(x,\lambda)\,\psi_1(x,\lambda) &= 0, \\ \lim_{x\to+\infty} \psi_2^*(x,\lambda)\,\psi_2(x,\lambda) &= 0. \end{aligned}$$

The last equality yields $\lim\limits_{x\to-\infty} \psi_1(x,\lambda) = 0$ and $\lim\limits_{x\to+\infty} \psi_2(x,\lambda) = 0$. Since the solution of the problem with boundary conditions $(2.1-a)$, $(2.1-b)$ or $(2.1-c)$, $(2.1-d)$ is unique, the equality $\psi(x,\lambda) = 0$ is true.

Because the system (3.1) is the system of Fredholm integral equation in the real λ, the existence of the solution follows from its uniqueness.

The uniqueness of the bounded solution of the problem (1.1), $(2.1-a)$, $(2.1-d)$ is analogously proved. □

Let us show that there exist matrix function $C(x,t)$ such that

$$[I_n + B_+(x,\lambda)]^{-1} = I_n + \int_0^{+\infty} C(x,t)e^{i\lambda t}dt$$

for arbitrary x and λ ($\mathrm{Im}\lambda \geq 0$). First, show that the matrix $I_n + B_+(x,\lambda)$ is invertible. Let

$$\det\left[I_n + \int_0^{+\infty}\begin{bmatrix} B_{11}(x_0, x_0+t) & A_{12}(x_0, x_0-t) \\ B_{21}(x_0, x_0+t) & A_{22}(x_0, x_0-t)\end{bmatrix} e^{i\lambda_0 t}dt\right] = 0 \tag{3.2}$$

for some x_0 and λ_0.

It is easy to verify that $\psi^{(k)}(x,\lambda_0) = \begin{bmatrix} \psi_1^{(k)}(x,\lambda_0) \\ \psi_2^{(k)}(x,\lambda_0)\end{bmatrix}$, $k = 1,\ldots,n$, with the components

$$\begin{aligned}\left[\psi_1^{(k)}\right]_j(x,\lambda_0) &= \delta_{kj}e^{i\lambda_0 x} + \int_0^{+\infty}[B_{11}]_{kj}(x,t+x)\,e^{i\lambda_0(t+x)}dt, \\ j &= 1,\ldots,n_1, \\ \left[\psi_2^{(k)}\right]_j(x,\lambda_0) &= \int_0^{+\infty}[B_{21}]_{k,j-n_1}(x,t+x)\,e^{i\lambda_0(t+x)}dt,\ j = n_1+1,\ldots,n,\end{aligned}$$

for $k = 1,\ldots,n_1$ and

$$\left[\psi_1^{(k)}\right]_j(x,\lambda_0) = \int_0^{+\infty}[A_{12}]_{k-n_1,j}(x,x-t)\,e^{i\lambda_0(t-x)}dt,\ j = 1,\ldots,n_1,$$

$$\left[\psi_2^{(k)}\right]_j(x,\lambda) = \delta_{kj}e^{-i\lambda_0 x} + \int_0^{+\infty}[A_{22}]_{k-n_1,j-n_1}(x,x-t)\,e^{i\lambda_0(t-x)}dt,$$
$$j = n_1+1,\ldots,n,$$

for $k = n_1+1,\ldots,n$, are solutions of the system (1.1) with the conditions

$$\lim_{x\to+\infty}\left[\psi_1^{(k)}\right]_j(x,\lambda_0)e^{-i\lambda_0 x} = \delta_{kj},\ j = 1,\ldots,n_1,$$

$$\lim_{x\to+\infty}\left[\psi_2^{(k)}\right]_j(x,\lambda_0)e^{-i\lambda_0 x} = 0,\ j = n_1+1,\ldots,n,$$

for $k = 1,\ldots,n_1$ and

$$\lim_{x\to-\infty}\left[\psi_1^{(k)}\right]_j(x,\lambda_0)e^{i\lambda_0 x} = 0,\ j = 1,\ldots,n_1,$$

$$\lim_{x\to-\infty}\left[\psi_2^{(k)}\right]_j(x,\lambda_0)e^{i\lambda_0 x} = \delta_{kj},\ j = n_1+1,\ldots,n,$$

for $k = n_1+1,\ldots,n$, respectively. Here $[\psi]_j$ denotes j'st component of the vector ψ, $[B]_{kj}$ denotes (k,j)'st component of the matrix B, δ_{kj} is the Kronecker delta symbol.

From (3.2), it follows that $c_1\psi^{(1)}(x_0,\lambda_0) + \cdots + c_n\psi^{(n)}(x_0,\lambda_0) = 0$ for some constants c_1 ,..., c_n which at least one of these constants is not equal to zero. Without loss of generality suppose that $c_1 \neq 0$.

Let $\varphi(x,\lambda_0) = c_1\psi^{(1)}(x,\lambda_0) + \cdots + c_n\psi^{(n)}(x,\lambda_0)$. It is clear to see that $\varphi(x,\lambda_0)$ is solution of (1.1). Because the solution of Cauchy problem is unique the following equality is satisfied:

$$c_1\psi^{(1)}(x_0,\lambda_0) + \cdots + c_n\psi^{(n)}(x_0,\lambda_0) = 0.$$

Denoting

$$\begin{aligned}\omega(x,\lambda_0) &= c_1\psi^{(1)}(x,\lambda_0) + \cdots + c_{n_1}\psi^{(n_1)}(x,\lambda_0) \\ &= -c_{n_1+1}\psi^{(n_1+1)}(x,\lambda_0) - \cdots - c_n\psi^{(n)}(x,\lambda_0)\end{aligned}$$

the function $\omega(x,\lambda_0)$ is bounded solution of (1.1), since $c_1\psi^{(1)}(x,\lambda_0)+\cdots+c_{n_1}\psi^{(n_1)}(x,\lambda_0)$ is bounded at $x\to+\infty$ and $c_{n_1+1}\psi^{(n_1+1)}(x,\lambda_0)+\cdots+c_n\psi^{(n)}(x,\lambda_0)$ is bounded at $x\to-\infty$. In addition $\omega(x,\lambda_0)$ satisfies the boundary condition $\lim\limits_{x\to-\infty}\omega_1(x,\lambda_0)e^{-i\lambda_0 x}=0$, $\lim\limits_{x\to+\infty}\omega_2(x,\lambda_0)e^{i\lambda_0 x}=0$. Since the bounded solution of system (1.1) with such boundary conditions is unique by the lemma 3.1, it implies that

$$\begin{aligned} c_1\psi^{(1)}(x,\lambda_0)+\cdots+c_{n_1}\psi^{(n_1)}(x,\lambda_0) &= \\ -c_{n_1+1}\psi^{(n_1+1)}(x,\lambda_0)-\cdots-c_n\psi^{(n)}(x,\lambda_0)=0. \end{aligned}$$

Last equalities yield

$$c_1\left(1+\int_0^{+\infty}\left[B_{11}\right]_{11}(x,t+x)\,e^{i\lambda_0 t}dt\right)+c_2\int_0^{+\infty}\left[B_{11}\right]_{21}(x,t+x)\,e^{i\lambda_0 t}dt$$

$$+\cdots+c_{n_1}\int_0^{+\infty}\left[B_{11}\right]_{n_1 1}(x,t+x)\,e^{i\lambda_0 t}dt=0.$$

It follows from the last equalities that $c_1=0$ by using (2.4). It becomes to contradiction. It means that matrix $I_n+B_+(x,\lambda)$ is invertible and is analytic in upper half-plane $\mathrm{Im}\lambda\geq 0$. Then there exists a matrix $C(x,t)$ with the components belonging to L_1 for arbitrary x, such that $\left[I_n+B_+(x,\lambda)\right]^{-1}=I_n+\int_0^{+\infty}C(x,t)e^{i\lambda t}dt$, by matrix analogue of the Wiener theorem [14, p. 60-63].

It is proved the uniqueness of the inverse scattering problem for the system (1.1) with self-adjoint potential.

Theorem 3.1. *Let $S(\lambda)$ be a scattering matrix for the system of equation* (1.1) *with the self-adjoint potential satisfying condition* (1.2). *Then coefficients of the system* (1.1) *on the whole axis are uniquely determined from $S(\lambda)$.*

4. Conclusion

There are many publications on the ISP on the semi-axis for two component Dirac systems and their $2n$ generalizations. In contrast to these cases, the ISP in semi-axis for the system (1.1) is not intensively investigated for $n_1\neq n_2$. The principal difficulty is to determine the sufficiently many scattering problems to ensure the unique solution of the ISP under the consideration. In the paper [8], the case of the non-stationary potential on the semi-plane has been studied for $n_1=2$, $n_2=1$. It gives an idea of reducing the solution of the ISP in the half-axis to the solution of the ISP in whole axis for the equation (1.1) for the future works.

References

[1] T. Aktosun, M. Klaus and C. Mee, Direct and inverse scattering for selfadjoint Hamiltonian systems on the line, *Integral Equations and Operator Theory*, **38** (2000), 129–171.

[2] S. Albeverio, R. Hryniv and Ya. Mykytyuk, Inverse spectral problems for Dirac operators with summable potentials, *Russ. J. Math. Phys.*, **12(4)** (2005), 406–423.

[3] F. Demontis and C. Mee, Scattering operators for matrix Zakharov-Shabat systems, *Integral Equations and Operator Theory*, **62** (2008), 517–540.

[4] I.S. Frolov, An inverse scattering problem for the Dirac system on the entire axis, *Soviet Math. Dokl.*, **13** (1972), 1768–4172. also: Dokl. Akad. Nauk SSSR 207 (1972) 44–47 (in Russian).

[5] M.G. Gasymov and B.M. Levitan, The inverse problem for a Dirac system, *Dokl. Akad. Nauk SSSR*, **167** (1966), 967–970; English transl. in Soviet Math. Dokl. 7 (1966).

[6] M.G. Gasymov, The inverse scattering problem for a system of Dirac equations of order 2n, *Trudy Moscov. Mat. Obşç*, **19** (1968), 41-112 (in Russian); English trans. Moscow Math. Soc. 19 (1968), 41–119.

[7] M.I. Ismailov, Inverse scattering problem for nonstationary Dirac-type systems on the plane, *J. Math. Anal. Appl.* **365** (2010), 498–509

[8] M.I. Ismailov, Inverse nonstationary scattering for the linear system of the 3-wave interaction problem in the case of two incident waves with the same velocity, *Wave Motion*, **47** (2010), 205–216.

[9] M.I. Ismailov, The uniqueness of solution of inverse scattering problem for the system of ordinary differential equations on the whole axis, *Transactions of NAS of Azerbaijan*, **22 (4)** (2002), 139–145.

[10] M.G. Krein, Topics in differential and integral equations and operator theory, *Operator Theory: Advances and Applications*, **7**, Birkhäuser Verlag, Basel, 1983.

[11] B.M. Levitan and I.S. Sargsjan, Sturm-Liouville and Dirac operators, *Mathematics and its Applications,* **59**, Kluwer, 1991.

[12] S.V. Manakov, On the theory of two-dimensional stationary self-focusing of electromagnetig waves, *Sov. Phys. JETP* **38** (1974), 248–253.

[13] L.P. Nizhnik and P.L. Vu, An inverse scattering problem on the semi-axis with a nonselfadjoint potential matrix, *Ukrain. Mat. Z.*, **26(4)** (1974), 469–486 (in Russian); *Ukrainian Math. J.*, **26** (1975), 384–398.

[14] R.C. Paley and N. Wiener, Fourier transforms in the complex domain, *Amer. Math. Soc., New York,* 1943.

[15] A.L. Sakhnovich, Dirac type systems on the axis: explicit formulae for matrix potentials with singularities and soliton-positon interactions, *Inverse Problems*, **19** (2003), 845–854.

[16] X. Zhou, The Riemann-Hilbert problem and inverse scattering, *SIAM J. Math. Anal.*, **20(4)** (1989), 966–986.

Mansur I. Ismailov

Department of Mathematics, Gebze Institute of Technology, Gebze-Kocaeli 41400, Turkey.

E-mail address: `mismailov@gyte.edu.tr`

Received: July 05, 2014; Accepted: July 18, 2014