Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 40, Special Issue, 2014, Pages 283–289

ON THE SPECTRUM OF A CLASS OF NON-SELF-ADJOINT "WEIGHTED" OPERATOR WITH POINT δ-INTERACTIONS

MANAF DZH. MANAFOV

In memory of M. G. Gasymov on his 75th birthday

Abstract. We obtain the spectrum structures of a non-self-adjoint operator. Notice that the given "weig" is a sum of finite zero order generalized functions.

1. Introduction, Definition Of The Operator

We use the following notation: $\mathbb{C}^{(n)}(a,b)$ is a linear space of scalar complex-valued functions which are n-times continuously differentiable on (a,b). Let $L_2(a,b)$ be a linear space of scalar complex-valued functions on (a,b), which has square summable modules $W_2^j(a,b)$, $j=0,1,2$, stands for the Sobolev space of functions defined on (a,b) that belong to $L_2(a,b)$ together with their derivatives up to order j, m be a fixed number in $\mathbb{N}$, $x_0=-\infty$, and $x_{m+1}=+\infty$.

The paper is devoted to study of the spectral properties of "weighted" one-dimensional equation

$$\ell[y] \equiv -\frac{1}{\rho(x)}\frac{d}{dx}\left(\rho(x)\frac{dy}{dx}\right) + q(x)y = \lambda^2 y, \tag{1.1}$$

in the space $L_2(\mathbb{R})$, where "weighted" function is $\rho(x) = 1 + \sum\limits_{k=1}^{m} \alpha_k \delta(x-x_k)$ and coefficient is

$$q(x) = x^{\gamma} \sum_{\beta=1}^{\infty} q_\beta e^{i\beta x},$$

where $q = \sum_{\beta=1}^{\infty} |q_\beta|$ converges $(\gamma \geq 0)$.

In this formula, $\alpha_k > 0$, x_k $(x_1 < x_2 < ... < x_m)$ $(k = 1, 2, ..., m = \overline{1,m})$ are real numbers.

Notice that, the problems on the studies of one and multi-dimensional Schrödinger operators with singular potentials (that is, e.g., point interactions, measures, or distributions) have appeared in physical literature. Mathematical investigations of appropriate physical models were initiated at the beginning of the last century in the papers [5, 20]. This theme intensively has developed in the last

---

2010 *Mathematics Subject Classification.* 47A10, 47E05, 34L05.
Key words and phrases. non-self-adjoint "weighted" operator, spectral singularities, point δ-interactions, negative eigenvalues.

three decades (see [4, 9, 13, 14, 15, 18, 19]) and to the books [2, 3], where additional references can be found. Spectral problems for singular potentials from $W_2^{-1}(\mathbb{R})$ were recently studied in the papers [10, 23], see also the bibliography therein, where $W_2^{-1}(\mathbb{R})$ is the set of integrals (that means interactions) which in $L_2(\mathbb{R})$. The spectrum and Parseval formula are studied in the papers [16, 17] for $\alpha_k = 0$, $k = \overline{1, m}$.

Here, the approach is based on the idea of approximation of the generalized "weight" with smooth "weight"s.

Consider the differential expression

$$\ell_\varepsilon[y] \equiv -\frac{1}{\rho_\varepsilon(x)}\frac{d}{dx}\left(\rho_\varepsilon(x)\frac{dy}{dx}\right) + q(x)y,$$

where the density function

$$\rho_\varepsilon(x) = 1 + \frac{1}{\varepsilon}\sum_{k=1}^{m}\alpha_k\chi_\varepsilon(x - x_k),$$

is defined using the characteristic function

$$\chi_\varepsilon(x) = \begin{cases} 1, & \text{for } \ x \in [0, \varepsilon], \\ 0, & \text{for } x \notin [0, \varepsilon],\ \varepsilon < \min\limits_{i=\overline{2,m}}\{x_i - x_{i-1}\}. \end{cases}$$

The density function $\rho_\varepsilon(x)$ is chosen so that it converges to the $\rho(x)$ as $\varepsilon \to 0^+$ (see [24]). Therefore, the approximation equation is of the form:

$$\ell_\varepsilon[y] = \lambda^2 y. \tag{1.2}$$

Agree that the solution of equation (1.2) is any function $y(x)$ determined on $\mathbb{R}$ for which the following conditions are fulfilled:

1) $y(x) \in \mathbb{C}^2(x_k, x_k + \varepsilon) \cap \mathbb{C}^2(x_k + \varepsilon, x_{k+1})$ for $k = \overline{0, m}$;
2) $-y''(x) + q(x)y(x) = \lambda^2 y(x)$ for $x \in (x_k, x_k + \varepsilon) \cup (x_k + \varepsilon, x_{k+1})$, $k = \overline{0, m}$;
3) $y(x_k^+) = y(x_k^-)$, $(1 + \alpha_k\frac{1}{\varepsilon})y'(x_k^+) = y'(x_k^-)$ for $k = \overline{1, m}$;
4) $y((x_k + \varepsilon)^+) = y((x_k + \varepsilon)^-)$, $y'((x_k + \varepsilon)^+) = (1 + \alpha_k\frac{1}{\varepsilon})y'((x_k + \varepsilon)^-)$ for $k = \overline{1, m}$.

These conditions guarantee that the functions $y(x)$ and $\rho_\varepsilon(x)y'(x)$ are continuous at the points x_k and $x_k + \varepsilon$ ($k = \overline{1, m}$).

Define the operator L_ε generated in the Hilbert space $L_2(\mathbb{R})$ by the differential expression $\ell_\varepsilon[y]$. The domain of definition of the operator L_ε is the set of all function belonging to $L_2(\mathbb{R})$ with satisfying conditions 1) - 4).

Let R_λ^ε be resolvent of the operator L_ε and R_λ be a resolvent of the operator $L_0(\alpha_k \equiv 0, k = \overline{1, m})$.

Now, define the operator according to the differential expression (1.1). Obviously the extended operator is not uniquely defined. One such self-adjoint operator can be constructed using the method of generalized point interactions (see [3, 22]). Consider the Hilbert space $H = L_2(\mathbb{R}) \oplus l_2$ and the non-self-adjoint operator L defined by following formula

$$L\binom{y}{h} \equiv \begin{pmatrix} -\frac{d^2}{dx^2}y + q(x)y \\ \{[y']/\sqrt{\alpha}\} \end{pmatrix}$$

on functions $(y,h) \in W_2^2(\mathbb{R}/\{x_1,x_2,...,x_m\}) \oplus l_2$ satisfying the boundary conditions

$$y\left(x_k^+\right) = y\left(x_k^-\right),$$
$$h_k = -\sqrt{\alpha_k}\{y\}_k, \qquad k = \overline{1,m}\,,$$

where $\{[y']\,/\sqrt{\alpha}\}$ and $\{y\}$ denote the vectors from l_2 with the coordinates $[y'\left(x_k^+\right) - y'\left(x_k^-\right)]/\sqrt{\alpha_k}$ and $y(x_k)$, respectively. The resolvent of the operator L restricted to the space $L_2(\mathbb{R})$ coincides with the resolvent of differential operator L_0 with the boundary conditions at the points x_k

$$\begin{pmatrix} y\left(x_k^+\right) \\ y'\left(x_k^+\right) \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ -\alpha_k\lambda^2 & 1 \end{pmatrix} \begin{pmatrix} y\left(x_k^-\right) \\ y'\left(x_k^-\right) \end{pmatrix}, \qquad k = \overline{1,m}.$$

This paper comprises three sections. In section 2 we prove that norm resolvent convergent , as $\varepsilon \to 0^+$. In section 3 the spectrum operator L is determined.

2. On Norm Resolvent Convergence

Let's study the norm resolvent convergence of the operator sequence L_ε with respect to ε. One can easily prove that the resolvents of L_ε do not converge to a resolvent of any operator acting in the Hilbert space $L_2(\mathbb{R})$. The limit is called generalized resolvent and it is restriction to $L_2(\mathbb{R})$ of the resolvent of a certain self-adjoint operator acting in a certain extended Hilbert space [1]. Recently, norm resolvent convergence was investigated with different conditions in [7, 8].

Theorem 2.1. *Let $\alpha_k > 0$ $(k = \overline{1,m})$, then the resolvents of the operators L_ε converge to the restriction to $L_2(\mathbb{R})$ of the resolvent of L as $\varepsilon \to 0^+$.*

Proof. We construct the resolvent of the operator L_ε for $Im\lambda \neq 0$. For that we solve in $L_2(\mathbb{R})$ the problem

$$\left\{\begin{array}{c} -y''(x) + q(x)y(x) = \lambda^2 y(x) + F(x),\ x \neq x_k, x_k+\varepsilon \ \ \left(k = \overline{1,m}\right), \\ y(x_k^+) = y(x_k^-),\ (1+\alpha_k\frac{1}{\varepsilon})y'(x_k^+) = y'(x_k^-) \ \ (k = \overline{1,m}), \\ y((x_k+\varepsilon)^+) = y((x_k+\varepsilon)^-) \ \left(k = \overline{1,m}\right), \\ y'((x_k+\varepsilon)^+) = (1+\alpha_k\frac{1}{\varepsilon})y'((x_k+\varepsilon)^-) \ \left(k = \overline{1,m}\right), \end{array}\right. \tag{2.1}$$

where $F(x)$ is an arbitrary function belonging to $L_2(\mathbb{R})$.

It is well-known (see [16]) that the equation

$$-y''(x) + q(x)y(x) = \lambda^2 y(x), \quad x \in (-\infty,\infty),$$

has two linear independent solutions $f(x,\lambda)$, $f(x,-\lambda)$. Any solution of the equation $y(x,\lambda)$ has the following representation

$$y(x,\lambda) = C_1 f(x,\lambda) + C_2 f(x,-\lambda),$$

where C_1, C_2 are some numbers.

By the Lagrange method (see [21]) the solution of problem (2.1) takes the form

$$y^\varepsilon(x,\lambda) = -\frac{1}{W[\varphi_1,\varphi_2]}\int\limits_{-\infty}^{\infty} R(x,t;\lambda)F(t)dt$$

$$-\frac{1}{W[\varphi_1,\varphi_2]}\left\{\begin{array}{cc} b_2^{\varepsilon} f(x,-\lambda), & -\infty<x<x_1, \\ b_{4k-1}^{\varepsilon} f(x,\lambda)+b_{4k}^{\varepsilon} f(x,-\lambda), & x_k<x<x_k+\varepsilon \quad \left(k=\overline{1,m}\right), \\ b_{4k+1}^{\varepsilon} f(x,\lambda)+b_{4k+2}^{\varepsilon} f(x,-\lambda), & x_k+\varepsilon<x<x_{k+1} \quad \left(k=\overline{1,m-1}\right), \\ b_{4m+1}^{\varepsilon} f(x,\lambda), & x_m<x<\infty, \end{array}\right.$$

where

$$R(x,t;\lambda)=\left\{\begin{array}{ll} f(x,\lambda)f(t,-\lambda), & t\le x, \\ f(t,\lambda)f(x,-\lambda), & t\ge x, \end{array}\right.$$

and b_j^{ε} $(j=\overline{2,4m+1})$ are arbitrary numbers.

Denote

$$f_{k,\pm}=f(x_k,\pm\lambda),\quad f'_{k,\pm}=f'(x_k,\pm\lambda),\quad f_{k+\varepsilon,\pm}=f(x_k+\varepsilon,\pm\lambda),\quad f'_{k+\varepsilon,\pm}=f'(x_k+\varepsilon,\pm\lambda);$$

$$R'_h(F)=\left\{\begin{array}{ll} \int\limits_{-\infty}^{\infty} R(x_k,t;\lambda)F(t)dt, & \text{if } h=2k-1, \\ \int\limits_{-\infty}^{\infty} R(x_k+\varepsilon,t;\lambda)F(t)dt, & \text{if } h=2k, \end{array}\right.$$

$$A_h=\left\{\begin{array}{ll} -\alpha_k, & h=2k-1, \\ \alpha_k, & h=2k, \end{array}\right. \left(k=\overline{1,m}\right);\ D^{\varepsilon}(\lambda)=\det(M_{4m}^{\varepsilon}(\lambda)),\ \text{where } M_{4m}^{\varepsilon}(\lambda)=$$

$$\left[\begin{array}{cccccc} -f_{1,-} & f_{1,+} & f_{1,-} & 0 & 0 & \cdots \\ -f'_{1,-} & (1+\frac{\alpha_1}{\varepsilon})f'_{1,+} & (1+\frac{\alpha_1}{\varepsilon})f_{1,-} & 0 & 0 & \cdots \\ 0 & -f_{1+\varepsilon,+} & -f_{1+\varepsilon,-} & f_{1+\varepsilon,+} & f_{1+\varepsilon,-} & \cdots \\ 0 & -(1+\frac{\alpha_1}{\varepsilon})f'_{1+\varepsilon,+} & -(1+\frac{\alpha_1}{\varepsilon})f'_{1+\varepsilon,-} & f'_{1+\varepsilon,+} & f'_{1+\varepsilon,-} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & -f_{m,+} & -f_{m,-} & f_{m,+} & f_{m,-} & 0 \\ \cdots & -f'_{m,+} & -f'_{m,-} & (1+\frac{\alpha_m}{\varepsilon})f'_{m,+} & (1+\frac{\alpha_m}{\varepsilon})f'_{m,-} & 0 \\ \cdots & 0 & 0 & -f_{m+\varepsilon,+} & -f_{m+\varepsilon,-} & f_{m+\varepsilon,+} \\ \cdots & 0 & 0 & -(1+\frac{\alpha_m}{\varepsilon})f'_{m+\varepsilon,+} & -(1+\frac{\alpha_m}{\varepsilon})f'_{m+\varepsilon,-} & f'_{m+\varepsilon,+} \end{array}\right]$$

Then for defining the number b_j^{ε}, from the conditions of problem (2.1) we get the system

$$M_{4m}^{\varepsilon}(\lambda)B^{\varepsilon}=\frac{1}{\varepsilon}AR',$$

where $B^{\varepsilon}=col(b_1^{\varepsilon},b_2^{\varepsilon},...,b_{4m}^{\varepsilon})$, $\qquad AR'=col(0,A_1R'_1,0,A_2R'_2,...,0,A_{2m}R'_{2m})$.

Define the set $\Gamma=\{\lambda: Im\lambda\neq 0, D^{\varepsilon}(\lambda)=0\}$. For $\lambda\notin\Gamma$ we have

$$b_j^{\varepsilon}=\frac{1}{\varepsilon D^{\varepsilon}(\lambda)}\sum_{p=1}^{2m}A_pR'_pM_{4m,2p,j}^{\varepsilon}(\lambda),$$

where $M_{4m,2p,j}^{\varepsilon}(\lambda)$ is an algebraic complement of the element $m_{i,j}$ of the matrix $M_{4m}^{\varepsilon}(\lambda)=(m_{i,j})_{4m\times 4m}$. If we introduce the denotation

$$X_p^{\varepsilon}(x,\lambda)=$$

$$\left\{\begin{array}{cc} A_1M_{4m,2p,1}^{\varepsilon}(\lambda)f(x,-\lambda), & x\in(-\infty,x_1), \\ A_k\left[M_{4m,2p,4k-2}^{\varepsilon}(\lambda)f(x,\lambda)+M_{4m,2p,4k-1}^{\varepsilon}(\lambda)f(x,-\lambda)\right], & x\in(x_k,x_k+\varepsilon)\left(n=\overline{1,m}\right), \\ A_k\left[M_{4m,2p,4k}^{\varepsilon}(\lambda)f(x,\lambda)+M_{4m,2p,4k+1}^{\varepsilon}(\lambda)f(x,-\lambda)\right], & x\in(x_k+\varepsilon,x_{k+1})\left(n=\overline{1,m-1}\right), \\ A_mM_{4m,2p,4m}^{\varepsilon}(\lambda)f(x,\lambda), & x\in(x_m+\varepsilon,\infty), \end{array}\right.$$

for $p = \overline{1, m}$, then the solution of problem (2.1) takes the form

$$R_\lambda^\varepsilon(F) \equiv y^\varepsilon(x, y) =$$

$$-\frac{1}{W[\varphi_1, \varphi_2]}\left[\int_{-\infty}^{\infty} R(x, t; \lambda) F(t)\, dt + \frac{1}{\varepsilon D^\varepsilon(\lambda)} \sum_{p=1}^{2m} X_p^\varepsilon(x, \lambda) R_p(F)\right]$$

$$\equiv R_\lambda(F) - \frac{1}{W[\varphi_1, \varphi_2]} \cdot \frac{1}{\varepsilon D^\varepsilon(\lambda)} \sum_{p=1}^{2m} X_p^\varepsilon(x, \lambda) R_p(F), \tag{2.2}$$

where

$$X_p^\varepsilon(., \lambda) \in L^2(\mathbb{R}) \left(p = \overline{1, 2m}\right), Im\lambda \neq 0, \lambda \notin \Gamma.$$

As $\varepsilon \to 0^+$; from expression (2.2) it follows that the resolvent R_λ^ε of L_ε converges in the operator norm the resolvent of the operator L restricted to the subspace $L_2(\mathbb{R}) \subset H$. □

3. Nature of The Spectrum of the Operator L

Now let's cite a theorem on the spectrum of the operator L.

Theorem 3.1. *Let all intensities of the δ- interactions be $\alpha_k > 0$, $k = \overline{1, m}$ .Then the spectrum of the operator L consists of the absolutely continuous part $[0, +\infty)$ on the continuous spectrum there are spectral singularities at the points $\lambda_n^2 = \left(\frac{n}{2}\right)^2$ $(n = 1, 2, 3...)$ of multiplicity $mn+1$, and has exactly m distinct eigenvalues on the negative half-line, that are determined as roots of the equation $\varepsilon D^\varepsilon(\lambda) = 0\,(\varepsilon \to 0^+)$.*

Proof. By the spectrum of the operator $L_0\left(\alpha_k \equiv 0, k = \overline{1, m}\right)$ is absolutely continuous and coincides with the set $[0, +\infty)$ on the continuous spectrum there are spectral singularities at the points $\lambda_n^2 = \left(\frac{n}{2}\right)^2$ $(n = 1, 2, 3...)$ of multiplicity $mn + 1$.Since the operator $(R_\lambda^\varepsilon - R_\lambda)(F)$ $(\varepsilon \to 0^+)$ is finite dimensional according to the known results of [6, 11], the absolutely continuous part of the spectrum of the operator L coincides with the absolutely continuous part of the spectrum of the operator $L_0\left(\alpha_k \equiv 0, k = \overline{1, m}\right)$, i.e. with $[0, +\infty)$. According to [12], the spectrum of the operator L may differ from the spectrum of the operator $L_0\left(\alpha_k \equiv 0, k = \overline{1, m}\right)$ unless by the finitely many negative eigenvalues. Furthermore, the number of these eigenvalues exactly m. □

Example 3.1. *Let $q(x) \equiv 0$, $m = 1$. Then the equation $\varepsilon D^\varepsilon(\lambda) = 0$ is of the form*

$$\varepsilon \begin{vmatrix} -e^{\lambda x_1} & e^{-\lambda x_1} & e^{\lambda x_1} & 0 \\ -\lambda e^{\lambda x_1} & -(1 + \frac{\alpha_1}{\varepsilon})e^{-\lambda x_1} & (1 + \frac{\alpha_1}{\varepsilon})e^{\lambda x_1} & 0 \\ 0 & -e^{-\lambda(x_1+\varepsilon)} & -e^{\lambda(x_1+\varepsilon)} & e^{-\lambda(x_1+\varepsilon)} \\ 0 & (1 + \frac{\alpha_1}{\varepsilon})e^{-\lambda(x_1+\varepsilon)} & -(1 + \frac{\alpha_1}{\varepsilon})e^{\lambda(x_1+\varepsilon)} & -\lambda e^{-\lambda(x_1+\varepsilon)} \end{vmatrix} = 0,$$

where $\lambda^2 < 0$.

Decomposing the determinant $D^{\varepsilon}(\lambda)$ into two rows, we obtain

$$-\varepsilon e^{\lambda x_1}\left[\begin{array}{c} -\lambda^2(2+\frac{\alpha_1}{\varepsilon})e^{-\lambda x_1}+\lambda^2(1+\frac{\alpha_1}{\varepsilon})\frac{\alpha_1}{\varepsilon}e^{-\lambda(x_1+2\varepsilon)}-\frac{\alpha_1^2}{\lambda}e^{-\lambda(x_1+2\varepsilon)} \\ -(2+\frac{\alpha_1}{\varepsilon})(1+\frac{\alpha_1}{\varepsilon})\lambda^2 e^{-\lambda x_1} \end{array}\right]=0.$$

Hence, we obtain that $\lambda^2 < 0$ is the eigenvalue of the operator $L(q(x) \equiv 0, m = 1)$ if λ is the solution of the equation $\alpha_1^2\ e^{-2\lambda\varepsilon} = (2\varepsilon + \alpha_1)^2$.

As $\varepsilon \to 0^+$, it follows that the operator L has exactly one eigenvalue in the form

$$\lambda^2 = -\frac{4}{\alpha_1^2}.$$

References

[1] N.I. Akhiezer and I.M. Glazman, Theory of linear operators in Hilbert Space, *Vol I and II. Ungar Pub. Co., New York,* 1961 and 1963.

[2] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, H. Holden with an appendix by P. Exner, Solvable Models in Quantum Mechanics (second edition), *AMS Chelsea Publ.*, (2005).

[3] S. Albeverio and P. Kurasov, Singular Perturbations of Differential Operators and Schrödinger Type Operators, *Cambridge Univ. Press*, (2000).

[4] S. Albeverio and L. Nizhnik, A Schrödinger operator with a δ'- interaction on a Cantor set and Krein- Feller operators, *Math. Nachr.*, **279** (2006), 467–476.

[5] F.A. Berezin and L.D. Faddeev, Remarks on Schrödinger Equation with Singular Potential, *Dokl. Akad. Nauk.*, **137(7)** (1961), 1011–1014 (in Russian).

[6] M.Sh. Birman and M.Z. Solomyak, Spectral Theory of Self-adjoint Operators in Hilbert Space, *D. Reidel Pulb.Co., Dordrecht, Holland,* (1987).

[7] Yu.D. Golovaty and R.O. Hryniv, On norm resolvent convergence of Schrödinger operators with δ'- like potentials, *J. Phys. A: Math. Theor.* **43(15)** (2010), 14 pp.

[8] A. Goriunov and V. Mikhallets, Resolvent convergence of Sturm-Liouville operators with singular potentials, *Mat. Zametki*, **87(2)**(2010), 311–315 (in Russian); *Engl. transl. in Math. Notes* **87(2)** (2010), 287–292.

[9] R.O. Hryniv and Ya.V. Mykytyuk, Eigenvalue asymptotics for Sturm-Liouville operators with singular potentials, *J. Funct. Anal.* **238** (2006), 27–57.

[10] T. Kappeler and C. Möhr, Estimates for periodic and Dirichlet eigenvalues of the Schrödinger operator with singular potentials, *J. Funct. Anal.*, **186** (2001), 62–91.

[11] T. Kato, Theory of Perturbation of Linear Operators, *M.:Mir,* (1972), 740 p.

[12] P.N. Knyazev, On the use of minimax properties in perturbation theory, *Russian Mathematics (İzvestiya VUZ. Matematika),* **2** (1959), 94–100.

[13] A.N. Kochubei, One-dimensional point interactions, *Ukr. Math. J.,* **41** (1989), 1391–1395.

[14] A.S. Kostenko and M.M. Malamud, 1-D Schrödinger operators with local point interactions on a discrete set, *J. Differential Equations* **249(2)** (2010), 253–304.

[15] P. Kurasov and J. Larson, Spectral asymptotics for Schrödinger operators with periodic point interactions, *J. Math. Anal. Appl.,* **266(1)** (2002), 127–148.

[16] M.Dzh. Manafov and M.G. Gasymov, Spectral analysis of a class of non-selfadjoint differential operators, *J. Spectral Math. Appl.,* **1** (2006), 8 p.

[17] M.Dzh. Manafov, Spectrum and spectral decomposition of a non-self-adjoint differential operator, *Math. Zametki,* **82 (1)** (2007), 58–63 (in Russian); *Eng. transl. in Math. Notes* **82 (1-2)** (2007), 52–56.

[18] V.A. Mikhailets, One-dimensional Schrödinger operator with point interactions, *Dokl. Math.,* **335(4)** (1994), 421–423.

[19] V.A. Mikhailets, Schrödinger operator with point δ'- interactions, *Dokl. Math.,* **348(6)** (1996), 727–730.

[20] R.A. Minlos and L.D. Faddeev, On pointwise interaction for a system of three particles in Quantum Mechanics, *Dokl. Akad. Nauk.,* **141(6)** (1961), 1335–1338 (in Russian).

[21] M.A. Naimark, Linear Differential operators, *Frederick Ungar Publ.Co., New York,* (1968).

[22] B.S. Pavlov, The theory of extensions and explicitly solvable models, *Uspekhi Mat. Nauk* **42** (1987), 99–131 (in Russian); *Engl. transl. in Russian Math. Surveys,* **42;6** (1987, 127–168.

[23] A.M. Savchuk and A.A. Shkalikov, Strum-Liouville operators with distributional potential, *Trudy Mosk. Math. Ob-va (Transl. Moscow Math. Soc.)* **64** (2003), 159–212 (in Russian).

[24] V.S. Vladimirov, Generalized Functions in Mathematical Physics, *M.: Nauka,* (1976), 280 p.(in Russian).

Manaf Dzh. Manafov

Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, Baku-Azerbaijan.

Department of Mathematics, Faculty of Arts and Sciences, Adıyaman University, Adıyaman-Turkey.

E-mail address: `mmanafov@adiyaman.edu.tr`

Received: June 15, 2014; Accepted: July 15, 2014