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SPECTRAL EXPANSION FORMULA FOR A DISCONTINUOUS STURM-LIOUVILLE PROBLEM

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In memory of M. G. Gasymov on his 75th birthday

Abstract. In this paper, a self adjoint boundary value problem with a piecewise continuous coefficient on the positive half line $[0,\infty)$ is considered. The resolvent operator is constructed and the expansion formula with respect to eigenfunctions is obtained.

1. Introduction

On the semi axis $0 < x < \infty$ we consider the boundary value problem by the differential equation

$$-y'' + q(x)y = \lambda^2\rho(x)y \tag{1.1}$$

and the boundary condition

$$-\left[\alpha_1 y(0) - \alpha_2 y'(0)\right] = \lambda^2\left[\beta_1 y(0) - \beta_2 y'(0)\right], \tag{1.2}$$

where λ is the spectral parameter and $q(x)$ is the real function satisfying the condition

$$\int_0^\infty (1+x)\,|q(x)|\,dx < \infty \tag{1.3}$$

and

$$\rho(x) = \begin{cases} \alpha^2, & 0 \le x < a, \\ 1, & x > a. \end{cases}$$

Here α_i, β_i $(i = 1, 2)$ are real numbers and $\gamma = \alpha_1\beta_2 - \beta_1\alpha_2 > 0$.

In this paper, we obtain the expansion formula for the (1)-(3) boundary value problem in the half line. Similar problem for classical Sturm-Liouville equation, i.e. when $\rho(x) = 1$ is studied in [15, 6, 13, 14, 2, 3]. When $\rho(x) \neq 1$ direct and inverse problems of spectral anlaysis are investigated in Gasymovs' (see [4]) and his students works (see [1, 5, 7, 8, 9]).

In this study we used the integral expression for the Jost solution of the equation (1.1) as in [5] and differently from the classical boundary conditions, a boundary condition with dependence of eigenvalue has taken into account.

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In this work, operator theoratic formulation is given for the boundary value problem (1.1), (1.2); resolvent operator is constructed and expansion formula with respect to eigenfunctions is obtained.

It is known from [5] that for all λ from the closed upper half plane equation (1.1) has a unique solution $f(x,\lambda)$ which satisfies the condition

$$\lim_{x\to\infty} f(x,\lambda)e^{-i\lambda x} = 1$$

and that can be represented in the form

$$f(x,\lambda) = f_0(x,\lambda) + \int_{\mu^+(x)}^{+\infty} K(x,t)e^{i\lambda t}dt, \tag{1.4}$$

where

$$f_0(x,\lambda) = \begin{cases} e^{i\lambda x}, & x \ge a, \\ \frac{1}{2}\left(1 + \frac{1}{\sqrt{\rho(x)}}\right) e^{i\lambda\mu^+(x)} + \frac{1}{2}\left(1 - \frac{1}{\sqrt{\rho(x)}}\right) e^{i\lambda\mu^-(x)}, & 0 \le x \le a, \end{cases}$$

is the Jost solution of equation (1.1) when $q(x) \equiv 0$ and

$$\mu^{\pm}(x) = \pm x\sqrt{\rho(x)} + a(1 \mp \sqrt{\rho(x)}).$$

Moreover, the kernel $K(x,t)$ satisfies the estimate

$$\int_{\mu^+(x)}^{+\infty} |K(x,t)|\, dt \le c \int_x^{+\infty} t\,|q(t)|\, dt, \qquad 0 < c = const.$$

For real $\lambda \ne 0$, the functions $f(x,\lambda)$, $\overline{f(x,\lambda)}$ form the fundamental system of solutions of equation (1.1) and the Wronskian of this system is equal to $2i\lambda$:

$$W\left\{f(x,\lambda), \overline{f(x,\lambda)}\right\} = f'(x,\lambda)\overline{f(x,\lambda)} - f(x,\lambda)\overline{f'(x,\lambda)} = 2i\lambda.$$

By $w(x,\lambda)$ we denote the solutions of equation (1.1) satisfying the initial data

$$w(0,\lambda) = \alpha_2 + \lambda^2\beta_2, \quad w'(0,\lambda) = \alpha_1 + \lambda^2\beta_1$$

and define

$$\varphi(\lambda) \equiv \left[\alpha_1 f(0,\lambda) - \alpha_2 f'(0,\lambda)\right] - \lambda^2\left[\beta_1 f(0,\lambda) - \beta_2 f'(0,\lambda)\right].$$

Using (1.4), it can be seen that $\varphi(\lambda)$ may have only a finite number of zeros in the half plane $Im\lambda > 0$. Moreover, all these zeros are simple and lie on the imaginary axis. These assertions can be proved with the same method in Lemma 3.1.6 in [13] (see [9]).

2. The operator theoretic formulation of the boundary value problem (1.1), (1.2)

In the Hilbert space $H_\rho \equiv L_2(0,\infty;\rho(x)) \times \mathbb{C}$, let us define an inner product by

$$(F,G) := \int_0^\infty F_1(x)\overline{G_1(x)}\rho(x)dx + \frac{1}{\gamma}F_2G_2$$

for

$$F = \begin{pmatrix} F_1(x) \\ F_2 \end{pmatrix}, \qquad G = \begin{pmatrix} G_1(x) \\ G_2 \end{pmatrix} \in H_\rho.$$

For convenience we put

$$F_2 = \beta_1 F_1(0) - \beta_2 F_1'(0) \equiv R_0' [F_1]$$

and define the operator L with

$$LF = \begin{pmatrix} l(F_1) \\ R_0(F_1) \end{pmatrix},$$

in the domain

$$D(L) = \{F \in H_\rho : F_1(x), F_1'(x) \in AC[0,b] \text{ for every } b, l(F_1) \in L_2(0,+\infty), \\ F_2 = \beta_1 F_1(0) - \beta_2 F_1'(0) \equiv R_0(F_1)\},$$

where

$$l(F_1) = \frac{1}{\rho(x)} \{-F_1'' + q(x)F_1\}.$$

We note that, the operator L is selfadjoint in the Hilbert space H_ρ and the boundary value problem (1.1), (1.2) is equivalent to the equation $LF = \lambda^2 F$.

3. Resolvent operator

If we assume that λ^2 is not a spectrum point of the operator L we can easily say that the resolvent operator $R_{\lambda^2}(L) = (L - \lambda^2 I)^{-1}$ exists. Now, let us find the expression of the operator $R_{\lambda^2}(L)$.

Theorem 3.1. *All numbers of the form λ^2 ($Im\lambda > 0$, $\varphi(\lambda) \neq 0$) belong to the resolvent set of the operator L. The resolvent R_{λ^2} is the integral operator*

$$R_{\lambda^2}(L)F = \begin{pmatrix} \left(\tilde{G}_{x,x}, \bar{F}\right)_{H_\rho} \\ R_0'\left[\left(\tilde{G}_{x,\lambda}, \bar{F}\right)_{H_\rho}\right] \end{pmatrix}, \quad F \in H_\rho,$$

where

$$\tilde{G}_{x,\lambda} := \begin{pmatrix} G(x,\cdot,\lambda) \\ R_0'[G(x,\cdot,\lambda)] \end{pmatrix} = \begin{pmatrix} G(x,\cdot,\lambda) \\ -\frac{\gamma}{\varphi(\lambda)} f(x,\lambda) \end{pmatrix},$$

$$G(x,y,\lambda) = -\frac{1}{\varphi(\lambda)} \begin{cases} w(x,\lambda) f(y,\lambda), & x \leq y < +\infty, \\ f(x,\lambda) w(y,\lambda), & 0 \leq y < x. \end{cases}$$

Proof. Assume that $F = \binom{F_1(x)}{F_2} \in D(L)$ and $F_1(x)$ is zero in exterior of every interval. To construct the resolvent operator of L we need to solve the boundary problem

$$-y'' + q(x)y = \lambda^2 \rho(x) y + \rho(x) F_1(x), \tag{3.1}$$

$$-\left[\alpha_1 y(0) - \alpha_2 y'(0)\right] = \lambda^2 \left[\beta_1 y(0) - \beta_2 y'(0)\right] + F_2. \tag{3.2}$$

Let us find the solutions of the problem (3.1), (3.2) which have the form

$$y(x,\lambda) = c_1(x,\lambda) w(x,\lambda) + c_2(x,\lambda) f(x,\lambda) \tag{3.3}$$

by noting that the functions $w(x,\lambda)$ and $f(x,\lambda)$ are the solutions of homogeneous problem for $Im\lambda > 0$.

By applying the method of variation of constants, we have the system of equations

$$\begin{cases} c_1'(x,\lambda)w(x,\lambda)+c_2'(x,\lambda)f(x,\lambda)=0, \\ c_1'(x,\lambda)w'(x,\lambda)+c_2'(x,\lambda)f'(x,\lambda)=-\rho(x)F_1(x). \end{cases} \tag{3.4}$$

Since $y(x,\lambda)\in L_{2,\rho}(0,\infty)$, then $c_1(\infty,\lambda)=0$. Using this relation and the system of equations (3.4) we get

$$c_1(x,\lambda)=-\int_x^\infty \frac{f(t,\lambda)}{\varphi(\lambda)}F_1(t)\rho(t)dt,$$

$$c_2(x,\lambda)=c_2(0,\lambda)-\int_0^x \frac{w(t,\lambda)}{\varphi(\lambda)}F_1(t)\rho(t)dt. \tag{3.5}$$

Taking into consideration (3.5) into (3.3) we obtain

$$y(x,\lambda)=\int_0^\infty G(x,t,\lambda)F_1(t)\rho(t)dt+c_2(0,\lambda)f(x,\lambda),$$

where

$$G(x,t,\lambda)=\frac{1}{\varphi(\lambda)}\begin{cases} w(x,\lambda)f(y,\lambda), & x\le y<+\infty, \\ f(x,\lambda)w(y,\lambda), & 0\le y<x. \end{cases}$$

Using formula (3.2) we have $c_2(0,\lambda)=\frac{F_2}{\varphi(\lambda)}$, thus

$$y(x,\lambda)=\int_0^\infty G(x,t,\lambda)F_1(t)\rho(t)dt+\frac{F_2}{\varphi(\lambda)}f(x,\lambda)=\left(\tilde{G}_{x,\lambda},\bar{F}\right)_{H_\rho},$$

where

$$\tilde{G}_{x,\lambda}=\begin{pmatrix} G(x,\cdot,\lambda) \\ R_0'\left[G(x,\cdot,\lambda)\right] \end{pmatrix}.$$

Theorem is proved. □

Lemma 3.1. *Let the function $F_1(x)$ is continuosly differentiable two times and finite at infinity. Then as $|\lambda|\to\infty, Im\lambda>0$ the following equation holds:*

$$\left(\tilde{G},\tilde{F}\right)=\int_0^\infty G(x,t,\lambda)F_1(t)\rho(t)dt+\frac{F_2}{\varphi(\lambda)}f(x,\lambda)=-\frac{F_1(x)}{\lambda^2}+O\left(\frac{1}{\lambda^2}\right). \tag{3.6}$$

Proof. Using Theorem 1 and integrating by parts, we write

$$\begin{aligned}\int_0^\infty G(x,t,\lambda)F_1(t)\rho(t)dt &= -\int_x^\infty \frac{w(x,\lambda)f(t,\lambda)}{\varphi(\lambda)}F_1(t)\rho(t)dt- \\ &- \int_x^\infty \frac{w(t,\lambda)f(x,\lambda)}{\varphi(\lambda)}F_1(t)\rho(t)dt= \\ &= \frac{1}{\lambda^2}\int_x^\infty \frac{w(x,\lambda)}{\varphi(\lambda)}\left\{f''(t,\lambda)-q(t)f(t,\lambda)\right\}F_1(t)dt+ \\ &+ \frac{1}{\lambda^2}\int_x^\infty \frac{f(x,\lambda)}{\varphi(\lambda)}\left\{w''(t,\lambda)-q(t)w(t,\lambda)\right\}F_1(t)dt= \\ &= \frac{W\left\{w(x,\lambda),f(x,\lambda)\right\}}{\lambda^2\varphi(\lambda)}F_1(x)+ \\ &+ \frac{1}{\lambda^2}\left[w(0,\lambda)F_1'(0)-w'(0,\lambda)F_1(0)\right]- \\ &- \frac{1}{\lambda^2}\int_0^\infty G(x,t,\lambda)\tilde{F}_1(t)dt,\end{aligned}$$

where

$$\tilde{F}_1(t) = F_1''(t) + q(t)F_1(t).$$

Thus (3.6) holds. Lemma is proved. □

4. Expansion formula

Put

$$F(x,\lambda) = \begin{cases} \left(\tilde{G}_{x,\lambda}, \bar{F}\right), & Im\lambda \geq 0, \\ \left(\tilde{G}_{x,\bar{\lambda}}, \bar{F}\right), & Im\lambda < 0. \end{cases}$$

Let us Γ_R denote the circle of radius R and center is zero which boundary contour is positive oriented. Let us $\Gamma_{R,\epsilon}$ denote boundary contour positive orinented in plane $D = \{z : |z| \leq R,\ Imz \geq \epsilon\}$ and $\Gamma_{R',\epsilon}$ denote boundary contour negative oriented in the plane $D = \{z : |z| \leq R, Imz < \epsilon\}$. Then we can use the property of the integration

$$\int_{\Gamma_{R,\epsilon}} = \int_{\Gamma_R} + \int_{\Gamma_{R',\epsilon}}. \tag{4.1}$$

Now multiplying both sides of equality (3.6) by $\frac{1}{2\pi i}\lambda$ and integrating over λ the contour $\Gamma_{R,\epsilon}$ we obtain

$$\frac{1}{2\pi i}\int_{\Gamma_{R,\epsilon}} \lambda F(x,\lambda)d\lambda = -\frac{1}{2\pi i}\int_{\Gamma_R} \frac{F_1(x)}{\lambda}d\lambda + \frac{1}{2\pi i}\int_{\Gamma_{R',\epsilon}} O\left(\frac{1}{\lambda}\right)d\lambda. \tag{4.2}$$

According to the equation (4.1) we get

$$\frac{1}{2\pi i}\int_{\Gamma_{R,\epsilon}} \lambda F(x,\lambda)d\lambda = \frac{1}{2\pi i}\int_{\Gamma_R} \lambda F(x,\lambda)d\lambda + \frac{1}{2\pi i}\int_{\Gamma_{R',\epsilon}} \lambda F(x,\lambda)d\lambda. \tag{4.3}$$

Using (4.2) let us calculate the integral on the right hand side

$$\frac{1}{2\pi i}\int_{\Gamma_R} \lambda F(x,\lambda)d\lambda = -\frac{1}{2\pi i}\int_{\Gamma_R} \frac{F_1(x)}{\lambda}d\lambda + \frac{1}{2\pi i}\int_{\Gamma_R} O\left(\frac{1}{\lambda}\right)d\lambda \to -F_1(x) \quad (R\to\infty),$$

taking into account (4.3) we have

$$\begin{aligned} lim_{\epsilon\to 0, R\to\infty}\frac{1}{2\pi i}\int_{\Gamma_{R,\varepsilon}} \lambda F(x,\lambda)d\lambda &= -F_1(x) \\ &+ \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \lambda\left[F(x,\lambda+i0) - F(x,\lambda-i0)\right]d\lambda. \end{aligned}$$

On the other hand using the residue calculus we get

$$\frac{1}{2\pi i}\int_{\Gamma_{R,\epsilon}} \lambda F(x,\lambda)d\lambda = \sum_{j=1}^{n} \underset{\lambda=i\lambda_j}{Res}\left[\lambda F(x,\lambda)\right] + \sum_{j=1}^{n} \underset{\lambda=-i\lambda_j}{Res}\left[\lambda F(x,\lambda)\right].$$

From the last two relations we get

$$\begin{aligned} F_1(x) = &- \sum_{j=1}^{n} \underset{\lambda=i\lambda_j}{Res}\left[\lambda F(x,\lambda)\right] + \sum_{j=1}^{n} \underset{\lambda=-i\lambda_j}{Res}\left[\lambda F(x,\lambda)\right] + \\ &+ \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \lambda\left[F(x,\lambda+i0) - F(x,\lambda-i0)\right]d\lambda. \end{aligned} \tag{4.4}$$

Let $\psi(x,\lambda)$ be the solution of (1.1) satisfying the initial conditions

$$\psi(0,\lambda)=\beta_2, \quad \psi'(0,\lambda)=\beta_1.$$

It is clear that $W\{w(x,\lambda),\psi(x,\lambda)\}=\gamma>0$. Then for $Im\lambda>0$, $f(x,\lambda)$ is written as the linear combination of the solutions $w(x,\lambda)$ and $\psi(x,\lambda)$, i.e.

$$f(x,\lambda)=c_1w(x,\lambda)+c_2\psi(x,\lambda).$$

Since $W\{w(x,\lambda),\psi(x,\lambda)\}=c_1\gamma, W\{w(x,\lambda),f(x,\lambda)\}=c_2\gamma$

$$f(x,\lambda)=\frac{\beta_2 f'(0,\lambda)-\beta_1 f(0,\lambda)}{\gamma}w(x,\lambda)+\frac{\varphi(\lambda)}{\gamma}\psi(x,\lambda).$$

Thus

$$\begin{aligned} G(x,t,\lambda) &= -\frac{\beta_2 f'(0,\lambda)-\beta_1 f(0,\lambda)}{\gamma\varphi(\lambda)}w(x,\lambda)w(t,\lambda)- \\ &- \frac{1}{\gamma}\begin{cases} w(x,\lambda)\psi(t,\lambda), & x\le t<+\infty, \\ \psi(x,\lambda)w(t,\lambda), & 0\le t<x. \end{cases} \end{aligned}$$

Then for $Im\lambda>0$ we obtain

$$\begin{aligned} F(x,\lambda) &= -\frac{\beta_2 f'(0,\lambda)-\beta_1 f(0,\lambda)}{\gamma\varphi(\lambda)}w(x,\lambda)\int_0^\infty w(t,\lambda)F_1(t)\rho(t)dt+ \\ &+ \frac{\psi(x,\lambda)}{\gamma}\int_0^x w(t,\lambda)F_1(t)\rho(t)dt+\frac{1}{\gamma}\psi(x,\lambda)\int_0^x \psi(t,\lambda)F_1(t)\rho(t)dt+ \\ &+ \frac{F_2}{\gamma\varphi(\lambda)}\left[\beta_2 f'(0,\lambda)-\beta_1 f(0,\lambda)\right]w(x,\lambda)-\frac{F_2}{\gamma}\psi(x,\lambda). \end{aligned} \tag{4.5}$$

It follows that

$$\begin{aligned} \operatorname*{Res}_{\lambda=i\lambda_j}\left[\lambda F(x,\lambda)\right] &= -\frac{i\lambda_j}{\varphi'(i\lambda_j)}f(0,i\lambda_j)w(x,i\lambda_j)\int_0^\infty w(t,i\lambda_j)F_1(t)\rho(t)dt+ \\ &+ \frac{F_2 i\lambda_j}{\gamma\varphi'(i\lambda_j)}\left[\beta_2 f'(0,i\lambda_j)-\beta_1 f(0,i\lambda_j)\right]w(x,i\lambda_j). \end{aligned}$$

With the help of equalities (1.5) and (1.8) we get

$$\begin{aligned} \sum_{j=1}^n \operatorname*{Res}_{\lambda=i\lambda_j}\left[\lambda F(x,\lambda)\right] &+ \sum_{j=1}^n \operatorname*{Res}_{\lambda=-i\lambda_j}\left[\lambda F(x,\lambda)\right]= \\ &- m_j^2 f\left(x,i\lambda_j\right)\int_0^\infty f\left(t,i\lambda_j\right)F_1(t)\rho(t)dt+ \\ &+ \frac{1}{\gamma}\left[\beta_2 f'(0,i\lambda_j)-\beta_1 f(0,i\lambda_j)\right]m_j^2 f\left(x,i\lambda_j\right)F_2= \\ &= -\left(F,U_j\left(\lambda\right)\right)_{H_\rho} m_j f\left(x,i\lambda_j\right), \end{aligned} \tag{4.6}$$

where

$$F=\begin{pmatrix} F_1(x) \\ F_2 \end{pmatrix}, \quad U_j\left(x\right)=m_j\begin{pmatrix} f\left(x,i\lambda_j\right) \\ \beta_2 f'(0,i\lambda_j)-\beta_1 f(0,i\lambda_j) \end{pmatrix}, \quad j=1,\cdots,n. \tag{4.7}$$

Now let us calculate

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\lambda\left[F\left(x,\lambda+i0\right)-F(x,\lambda-i0)\right]d\lambda.$$

From formula (3.5) and equality $F\left(x,\lambda-i0\right)=\overline{F\left(x,\lambda+i0\right)}$ we have

$$F\left(x,\lambda+i0\right)-F\left(x,\lambda-i0\right)=$$

$$\left[\frac{\overline{\beta_2 f'(0,\lambda)-\beta_1 f(0,\lambda)}}{\gamma\overline{\varphi(\lambda)}}-\frac{\beta_2 f'(0,\lambda)-\beta_1 f(0,\lambda)}{\gamma\varphi(\lambda)}\right]w(x,\lambda)\int_0^\infty w(t,\lambda)F_1(t)\rho(t)dt+$$

$$+\frac{F_2}{\gamma}\left[\frac{\beta_2 f'(0,\lambda)-\beta_1 f(0,\lambda)}{\gamma\varphi(\lambda)}-\frac{\overline{\beta_2 f'(0,\lambda)-\beta_1 f(0,\lambda)}}{\gamma\overline{\varphi(\lambda)}}\right]w(x,\lambda)=$$

$$=\frac{1}{\gamma}\left[\frac{(\alpha_1\beta_2-\beta_1\alpha_2)\left[f'\left(0,\lambda\right)\overline{f(0,\lambda)}-f\left(0,\lambda\right)\overline{f'(0,\lambda)}\right]}{\left|\varphi(\lambda)\right|^2}\right]w(x,\lambda)\times$$

$$\times\int_0^\infty w(t,\lambda)F_1(t)\rho(t)dt+$$

$$+\frac{F_2}{\gamma}\frac{(\alpha_1\beta_2-\beta_1\alpha_2)}{\left|\varphi(\lambda)\right|^2}\left[f\left(0,\lambda\right)\overline{f'(0,\lambda)}-f'\left(0,\lambda\right)\overline{f(0,\lambda)}\right]w(x,\lambda)=$$

$$=\frac{2i\lambda}{\left|\varphi(\lambda)\right|^2}w(x,\lambda)\int_0^\infty w(t,\lambda)F_1(t)\rho(t)dt-\frac{2i\lambda F_2}{\left|\varphi(\lambda)\right|^2}w(x,\lambda).$$

It follows that

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\lambda\left[F\left(x,\lambda+i0\right)-F(x,\lambda-i0)\right]d\lambda=$$

$$=\frac{2}{\pi}\int_0^\infty\frac{\lambda^2}{\left|\varphi(\lambda)\right|^2}w(x,\lambda)\int_0^\infty w(t,\lambda)F_1(t)\rho(t)dtd\lambda-$$

$$-\frac{2F_2}{\pi}\int_0^\infty\frac{\lambda}{\left|\varphi(\lambda)\right|^2}w(x,\lambda)d\lambda=\int_0^\infty T\left(F,\lambda\right)\frac{1}{\sqrt{2\pi}}\left[\overline{f(x,\lambda)}-S(\lambda)f(x,\lambda)\right]d\lambda, \tag{4.8}$$

where

$$T\left(F,\lambda\right)\equiv-\frac{1}{\sqrt{2\pi}}\int_0^\infty\frac{2i\lambda}{\left|\varphi(\lambda)\right|^2}w(t,\lambda)F_1(t)\rho(t)dt+F_2\frac{2i\lambda\gamma^2}{\varphi(\lambda)}.$$

Taking (4.6) and (4.8) into (4.4) we get the expansion formula with respect to eigenfunctions as

$$F\equiv\begin{pmatrix}F_1(x)\\F_2\end{pmatrix}=\sum_{j=1}^{n}\left(F,U_j\left(x\right)\right)_{H_\rho}U_j\left(x\right)+\int_0^\infty T(F,\lambda)U(x,\lambda)d\lambda,$$

where

$$U(x,\lambda)\equiv\frac{1}{\sqrt{2\pi}}\begin{pmatrix}\overline{f(x,\lambda)}-S(\lambda)f(x,\lambda)\\-\frac{2i\lambda\gamma}{\varphi(\lambda)}\end{pmatrix}=\frac{1}{\sqrt{2\pi}}\begin{pmatrix}\frac{2i\lambda f(x,\lambda)}{\varphi(\lambda)}\\-\frac{2i\lambda\gamma}{\varphi(\lambda)}\end{pmatrix}. \tag{4.9}$$

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