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METHOD OF INTEGRATION OF THE VOLTERRA CHAIN WITH STEPWISE INITIAL CONDITION

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In memory of M. G. Gasymov on his 75th birthday

Abstract. The Cauchy problem is considered for the Volterra chain with initial condition providing the existence of only one-direction scattering. The global solvability of the problem in some class is established. The formulas permitting to find the solution are obtained by the method of the inverse scattering problem.

1. Introduction

The Volterra chain

$$\dot{a}_n(t) = \frac{1}{2} a_n(t)\left(a_{n-1}(t) - a_{n+1}(t)\right), \quad n \in Z,\ a_n(t) > 0,\ \cdot = \frac{d}{dt}, \tag{1}$$

undoubtedly represents a great applied interest (see e.i. [12] and references therein) and therefore it is a subject of active study during several years.

The constructions of the solutions of the system (1) in the periodic case [3] in the case of bounded (self-adjoint) operators [3, 2, 13] and in the presence of scattering in both directions [11] (see also [4]) are known well. But in the presence of one-direction scattering this problem has not been studied.

In the present paper we consider the following Cauchy problem for the chain (1):

$$a_n(0) = \hat{a}_n \to 0 \quad as \quad n \to +\infty, \ \sum_{n<0} |n| \left|\hat{a}_n - 1\right| < \infty. \tag{2}$$

We'll consider the problem (1)-(2) in the class of sequences $a_n(t)$, $b_n(t)$ such that

$$\left\|a_n(t)\right\|_{C[0,T]} \quad as \ \ n \to +\infty, \quad \left\|\sum_{n<0} |n| \left|a_n(t) - 1\right|\right\|_{C[0,T]} < \infty \tag{3}$$

for all $T > 0$.

Note that problem (1)-(2) was earlier studied in the class of bounded operators [2] and also in the class of completely continuous operators [13]. But the results of these papers don't permit to affirm the existence of the solution in the class

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(3). Furthermore, in our case the evolution of spectral data is described by more explicit and simple formulas than in [2], [13].

2. Main results

1. Let's consider Banach space B convergent to the zero of sequences $y = \{y_n\}_{-\infty}^{\infty}$ with the norm

$$\|y\|_B = \max_{n\geq 0}(|y_n|) + \sum_{n<0}|ny_n|.$$

Then the set $C([0,T];B)$ continuous on the interval $[0,T]$ with the values in B functions is also [8] a Banach space.

Assume

$$x_n = \begin{cases} a_n(t), & n \geq 0, \\ a_n(t) - 1, & n < 0. \end{cases}$$

Then the system (1) is equivalent to the system

$$\dot{x}_n = \frac{1}{2}x_n(x_{n-1} - x_{n+1}) + \frac{1}{2}\left(1 - \delta_{n,|n|}\right)(x_{n-1} - x_{n+1}),$$

where δ_{nm} is the Kronecker symbol. The right side of the last system generates in $C([0,T];B)$ a continuously differentiable operator. Using the principle of compressed mappings, we find that on some interval $[0,\delta]$ there exists a unique solution $x = \{x_n\}_{-\infty}^{+\infty}$ with the finite norm $\|x(t)\|_{C([0,T];B)} < \infty$. In what follows, as in [5] it is established that this solution is continuable to the whole of the interval $[0,T]$. By the same token the following theorem is valid.

Theorem 2.1. *Problem (1)-(2) has a unique solution in the class (3).*

2. It is known [2] that the system (1) is equivalent to the Lax equation

$$\dot{L} = [L,A] = AL - LA,$$

where $(Ly)_n = a_{n-1}(t)\,y_{n-1} + a_n(t)\,y_{n+1}$, $(Ay)_n = 2^{-1}(a_{n-2}(t)\,a_{n-1}(t)\,y_{n-2} - a_n(t)\,a_{n+1}(t)\,y_{n+2})$, $n \in Z$ are difference operators in l_2. As the Lax equation describes the isospectral deformation of the operator L, its spectrum is independent of t. As is shown in [7, 6], the operator L has a continuous spectrum filling the interval [-2,2]. In addition to continuous spectrum L may have finitely many prime eigen values $\mu_k, k = 1,...,N$.

Let's consider the discrete Sturm-Liouville equation

$$(Ly)_n = \lambda y_n, \quad n \in Z. \tag{4}$$

Denote by $P_n(\lambda,t)$ and $Q_n(\lambda,t)$ the solution of equation (4) with the conditions $P_{-1}(\lambda,t) = Q_0(\lambda,t) = 0$, $P_0(\lambda,t) = 1$, $Q_1(\lambda,t) = a_0^{-1}(t)$. Let $m(\lambda,t)$ be the Weil function of the operator (denote it by $L_0 = L_0(t)$) generated by equation (4) for $n \geq 0$ and boundary condition $y_{-1} = 0$. As it follows from [7, 6]the function $m(\lambda,t)$ for each fixed t is analytic with respect to λ except prime real poles concentrated to zero. At the points of λ where $m(\lambda,t)$ is analytic, the Weil solution $\psi_n(\lambda,t) = Q_n(\lambda,t) + m(\lambda,t)\,P_n(\lambda,t)$ belongs [6] to $l_2[0,\infty)$.

Let Γ be a complex plane with the cut on the segment [-2,2]. Consider the function $z = z(\lambda) = \frac{\lambda}{2} + \sqrt{\frac{\lambda^2}{4} - 1}$, where the regular branch of the radical is selected from the condition $\sqrt{\frac{\lambda^2}{4} - 1} < 1$ for $\lambda > 0$. It is known [6] that for the solution $f_n(\lambda, t)$ of the Iost type equation (4) it holds the representation

$$f_n(\lambda, t) = \alpha_n(t) z^{-n} \left(1 + \sum_{m \le -1} A_{nm}(t) z^{-2m} \right), \quad n \in Z. \tag{5}$$

On the spectrum of the operator L the following relation is valid

$$a^{-1}(\lambda, t) \psi_n(\lambda, t) = \overline{f_n(\lambda, t)} + S(\lambda, t) f_n(\lambda, t), \quad \lambda \in \partial\Gamma, \quad \lambda^2 \ne 2, \tag{6}$$

where $a^{-1}(\lambda, t)$ and $S(\lambda, t) = \frac{\overline{a(\lambda,t)}}{a(\lambda,t)}$ are the passsage factor and reflection factor of equation (4). As it follows from [6] the Weil function $m(\lambda, t)$ and the reflection coefficient $S(\lambda, t)$ are connected with the equality

$$a_{-1}(t) m(\lambda, t) = -\frac{\overline{f_0(\lambda, t)} + S(\lambda, t) f_0(\lambda, t)}{\overline{f_{-1}(\lambda, t)} + S(\lambda, t) f_{-1}(\lambda, t)}, \lambda \in \partial\Gamma. \tag{7}$$

Assume

$$M_k^{-2}(t) = \sum_{n \in Z} f_n^2(\mu_k, t), \quad k = 1, ..., N,$$

$$F_n(t) = \sum_{k=1}^{N} M_k^2(t) z^{-n}(\mu_k) + \frac{1}{2\pi i} \int_{\partial\Gamma} S(\lambda, t) \frac{z^{-n}}{z^{-1} - z} d\lambda.$$

The set of quantities $\{S(\lambda, t), \mu_k, M_k(t), k = 1, ..., N\}$ is called the scattering data for equation (4). The operator L is restored uniquely by the scattering data. More exactly, for $n < 0$ we find the coefficient $a_n(t)$ by the formula

$$a_n(t) = \frac{\alpha_n(t)}{\alpha_{n+1}(t)}, \tag{8}$$

where

$$\alpha_n^{-2}(t) = 1 + F_{2n}(t) + \sum_{k \le -1} A_{nk}(t) F_{2n+2k}(t), \quad n \le 0, \tag{9}$$

and $A_{nm}(t)$ is the solution of Marchenko type equation

$$F_{2n+2m}(t) + A_{nm}(t) + \sum_{k \le -1} A_{nk}(t) F_{2n+2m+2k}(t) = 0, \quad n \le 0, \quad m \le -1. \tag{10}$$

According to (5), (9), (10) we construct $f_{-1}(\lambda, t)$, $f_0(\lambda, t)$ and with their help from formula (7) we find the function $m(\lambda, t)$ for $\lambda \in [-2, 2]$. As $m(\lambda, t)$ outside of the segment [-2,2] may have only finitely many prime real poles, then it is restored entirely (see [10], chp. IV, 925, p. 191). By means of the Stieltjes-Perron inverse transformation restore the spectral measure $d\rho(\lambda, t)$ of the operator L_0. Then the coefficients $a_n(t), b_n(t)$ for $n \ge 0$ are found with respect to spectral measure $d\rho(\lambda, t)$ [2].

Find now the dynamics of scattering data. First of all study the differentiability properties of the function $m(\lambda, t)$ with respect to the variable t. Consider the operator $L_0 = L_0(t)$. Let $\pm\lambda_n(t), n = 1, 2, ...$ be its eigen values. Obviously, this operator is strongly continuously differentiable. Prove that for all λ lying outside of the spectra of the operator L_0 the resolvent $R_\lambda = R_\lambda(t)$ is also strongly continuously differentiable with respect to t. If $\operatorname{Im}\lambda \neq 0$, then the resolvent $R_\lambda = R_\lambda(t)$ for all t exists and is bounded. Therefore, the resolvent $R_\lambda = R_\lambda(t)$ is strongly continuously differentiable with respect to t and it is valid [8] the formula

$$\dot{R}_\lambda = -R_\lambda \dot{L}_0 R_\lambda. \tag{11}$$

Let now $\operatorname{Im}\lambda = 0$. Take some $t = t_0$ and prove that for $\lambda \neq \pm\lambda_n(t_0)$, $n = 1, 2, ...$, the resolvent $R_\lambda = R_\lambda(t)$ is strongly continuously differentiable at the point $t = t_0$. As the operator $L_0 = L_0(t)$ is continuous in the norm and its eigen values are prime, then $\lambda_n(t)$ for each n continuously depends on t (see [9]). Therefore if $\lambda_k(t_0) < \lambda < \lambda_{k-1}(t_0)$ for some k the last inequality at the rather small vicinity of the point $t = t_0$ will also be fulfilled: $\lambda_k(t_0) < \lambda < \lambda_{k-1}(t)$. Then at the same vicinity there exists a bounded operator $L_0 = L_0(t)$ that is strongly continuously differentiable (see [8]) at the point $t = t_0$ and formula (11) is valid. Consequently, the resolvent $R_\lambda = R_\lambda(t)$ of the operator $L_0 = L_0(t)$ and its Weil function $m(\lambda, t) = \langle R_\lambda(t)\delta, \delta\rangle$, where $\delta = (1, 0, 0, 0, ...) \in \ell_2[0, \infty)$ are differentiable with respect to t if $\lambda \neq \pm\lambda_n(t_0)$, $n = 1, 2, ...$ Furthermore, as is known ([11]) the solution $f_n(\lambda, t)$ of Iost type for all λ is differentiable with respect to t. Then formula (6) yields that the functions $a(\lambda, t)$ and $\overline{a(\lambda, t)}$ are differentiable with respect to t for $\lambda \in \partial\Gamma$, $\lambda^2 \neq 4$, $\lambda \neq \pm\lambda_n(t)$, $n = 1, 2, ...$

Theorem 2.2. *The following relations are valid:*

$$S(\lambda, t) = S(\lambda, 0)\exp\left\{\left(z^{-2} - z^2\right)t\right\}, \tag{11}$$

$$M_k^{-2}(t) = M_k^{-2}(0)\exp\left\{\left(z_k^{-2} - z_k^2\right)t\right\}, \tag{12}$$

$$\mu_k(t) = \mu_k(0), z_k = z_k(\mu_k), k = 1, ..., N. \tag{13}$$

Give the proof of theorem 2. It is known [2] that the operator $\frac{d}{dt} - A$ takes the solution of equation (4) to the solution of the same equation. Using (5), we find

$$\dot{\psi}_n(\lambda, t) - (A\psi(\lambda, t))_n = \left(\dot{a}(\lambda, t) + \frac{1}{2}\left(z^{-2} - z^2\right)a(\lambda, t)\right)\overline{f_n(\lambda, t)} +$$

$$+\left(\overline{\dot{a}(\lambda, t)} - \frac{1}{2}\left(z^{-2} - z^2\right)\overline{a(\lambda, t)}\right)f_n(\lambda, t). \tag{14}$$

On the other hand, from the representation of the solution $\psi_n(\lambda, t)$ we get

$$\dot{\psi}_n(\lambda, t) - (A\psi(\lambda, t))_n = \theta(\lambda, t)\psi_n(\lambda, t), \tag{15}$$

where

$$\theta(\lambda, t) = -a_{-1}^2(t)\lambda m(\lambda, t) + \frac{a_0^2(t) - a_{-2}^2(t) - a_{-1}^2(t) + a_{-1}(t)a_{-2}(t) - \lambda^2}{2}.$$

Associating (6), (14), (15) we come to equality (11). Identity (12) is established in the same way.

For constructing the solution of problem (1)-(2) by the initial data $a_n(0)$, $b_n(0)$ we calculate the scattering data $\{S(\lambda, 0), \mu_k, M_k(0), k = 1, ..., N\}$. Then the set $\{S(\lambda, t); \mu_k; M_k(t), k = 1, ..., N\}$ may be found by means of (11)-(13). Having solved the inverse problem according to the last set we get the solution $a_n(t)$.

Remark 2.1. By constructing the solution $a_n(t)$ we can avoid the renewal of the Weil function $m(\lambda, t)$ or the spectral measure $d\rho(\lambda, t)$. For the at each $k \in Z$ consider the operator

$$\left(L^{(k)}y\right)_n = a_{n+k-1}y_{n-1} + a_{n+k}y_{n+1}$$

one apply to it the above described procedure. Find $a_{n+k}(t)$ for $n < 0$ according to corresponding formulas (8), (9), (10).

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