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NEW LIMIT-POINT CRITERIA FOR STURM-LIOUVILLE OPERATOR

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In memory of M. G. Gasymov on his 75*th birthday*

Abstract. Let $I := (a, b)$ be a finite or infinite interval. We assume that $p_0(x), q_0(x)$ and $p_1(x)$ are real-valued measurable functions on I, $p_0, p_0^{-1}, p_1^2 p_0^{-1}$ and $q_0^2 p_0^{-1}$ are locally Lebesque-integrable, i.e. belong to $\mathcal{L}^1_{loc}(I)$, and $w(x)$ is a positive function almost everywhere on I. Consider the operators generated in $\mathcal{L}^2_w(I)$ by the formal differential expression

$$l[f] := w^{-1}\{-(p_0 f^{'})^{'} + i[(q_0 f)^{'} + q_0 f^{'}] + p_1^{'} f\},$$

where the derivatives are understood in the sense of distribution. The method described in this paper gives the ability to correctly define the minimal operator L_0 generated by $l[f]$ in the space $\mathcal{L}^2_w(I)$ and include it in the class of operators generated by the second order symmetric (formally self-adjoint) quasi-differential expressions with locally integrable coefficients. Thus, the well-developed spectral theory of second order quasi-differential operators is applied to the Sturm-Liouville operators with distribution coefficients. The main goal of this work is to construct the Titchmarsch-Weyl theory for such operators. The central problem here is to find the conditions of the coefficients p_0, q_0 and p_1 when the limit-point or limit-circle cases can be realized. The obtained results are applied to the Hamiltonian theory with δ-interactions, i.e. when

$$l[f] = -f^{''} + \sum_j h_j \delta(x - x_j) f,$$

where h_j is a strength of the interaction at the points x_j, and to the associated Jacobi matrices.

1. Introduction. Preliminaries

1. Let $I := (a, b) \subset \mathbb{R}$; $p(x), q(x)$ and $w(x)$ be real-valued functions such that $p(x) \neq 0$, $w(x) > 0$ a.e. on I, p^{-1} $(:= 1/p)$, q and w are locally Lebesque-integrable (i.e. p^{-1}, q, $w \in \mathcal{L}^1_{loc}(I)$), and $r(x)$ be a complex-valued function on

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I with $r \in \mathcal{L}^1_{loc}(I)$. We define the quasi-derivative $f^{[1]}$ of a locally absolutely continuous complex-valued function f on I by $f^{[1]} := p(f' - rf)$ and assume that the function $f^{[1]}$ is also locally absolutely continuous. Thus, we can construct the second quasi-derivative $f^{[2]}$ by $f^{[2]} := (f^{[1]})' + \overline{r} f^{[1]} - qf$ and the quasi-differential expression $l[f]$ by $l[f] := -w^{-1} f^{[2]}$. Therefore,

$$l[f](x) = w^{-1}(x)\{ - [p(f' - rf)]' - \overline{r}p(f' - rf) + qf\}(x). \tag{1.1}$$

The domain Δ of the expression $l[f]$ is the set of all complex-valued functions f such that f and $f^{[1]}$ are locally absolutely continuous on I and $l[f]$ $(\in \mathcal{L}^1_{loc}(I))$ is represented by (1.1) for $f \in \Delta$.

Following the terminology of the theory of linear quasi-differential equations of arbitrary order, we also say that the quasi-derivatives $f^{[0]}(:= f)$, $f^{[1]}$ and $f^{[2]}$ are generated by the matrix

$$F = \begin{pmatrix} r & p^{-1} \\ q & -\overline{r} \end{pmatrix}.$$

Let $\mathcal{L}^2_w(I)$ be a space of the equivalence classes of all complex-valued measurable functions f such that $|f|^2 w$ is Lebesque-integrable on I. It is well known that in the spectral theory of ordinary differential equations there is the procedure of constructing the minimal and maximal operators (L_0 and L_1 respectively) generated by $l[f]$ in the Hilbert space $\mathcal{L}^2_w(I)$. Operators L_0, L_1 and the ones related to them are called Sturm-Liouville operators. More information about them and the terminology used in this paper can be found in the author's recent article [9].

Recall that the expression $l[f]$ is said to be regular if the interval (a, b) is finite and the elements of the matrix F and the function w are summable in the whole $[a, b]$. Otherwise, the expression $l[f]$ is said to be singular. In particular, the left end-point a of the interval (a, b) is regular if $a > -\infty$ and if the elements of matrix F and the function w are summable in every $[a, \beta] \subset [a, b)$, $\beta < b$; otherwise we say that the end-point a is singular. Similarly we define the regularity or singularity for the right end-point b.

Let the end-point a be regular and the end-point b be singular points of the interval I. Following H.Weyl, we define $l[f]$ to be of limit-point type at b if not all solutions to $l[y] = \lambda y$, $\lambda \in \mathbb{C}$ lie in $\mathcal{L}^2_w(I)$ and of limit - circle type otherwise. In other words, the expression $l[f]$ is in the limit-point case at b if the deficiency numbers - d_+ and d_- - of the operator L_0 are equal to 1. If $d_+ = d_- = 2$, then $l[f]$ is in the limit-circle case at b.

The following theorem holds.

Theorem 1.1. *Let the end-point a be regular and the end-point b be singular points of the interval I. The quasi-differential expression $l[f]$ is in the limit-point case at b if and only if the condition*

$$\sum_{k=1}^{+\infty} \left(\int_{\alpha_k}^{\beta_k} w(x)dx \int_{\alpha_k}^{x} |K(x,t)|^2 w(t)dt \right)^{1/2} = \infty \tag{1.2}$$

be fulfilled for some sequence of disjoint intervals $(\alpha_k, \beta_k) \subset (a, b)$, $k = 1, 2, \ldots$, where $K(x, t)$ is the Cauchy function of the equation $l[f] = 0$, i.e. is the solution of this equation with respect to x satisfying the initial conditions $K(x, t)|_{x=t} = 0$, $K^{[1]}(x, t)|_{x=t} = 1$.

We note here that Theorem 1.1 is a special case of Theorem 1 in the author's paper [8]. Theorem 1 was formulated there for arbitrary quasi-differential operators of arbitrary order and spaces $\mathcal{L}^p_w(I)$.

The problem of finding additional conditions on the functions p, q, r and w, which would guarantee the limit-point case for the differential expressions has been studied by many mathematicians over the last 100 years, and several such conditions have already been found. Theorem 1.1 has been formulated and proven in [9] (see [9], Theorem 4) and it was shown there that this theorem really allows to obtain the most of known and some of new sufficient conditions for the limit-point case (see [9], Theorem 8).

In this work, as a corollary of Theorem 1.1, we present another sufficient condition for the limit-point case (Theorem 2.1).

2. Let p_0, q_0 and p_1 be real-valued Lebesgue-measurable functions such that $p_0^{-1}, p_1^2 p_0^{-1}, q_0^2 p_0^{-1}$ are locally Lebesgue-integrable (i.e. $p_0^{-1}, p_1^2 p_0^{-1}, q_0^2 p_0^{-1} \in \mathcal{L}^1_{loc}(I)$). Let $\varphi := p_1 + iq_0$. Consider the matrix

$$F = \frac{1}{p_0}\begin{pmatrix} \varphi & 1 \\ -|\varphi|^2 & -\overline{\varphi} \end{pmatrix}.$$

Using matrix F, we define the quasi-derivatives $f^{[0]}$, $f^{[1]}$, $f^{[2]}$, as follows (similar to Section 1)

$$f^{[0]} = f, \quad f^{[1]} = p_0 f' - \varphi f, \quad f^{[2]} = (f^{[1]})' + \frac{\overline{\varphi}}{p_0} f^{[1]} + \frac{|\varphi|^2}{p_0} f.$$

Assume further that p_0 also belongs to $\mathcal{L}^1_{loc}(I)$, it is easy to see that $\varphi \in \mathcal{L}^1_{loc}(I)$. It is possible to conclude from these assumptions that if $'$ is a distributional derivative then we can remove all parentheses in $f^{[2]}$ and this expression can be rewritten as follows

$$f^{[2]} = (p_0 f')' - i((q_0 f)' + q_0 f') - p_1' f.$$

It is necessary to emphasize that monomials $(p_0 f')'$, $(q_0 f)'$ and $p_1' f$ are singular generalized functions. The first two of them are the distributional derivatives of regular generalized functions, while $q_0 f'$ and $f^{[2]}$ are regular generalized functions.

Thus, the expression $l[f]$ (see (1.1)) can be formally rewritten in terms of generalized functions

$$l[f] = w^{-1}\{-(p_0 f')' + i((q_0 f)' + q_0 f') + p_1' f\}. \tag{1.3}$$

In particular, if $w(x) = 1$, $p_0(x) = 1$, $q_0(x) = 0$, $p_1(x) = \sigma(x)$, where $\sigma^2(x) \in \mathcal{L}^1_{loc}(I)$, then the quasi-differential expression $l[f]$ has the form

$$l[f] = -f'' + \sigma' f. \tag{1.4}$$

Let $I = [0, +\infty)$, x_n $(n = 1, 2, \dots)$ be an increasing sequence of positive numbers, $x_0 = 0$ and $\lim\limits_{n\to+\infty} x_n = +\infty$. If, in addition, the function $\sigma(x)$ is a step function with jumps $h_j \in \mathbb{R}$ at the points $x_j \in I$ then

$$l[f] = -f'' + \sum_{j=1}^{+\infty} h_j \delta(x - x_j) f, \tag{1.5}$$

where $\delta(x)$ is the Dirac δ-function.

Thus, the Hamiltonian theory with δ-interactions, i.e. the theory of operators generated by the expressions (1.5), is included in the theory of operators generated by the quasi-differential expressions of second order. In particular, there is Weyl's limit-point/limit-circle dichotomy for the corresponding expressions.

The following theorem holds.

Theorem 1.2. *The expression $l[f]$ (see (1.5)) is in the limit-circle case if and only if all solutions of the equation*

$$\frac{Z_{k+1}}{r_{k+1}r_{k+2}d_{k+1}} - \frac{1}{r_{k+1}^2}\left(h_k + \frac{1}{d_k} + \frac{1}{d_{k+1}}\right)Z_k + \frac{Z_{k-1}}{r_k r_{k+1} d_k} = 0, \; k = 1, 2, \ldots,$$

belong to the space l^2, where $d_k = x_k - x_{k-1}$, $r_{k+1} = \sqrt{d_k + d_{k+1}}$.

The authorship of this theorem is assigned to M.M. Malamud and A.S. Kostenko (see [4], [5] and also [9]).

3. In accordance with the Theorem 1.2, we will need the following facts about the theory of operators generated by the second order difference expressions in the Hilbert space l^2, where l^2 is the space of all sequences of vectors $u = (u_0, u_1, \ldots)$, $u_j \in C$ with the standard scalar product (see, e.g. [1]).

Let a_j and b_j be sequences of real numbers, $b_j \neq 0$ for $j = 0, 1, \ldots$. Consider the Jacobi matrix

$$J = \begin{pmatrix} a_0 & b_0 & 0 & 0 & \ldots \\ b_0 & a_1 & b_1 & 0 & \ldots \\ 0 & b_1 & a_2 & b_2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and the second order difference expression generated by the matrix J

$$(lu)_j = b_j u_{j+1} + a_j u_j + b_{j-1} u_{j-1}, \; j = 0, 1, \ldots,$$

where $u_{-1} = 0$. In the standard way we define the minimal closed symmetric operator L_0 with an everywhere dense domain. The deficiency index of L_0 is $(0,0)$ or $(1,1)$. Following the terminology of the classical moment problem, we say that there is the definite case for the operator L_0 if deficiency index is $(0,0)$ and there is the indefinite case for L_0 otherwise. Moreover these two cases correspond to the limit-point and the limit-circle cases for the Sturm-Liouville operators.

Theorem 1.2 shows that the deficiency number of the operator L_0 generated by (1.5) in the Hilbert space $\mathcal{L}^2(0, +\infty)$ is equal to 2 if and only if there is the indefinite case for the Jacobi matrix J, where a_0, $b_0 \neq 0$ are arbitrary real numbers and

$$a_j = -\frac{1}{r_{j+1}^2}\left[h_j + \frac{1}{d_j} + \frac{1}{d_{j+1}}\right], \quad b_j = \frac{1}{r_{j+1}r_{j+2}d_{j+1}}, \quad j = 1, 2, \ldots. \tag{1.6}$$

In [6], the analogue of Theorem 1.1 for the operators generated by the Jacobi matrices J is proved, namely,

Theorem 1.3. *For the realization of the indefinite case for l, it is necessary and sufficient that for any sequence of intervals of positive integers $[n_k, m_k]$ such that*

$m_k \le n_{k+1} \le m_{k+1}$ *$(k = 1, 2, \dots)$ the following condition be satisfied*

$$\sum_{k=1}^{+\infty} \left(\sum_{i=n_k}^{m_k} \sum_{j=n_k}^{i} |K_{ij}|^2 \right)^{1/2} < +\infty, \tag{1.7}$$

where K_{ij} is the Cauchy function of the operation l, i.e. it is the solution of difference equation $(lu)_j = 0$ with initial conditions $K_{jj} = 0$, $K_{j+1,j} = b_j^{-1}$.

2. Sufficient conditions for the limit point case

1. Further, we always assume that $I = [0, +\infty)$ and the functions $p(x)$, $q(x)$, $r(x)$, $w(x)$ satisfy the conditions in section 1 §1 on I (or the functions $p_0(x), q_0(x)$ and $p_1(x)$ satisfy the conditions in section 2 § 1). Let $(\alpha_k, \beta_k) \subset I$, $k = 1, 2, \dots$ be a sequence of disjoint intervals and $K(x,t)$ be the Cauchy function of the equation $l[f] = 0$. We note here that the function $K(x,t)$ is uniquely determined by the coefficients of l in the triangle $\{(x,t) | \alpha_k < t \le x < \beta_k\}$ for fixed k . Thus, if the condition (1.2) is fulfilled then independently of the values of these functions outside the set $\bigcup_{k=1}^{+\infty} [\alpha_k, \beta_k]$ the expression l is in the limit-point case at infinity. This fact we will use later.

The following lemma is valid.

Lemma 2.1. *Let a, b and c be positive numbers with $a < c < b$, h be any real number and $K(x,t)$ be the Cauchy function of the quasi-differential equation*

$$-y'' + h\delta(x - c)y = 0. \tag{2.1}$$

Then the inequality

$$\int_a^b dx \int_a^x K^2(x,t)dt \ge \frac{1}{3\sqrt{3}} (\rho s)^2 (\rho + s) \left| h + \frac{3}{2} \left(\frac{1}{\rho} + \frac{1}{s} \right) \right|,$$

is true, where $\rho = c - a$ and $s = b - c$.

Proof. In the definition of the quasi-differential expression $l[f]$ (1.4), assume

$$\sigma(x) = \begin{cases} 0, & \text{if } a \le x < c, \\ h, & \text{if } c \le x \le b. \end{cases}$$

Then the equation $l[f] = 0$ takes the form of (2.1). Let $t \in [a, b]$ be fixed. We calculate now the Cauchy function $K(x,t)$ using its definition. This function satisfies the equation (2.1) for $x > t$ and the initial conditions $K(x,t)|_{x=t} = 0$ and $K^{[1]}(x,t)|_{x=t} = 1$. It is easy to show that if $t \in [a, c]$, $x \ge t$, $x \le c$ or $t \in [c, b]$, $x \ge t$, $c \le x \le b$ then $K(x,t) = x - t$.

Now let $t \in (a, c)$ and $x \in (c, b)$. Then $K(x,t) = c_1(t) + c_2(t)x$ and consequently

$$K(c + 0, t) = c_1(t) + c_2(t)c,$$

$$K^{[1]}(c + 0, t) = \lim_{x \to c+0} [K'(x,t) - hK(x,t)] = c_2(t) - h(c_1(t) + c_2(t)c).$$

On the other hand,

$$K(c - 0, t) = c - t, \; K^{[1]}(c - 0, t) = 1.$$

Using the continuity of $K(x,t)$ and $K^{[1]}(x,t)$ at the point $x=c$, we obtain

$$\begin{cases} c_1(t)+c_2(t)c = c-t, \\ c_2(t)-h(c_1(t)+c_2(t)c)=1 \end{cases}$$

and hence, $c_1(t)=-t-h(c-t)c \ \ c_2(t)=1+h(c-t)$.

Thus,

$$K(x,t)=\begin{cases} x-t, & \text{if } t\in[a,c] \text{ and } x\in[t,c], \\ x-t+h(c-t)(x-c), & \text{if } t\in[a,c] \text{ and } x\in[c,b], \\ x-t, & \text{if } t\in[c,b] \text{ and } x\in[t,b]. \end{cases}$$

From this formula it follows that

$$J:=\int_a^b dx\int_a^x K^2(x,t)dt =: J_1+2hJ_2+h^2J_3,$$

where

$$J_1:=\int_a^b dx\int_a^x (x-t)^2dt,\ J_2:=\int_c^b dx\int_a^c (x-t)(x-c)(c-t)dt,$$

$$J_3:=\int_c^b dx\int_a^c (x-c)^2(c-t^2)dt.$$

Calculations show that

$$J_1=\frac{1}{12}(b-a)^4,\ J_2=\frac{1}{6}(b-c)^2(c-a)^2(b-a),\ J_3=\frac{1}{9}(b-c)^3(c-a)^3.$$

Thus,

$$J=\frac{h^2}{9}(\rho s)^3+\frac{h}{3}\rho^2s^2(\rho+s)+\frac{1}{12}(\rho+s)^4.$$

We rewrite now the integral J in the form

$$J=(\rho s)^3\left[\left[\frac{h}{3}+\frac{1}{2}\left(\frac{1}{\rho}+\frac{1}{s}\right)\right]^2+\left(\frac{1}{\rho}+\frac{1}{s}\right)^2\left[\frac{1}{12}\rho s\left(\frac{1}{\rho}+\frac{1}{s}\right)^2-\frac{1}{4}\right]\right]$$

and note that

$$\frac{1}{12}\rho s\left(\frac{1}{\rho}+\frac{1}{s}\right)^2-\frac{1}{4}\geq\frac{1}{12}.$$

Therefore, we obtain

$$J\geq(\rho s)^3\left[\left[\frac{h}{3}+\frac{1}{2}\left(\frac{1}{\rho}+\frac{1}{s}\right)\right]^2+\frac{1}{12}\left(\frac{1}{\rho}+\frac{1}{s}\right)^2\right].$$

Applying the inequality of arithmetic and geometric means once again, we conclude that Lemma 2.1 holds.

Using this lemma, we prove the following theorem.

Theorem 2.1. *Let $(\alpha_k,\beta_k)\subset[0,+\infty)$ $(k=1,2,\ldots)$ be a sequence of pairwise disjoint intervals, γ_k be a sequence of positive numbers such that $\alpha_k<\gamma_k<\beta_k$ and the equalities $w(x)=p(x)=1$, $r(x)=\sigma(x)$, $q(x)=\sigma^2(x)$ be fulfilled on every fixed interval $[\alpha_k,\beta_k]$, where $\sigma(x)$ is a piecewise continuous function with a* *single jump h_k at the point γ_k. Further, let the numbers $\rho_k = \gamma_k - \alpha_k$, $s_k = \beta_k - \gamma_k$ and h_k be such that*

$$\sum_{k=1}^{+\infty} \rho_k s_k \sqrt{\rho_k + s_k} \sqrt{\left| h_k + \frac{3}{2}\left(\frac{1}{\rho_k} + \frac{1}{s_k}\right)\right|} = +\infty. \tag{2.2}$$

Then the operator L_0 is of limit-point type.

Proof. Apply Lemma 2.1 with $a = \alpha_k$, $c = \gamma_k$, $b = \beta_k$. Then we have

$$\int_{\alpha_k}^{\beta_k} dx \int_{\alpha_k}^{x} K^2(x,t)dt \geq \frac{1}{3\sqrt{3}}(\rho_k s_k)^2(\rho_k + s_k)\left| h_k + \frac{3}{2}\left(\frac{1}{\rho_k} + \frac{1}{s_k}\right)\right|$$

for $k = 1, 2, \ldots$.
Taking the square root of both sides of this inequality and summing over k, we obtain

$$\sum_{k=1}^{+\infty}\left(\int_{\alpha_k}^{\beta_k} dx \int_{\alpha_k}^{x} K^2(x,t)dt\right)^{1/2} \geq$$

$$3^{-3/4}\sum_{k=1}^{+\infty} \rho_k s_k \sqrt{(\rho_k + s_k)} \sqrt{\left| h_k + \frac{3}{2}\left(\frac{1}{\rho_k} + \frac{1}{s_k}\right)\right|}.$$

Using now the condition (2.2) and applying Theorem 1.1 we completely prove Theorem 2.1.

2. Some corollaries of Theorem 2.1.

If we assume that the points γ_k are the mid-points of the intervals $[\alpha_k, \beta_k]$ $(k = 1, 2, \ldots)$ then $s_k = \rho_k$ and, of course, we can simplify the condition of Theorem 2.1 Thus, the following corollary of Theorem 2.1 is true.

Corollary 2.1. *Let the conditions of Theorem 2.1 be fulfilled and in addition $\gamma_k = \frac{\alpha_k + \beta_k}{2}$. Then the operator L_0 is of limit-point type if*

$$\sum_{k=1}^{+\infty} \rho_k^{5/2} \sqrt{\left| h_k + \frac{6}{\rho_k}\right|} = +\infty,$$

where $\rho_k = \beta_k - \alpha_k$.

Let the differential expression l be in the form of (1.5). Choose x_k as the points γ_k, $[x_k - \frac{d_k}{2}, x_k + \frac{d_{k+1}}{2}]$ as the intervals $[\alpha_k, \beta_k]$, where $d_k = x_k - x_{k-1}$ $(k = 1, 2, \ldots)$ and apply Theorem 2.1, we obtain the following corollary.

Corollary 2.2. *Let the differential expression $l[f]$ be as in* (1.5) *and*

$$\sum_{k=1}^{+\infty} d_k d_{k+1} r_{k+1} \sqrt{\left| h_k + \frac{3}{2}\left(\frac{1}{d_k} + \frac{1}{d_{k+1}}\right)\right|} = +\infty,$$

where $r_{k+1} = \sqrt{d_k + d_{k+1}}$. Then the operator L_0 is of limit-point type.

Using Theorem 1.2, we also note that if the condition of Corollary 2.2 is fulfilled, then there is the definite case for the Jacobi matrix J with the elements (1.6).

3. In sections 1,2 of this paragraph we have shown the way to use Theorem 1.1 to obtain the limit-point criteria for Sturm-Liouville operators generated by the expressions (1.1), (1.3) or (1.5). Some of these criteria, as we have already noted above, are the definite case criteria for the Jacobi matrix J with the elements (1.6). In this section, using Theorem 1.3, we obtain the definite case criteria for the Jacobi matrices J with the elements (1.6). We also note that according to Theorem 1.2, these criteria will be at the same time the limit-points criteria for Sturm-Liouville operators generated by (1.5).

The following theorem holds.

Theorem 2.2. *Let d_j and h_j $(j = 1, 2, \ldots)$ be number sequences such that*

$$\sum_{j=1}^{+\infty} d_j^{3/2} d_{j+1} \sqrt{\left| h_j + \frac{1}{d_j} + \frac{1}{d_{j+1}} \right|} = +\infty. \tag{2.3}$$

Then the Jacobi matrix J with the elements (1.6) *is in the definite case.*

Proof. Consider two series S_1 and S_2 of the form (1.7), taking the sequence of intervals $[n_k, m_k]$ with the end-points $n_k = 2k$, $m_k = 2k + 2$ in the first case and with the end-points $n_k = 2k + 1$, $m_k = 2k + 3$ in the second one.

In the first case we have

$$\left(\sum_{i=2k}^{2k+2} \sum_{j=2k}^{i} |K_{ij}|^2 \right)^{1/2} \geq \sqrt{2} |K_{2k+1,2k} K_{2k+2,2k}|^{1/2}.$$

Consequently,

$$S_1 \geq \sqrt{2} \sum_{k=0}^{+\infty} |K_{2k+1,2k} K_{2k+2,2k}|^{1/2}.$$

Similarly we obtain

$$S_2 \geq \sqrt{2} \sum_{k=0}^{+\infty} |K_{2k+2,2k+1} K_{2k+3,2k+1}|^{1/2}.$$

Adding together the obtained inequalities, we have

$$S_1 + S_2 \geq \sqrt{2} \sum_{j=1}^{+\infty} |K_{j,j-1} K_{j+1,j-1}|^{1/2}.$$

On the other hand, according to the definition of Cauchy sequence K_{ij}, it is known that $K_{jj} = 0$, $K_{j+1,j} = b_j^{-1}$. Moreover using the difference equation for K_{ij}, we obtain $K_{j+2,j} = -b_{j+1}^{-1} b_j^{-1} a_{j+1}$. Consequently,

$$|K_{j,j-1} K_{j+1,j-1}| = b_{j-1}^{-1} \sqrt{b_j^{-1} |a_j|} \geq d_j^{3/2} d_{j+1} \sqrt{\left| h_j + \frac{1}{d_j} + \frac{1}{d_{j+1}} \right|}.$$

Thus, from the conditions of Theorem 2.1 it follows that the series $S_1 + S_2$ is divergent. Hence, either S_1 or S_2 is divergent. Applying now Theorem 1.3, we complete the proof of Theorem 2.2.

We also note (in more details see [6]) that, using the ideas of the proof of Theorem 2.2 and Theorem 1.3, we can obtain other sufficient conditions for the definite case for the matrix J by choosing other sequences of intervals $[n_k, m_k]$. However, we have only restricted ourselves here to the above theorem.

3. Conclusion. Examples

In Theorem 8 of [9], it is assumed that the quasi-differential expression l has the form (1.4) on the sequence of disjoint intervals (α_k, β_k) $(k = 1, 2, \ldots)$, where the function $\sigma(x)$ is absolutely continuous on every $[\alpha_k, \beta_k]$ and satisfies some additional conditions which guarantee that (1.1) or (1.3) are in the limit-point case. More exactly, we assume in this theorem, in particular, that the expression l is a regular differential expression on $[\alpha_k, \beta_k]$, and the coefficients in (1.3) *can be* singular generalized functions only outside these intervals. In Theorem 2.1 of the present work we considers, for the first time, the case when the coefficients *are* the singular generalized functions on the intervals $[\alpha_k, \beta_k]$.

In [2], C.S. Christ and G. Stolz proved that if $d_j = \frac{1}{j}$ and $h_j = -2j - 1$ $(j = 1, 2, \ldots)$ then there is the limit-circle case for l in (1.5). They were probably the first to show that it is possible for such expression. Later, A.S. Kostenko and M.M. Malamud (see [4], [5]) and N.N. Konechnaya (see [3]) constructed numerous examples of the limit-point or limit-circle cases for (1.5). In particular, using Theorem 1 of [7], the following theorem was proved in [5].

Theorem 3.1. *Let* $d_j = x_j - x_{j-1}$ $(j = 1, 2, \ldots)$ *be sequence such that*

$$d_{j-1}d_{j+1} \geq d_j^2, \qquad \sum_{j=1}^{+\infty} d_j^2 < \infty, \qquad \sum_{j=1}^{+\infty} d_{j+1}|h_j + \frac{1}{d_j} + \frac{1}{d_{j+1}}| < \infty.$$

Then the expression l *in* (1.5) *is in the limit-circle case.*

Theorem 3.1 shows that if the condition (2.3) of Theorem 2.2 is not satisfied, then the limit-circle case for (1.5) is also possible.

Now consider the Jacobi matrix J with the elements a_j, b_j $(j = 0, 1, \ldots)$ defined by (1.6), and assume that $h_j = -\frac{1}{d_j} - \frac{1}{d_{j+1}}$. Using Theorem 1 of [6], it is easy to obtain that there is the indefinite case for this matrix if the series

$$\sum_{j=1}^{+\infty} \left\{ \left(r_{2j+1} \prod_{k=1}^{j} \frac{d_{2k}}{d_{2k-1}} \right)^2 + \left(r_{2j+2} \prod_{k=1}^{j} \frac{d_{2k+1}}{d_{2k}} \right)^2 \right\} \tag{3.1}$$

is convergent. On the other hand, it is easy to prove that if this series is divergent, then there is the definite case for the matrix J. So we have established the following generalization of the Christ-Stolz example.

Let the elements of the matrix J *be defined by the formulas* (1.6) *and* $h_j = -\frac{1}{d_j} - \frac{1}{d_{j+1}}$. *Then there is the indefinite case for this matrix if and only if the series* (3.1) *is convergent.*

Now let $h_j = -\frac{3}{2}\left(\frac{1}{d_j} + \frac{1}{d_{j+1}}\right)$. Then the series in Corollary 2.2 is convergent and therefore this corollary is not applicable. In this case, the elements b_j of the matrix J are the same as in (1.6) and the elements a_j are defined by the formula $a_j = \frac{1}{2r_{j+1}^2}\left[\frac{1}{d_j} + \frac{1}{d_{j+1}}\right]$. Now we apply again Theorem 1 of [6] and obtain that if the series (3.1) and

$$\sum_{j=1}^{+\infty} \left\{ \left(\prod_{k=1}^{j} \frac{d_{2k}}{d_{2k-1}} \right)^2 \left(\frac{1}{d_{2j}} + \frac{1}{d_{2j+1}} \right) + \left(\prod_{k=1}^{j} \frac{d_{2k+1}}{d_{2k}} \right)^2 \left(\frac{1}{d_{2j+1}} + \frac{1}{d_{2j+2}} \right) \right\}$$

are convergent then there is the indefinite case for the matrix J. On the other hand, the numbers d_j can be chosen in such a way that $\sum_{j=1}^{+\infty} d_j^2 = \infty$. Then, according to Carleman theorem, the matrix J is in the definite case. Thus, if $h_j = -\frac{3}{2}\left(\frac{1}{d_j} + \frac{1}{d_{j+1}}\right)$ then both are possible, namely, the definite and the indefinite cases for the matrix J.

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