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GENERALIZATION OF ONE M.G. GASYMOV THEOREM ON SOLVABILITY OF A BOUNDARY VALUE PROBLEM FOR SECOND ORDER OPERATOR-DIFFERENTIAL EQUATIONS OF ELLIPTIC TYPE

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In memory of M. G. Gasymov on his 75th birthday

Abstract. In the paper a generalization of one M. G. Gasymov theorem on solvability of a boundary value problem for a class of second order operator-differential equations of elliptic type is established. Therewith, the exact value of the norm of the intermediate derivative operator is found and its relation with solvability conditions is shown.

1. Introduction

In the separable Hilbert space H consider the boundary value problem

$$P\,(d/dt)\,u(t) = (d/dt - \omega_1 A)(d/dt - \omega_2 A)u(t) + A_1 \frac{du(t)}{dt} = f(t),\ t \in \mathbb{R}_+ = (0, +\infty), \tag{1.1}$$

$$u(0) = 0, \tag{1.2}$$

where the derivatives are understood in the sense of distributions theory [4], and the operator coefficients satisfy the conditions:

1. A is a positive-definite self-adjoint operator ($A = A^* \geq cE$, $c > 0$, E - is a unit operator) with domain of definition $D\,(A)$;
2. ω_1, ω_2 are complex numbers, $Re\omega_1 < 0$, $Re\omega_2 > 0$;
3. the operator $B_1 = A_1 A^{-1}$ is bounded in H.

As is known, the domain of definition of the operator A^γ $(\gamma \geq 0)$ becomes a Hilbert space H_γ with respect to the scalar product $(x, y)_\gamma = (A^\gamma x, A^\gamma y)$, $x, y \in D\,(A^\gamma)$. For $\gamma = 0$ we assume $H_0 = H$.

Denote by $L_2(\mathbb{R}_+; H)$ Hilbert space of all functions $f(t)$ determined in $\mathbb{R}_+ = (0,\ \infty)$ almost everywhere, with the values in H, with the norm

$$\|f\|_{L_2(\mathbb{R}_+;H)} = \left(\int_0^{+\infty} \|f(t)\|^2 \, dt \right)^{1/2}.$$

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Following the monograph [4], introduce the Hilbert space

$$W_2^2(\mathbb{R}_+;H)=\left\{u:\ u''\in L_2(\mathbb{R}_+;H)\,,A^2u\in L_2(\mathbb{R}_+;H)\right\}$$

with the norm

$$\|u\|_{W_2^2(\mathbb{R}_+;H)}=\left(\|u''\|^2_{L_2(\mathbb{R}_+;H)}+\left\|A^2u\right\|^2_{L_2(\mathbb{R}_+;H)}\right)^{1/2}.$$

The spaces $L_2(\mathbb{R};H)$ and $W_2^2(\mathbb{R};H)$ are determined in the similar way for $\mathbb{R}=(-\infty,\ \infty)$.

From the theorem on traces [4] it follows that the following linear sets are complete subspaces of the space $W_2^2(\mathbb{R}_+;H)$:

$$W_2^2(\mathbb{R}_+;H;0,1)=\left\{u:\ u\in W_2^2(\mathbb{R}_+;H),\, u(0)=u'(0)=0\right\},$$

$$W_2^2(\mathbb{R}_+;H;0)=\left\{u:\ u\in W_2^2(\mathbb{R}_+;H),\, u(0)=0\right\}.$$

Definition. Problem (1.1), (1.2) is said to be *regularly solvable* if for any function $f(t)\in L_2(\mathbb{R}_+;H)$ there exists the function $u(t)\in W_2^2(\mathbb{R}_+;H)$ that satisfies equation (1.1) almost everywhere in $\mathbb{R}_+$, boundary condition (1.2) in the sense of convergence $\lim\limits_{t\to+0}\|u(t)\|_{3/2}=0$ and it holds the estimation

$$\|u\|_{W_2^2(\mathbb{R}_+;H)}\le const\,\|f\|_{L_2(\mathbb{R}_+;H)}.$$

In the present paper we show sufficient conditions for regular solvability of problem (1.1), (1.2), expressed by its coefficients.

For the first time, the regular solvability of boundary value problem (1.1), (1.2) when $\omega_1=-1,\ \omega_2=1$ was investigated in the papers of M. G. Gasymov [1, 2] in connection with completeness of a part of eigen and associated vectors of the bundle $P(\lambda)$. Further this result was generalized in the paper [3], when the boundary condition contains some linear operator. For $Im\omega_1=Im\omega_2=0$ problem (1.1), (1.2) was studied in the paper [9]. Note that in the papers [5, 11] higher order boundary value problems were investigated.

Following the papers [3, 6-10], for obtaining solvability conditions we'll find the exact value of the norm of the intermediate derivative operator in the space $W_2^2(\mathbb{R}_+;H;0)$ and connect it with the regular solvability condition.

2. Main results

At first we consider the boundary value problem

$$P_0\,(d/dt)\,u\,(t)=(d/dt-\omega_1A)\,(d/dt-\omega_2A)\,u\,(t)=f(t),t\in\mathbb{R}_+,$$

$$u(0)=0.$$

Denote by P_0 an operator acting from the space $W_2^2(\mathbb{R}_+;H;0)$ to $L_2(\mathbb{R}_+;H)$ in the following way:

$$P_0u=P_0\,(d/dt)\,u,u\ \in W_2^2(\mathbb{R}_+;H;0).$$

Using the intermediate derivatives theorem [4], we get

$$\|P_0u\|^2_{L_2(\mathbb{R}_+;H)}\le$$

$$2\left(\|u''\|^2_{L_2(\mathbb{R}_+;H)}+|\omega_1+\omega_2|^2\,\|Au'\|^2_{L_2(\mathbb{R}_+;H)}+|\omega_1\omega_2|^2\,\|A^2u\|^2_{L_2(\mathbb{R}_+;H)}\right)\le$$

$$const\,\|u\|^2_{W_2^2(\mathbb{R}_+;H)},$$

i.e. P_0 is a bounded operator.

The following theorem is valid.

Theorem 1. *The operator P_0 realizes isomorphism from the space $W_2^2(\mathbb{R}_+;H;0)$ onto $L_2(\mathbb{R}_+;H)$.*

Proof. Since P_0 is a bounded operator, it suffices to show that $KerP_0 = \{0\}$, $JmP_0 = L_2(R_+;H)$. Since the homogeneous equation $P_0\,(d/dt)\,u\,(t) = 0$ has a general solution from the space $W_2^2(\mathbb{R}_+;H)$ in the form $u_0\,(t) = e^{\omega_1 tA}x$, $x \in H_{3/2}$, then from the condition $u(0) = 0$ it follows that $u_0(t) = 0$. Further, it is easy to see that the general solution of the equation $P_0\,(d/dt)\,u\,(t) = f(t)$ is represented in the form $u\,(t) = \alpha\,(t) + e^{\omega_1 tA}x$, where $x \in H_{3/2}$ is an unknown vector, and

$$\alpha(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} P_0^{-1}\,(i\zeta, A)\left(\int_0^{+\infty} f(s)e^{-i\zeta s}ds\right)e^{i\zeta t}d\zeta,\ t \in \mathbb{R}_+,$$

where

$$P_0^{-1}\,(i\zeta, A) = (i\zeta - \omega_2 A)^{-1}\,(i\zeta - \omega_1 A)^{-1},\ \zeta \in \mathbb{R}.$$

Belonging of $\alpha(t)$ to the space $W_2^2(\mathbb{R}_+;H)$ follows from the Parseval theorem for the Fourier integrals. From the theorem on traces [4] it follows that $\alpha\,(0) \in H_{3/2}$. Then the condition $u(0) = 0$ yields $u\,(t) = \alpha\,(t) - e^{\omega_1 tA}\alpha\,(0)$. Thus, $u \in W_2^2(\mathbb{R}_+;H;0)$ and $P_0 u = f$. The theorem is proved.

Now prove a conditional theorem on regular solvability of problem (1.1), (1.2).

Theorem 2. *Let $\|B_1\| \le N_1^{-1}\,(0)$, where*

$$N_1\,(0) = \sup_{0\ne u\,\in W_2^2(\mathbb{R}_+;H;0)} \left\|Au'\right\|_{L_2(\mathbb{R}_+;H)} \|P_0 u\|^{-1}_{L_2(\mathbb{R}_+;H)}.$$

Then boundary value problem (1.1), (1.2) *is regularly solvable.*

Proof. The finiteness of the norm $N_1\,(0)$ follows from theorem 1 and from the intermediate derivatives theorem [4]. Write problem (1.1), (1.2) in the form of the operator equation $P_0 u + P_1 u = f$, where $u \in W_2^2(\mathbb{R}_+;H;0)$, $f(t) \in L_2(\mathbb{R}_+;H)$, and the operator $P_1 u = A_1 u'$ whose boundedness follows from condition 3) and theorem 1. By theorem 1 P_0^{-1} is an isomorphism. Then after substitution of $u = P_0^{-1}\vartheta$ we get the solution of the equation $\left(E + P_1 P_0^{-1}\right)\vartheta = f$ in the space $L_2(\mathbb{R}_+;H)$. On the other hand, for any $\vartheta(t) \in L_2(\mathbb{R}_+;H)$ the following inequalities hold:

$$\left\|P_1 P_0^{-1}\vartheta\right\|_{L_2(\mathbb{R}_+;H)} = \|P_1 u\|_{L_2(\mathbb{R}_+;H)} \le \|B_1\|\,\left\|Au'\right\|_{L_2(\mathbb{R}_+;H)} \le$$
$$\|B_1\|\,N_1\,(0)\,\|P_0 u\|_{L_2(\mathbb{R}_+;H)} = \|B_1\|\,N_1\,(0)\,\|\vartheta\|_{L_2(\mathbb{R}_+;H)}.$$

Since $\|B_1\|\,N_1\,(0) < 1$, then the operator $E + P_1 P_0^{-1}$ is inversible in $L_2(\mathbb{R}_+;H)$, $u = P_0^{-1}\left(E + P_1 P_0^{-1}\right)^{-1} f$ and

$$\|u\|_{W_2^2(\mathbb{R}_+;H)} \le const\,\|f\|_{L_2(\mathbb{R}_+;H)}.$$

The theorem is proved.

From this theorem it follows that for finding the exact solvability conditions of boundary value problem (1.1), (1.2), it is necessary to find the norm $N_1\,(0)$.

At first we find the norm

$$\sup_{0\ne u\,\in W_2^2(\mathbb{R}_+;H;0,1)} \left\|Au'\right\|_{L_2(\mathbb{R}_+;H)} \|P_0 u\|^{-1}_{L_2(\mathbb{R}_+;H)}.$$

To this end, we consider a polynomial operator bundle of fourth order, dependent on a real parameter $\beta \in \mathbb{R}$:

$$\Phi(\lambda;\beta;A) = P_0(\lambda;A)\,P_0^*\left(-\bar{\lambda};A\right) + \beta\lambda^2 A^2,$$

where

$$P_0(\lambda;A) = (\lambda E - \omega_1 A)(\lambda E - \omega_2 A).$$

It holds

Lemma 1. *Let* $\beta \in \left[0,\, d^{-2}\right)$, *where*

$$d = \sup_{\zeta\in\mathbb{R}} \left|\zeta P_0^{-1}(i\zeta,1)\right|, P_0(i\zeta,1) = (i\zeta - \omega_1)(i\zeta - \omega_2).$$

Then the operator bundle $\Phi(\lambda;\beta;A)$ *is represented in the form*

$$\Phi\ (\lambda;\beta;A) = F(\lambda;\beta;A)\,F^*\left(-\bar{\lambda};\beta;A\right), \tag{2.1}$$

and

$$F(\lambda;\beta;A) = (\lambda E - \eta_1(\beta)A)(\lambda E - \eta_2(\beta)A) = \lambda^2 E + \alpha_1(\beta)\lambda A + \alpha_0(\beta)A^2. \tag{2.2}$$

Proof. Let $\mu \in \sigma(A)$, $\lambda = i\zeta$, $\zeta \in \mathbb{R}$. Then

$$\Phi\ \ (\lambda;\beta;\mu) = P_0(i\zeta;\mu)\,\overline{P_0\left(-\overline{i\zeta};\mu\right)} - \beta\zeta^2\mu^2 = |P_0(i\zeta;\mu)|^2 - \beta\zeta^2\mu^2$$

$$\geq |P_0(i\zeta;\mu)|^2\left(1 - \beta\sup_{\zeta\in\mathbb{R}}\zeta^2\mu^2\,|P_0(i\zeta,\mu)|^{-2}\right) = |P_0(i\zeta;\mu)|^2\left(1-\beta d^2\right) > 0.$$

Consequently, $\Phi(\lambda;\beta;A)$ has no toots on the imaginary axis, and it follows from the equality $\Phi\ \ (\lambda;\beta;A) = \overline{\Phi\ \ \left(-\bar{\lambda};\beta;A\right)} = 0$ that its two roots lie in the left half-plane, two roots $\eta_1(\beta)\mu$ and $\eta_2(\beta)\mu$ lie in the right half-plane $-\overline{\eta_1(\beta)}\mu$ and $-\overline{\eta_2(\beta)}\mu$. Then denote $F(\lambda;\beta;\mu) = (\lambda - \eta_1(\beta)\mu)(\lambda - \eta_2(\beta)\mu)$, where $Re\eta_1(\beta) < 0$, $Re\eta_2(\beta) < 0$. We get $\Phi\ \ (\lambda;\beta;\mu) = F(\lambda;\beta;\mu)\,\overline{F\left(-\bar{\lambda};\beta;\mu\right)}$. Further, using the spectral expansion of the operator A, hence we get the validity of equality (2.1). The lemma is proved.

Remark. Obviously, there exists a point ζ_0 such that $d = \left|\zeta_0 P_0^{-1}(i\zeta_0,1)\right|$.

Denote

$$p = -(\omega_1 + \omega_2), q = \omega_1\omega_2.$$

It holds

Corollary 1. *The coefficients of the quadratic bundle* $F(\lambda;\beta;A)$ *satisfy the following relations:*

$$1)\ Re\alpha_1(\beta) > 0;\ 2)\ Im\alpha_1(\beta) = Imp;\ 3)\ 2Re\alpha_0(\beta) - |\alpha_1(\beta)|^2 = 2Req - |p|^2 + \beta;$$

$$4)\ Imp\bar{q} = Im\alpha_1(\beta)\,\overline{\alpha_0(\beta)};\ 5)\ |\alpha_0(\beta)| = |q|.$$

Indeed, $Re\alpha_1(\beta) = -Re\eta_1(\beta) - Re\eta_2(\beta) > 0$. The relations 2)-5) are obtained from equality (2.1) with regard to (2.2) by associating the same coefficients λ.

Lemma 2. *Let* $\beta \in \left[0,\, d^{-2}\right)$. *Then for any* $u \in W_2^2(\mathbb{R}_+;H;0)$ *it holds the equality*

$$\|P_0 u\|^2_{L_2(\mathbb{R}_+;H)} - \beta\left\|Au'\right\|^2_{L_2(\mathbb{R}_+;H)} = Re\left(\alpha_1(\beta) - p\right) + \|F(d/dt;\beta;A)\,u\|^2_{L_2(\mathbb{R}_+;H)}. \tag{2.3}$$

Proof. For $\beta \in \left[0,\, d^{-2}\right)$ and $u \in W_2^2(\mathbb{R}_+;H;0)$ we have:

$$\|P_0 u\|^2_{L_2(\mathbb{R}_+;H)} = \left\|u'' + pAu' + qA^2u\right\|^2_{L_2(\mathbb{R}_+;H)} = \|u''\|^2_{L_2(\mathbb{R}_+;H)} + \\ |p|^2 \|Au'\|^2_{L_2(\mathbb{R}_+;H)} + |q|^2 \left\|A^2u\right\|^2_{L_2(\mathbb{R}_+;H)} + 2Re\,(u'', pAu')_{L_2(\mathbb{R}_+;H)} + \\ 2Re\left(u'', qA^2u\right)_{L_2(\mathbb{R}_+;H)} + 2Re\left(pAu', qA^2u\right)_{L_2(\mathbb{R}_+;H)}. \tag{2.4}$$

Integrating by parts, we get:

$$2Re\,(u'', Au')_{L_2(\mathbb{R}_+;H)} = -\|u'(0)\|^2_{1/2}, \ \ \left(u'', A^2u\right)_{L_2(\mathbb{R}_+;H)} = -\|Au'\|^2_{L_2(\mathbb{R}_+;H)}, \\ 2Re\left(Au', A^2u\right)_{L_2(\mathbb{R}_+;H)} = 0.$$

Taking into account these equalities in (2.4), we get

$$\|P_0 u\|^2_{L_2(\mathbb{R}_+;H)} = \|u''\|^2_{L_2(\mathbb{R}_+;H)} + \left(|p|^2 - 2Req\right)\|Au'\|^2_{L_2(\mathbb{R}_+;H)} + \\ |q|^2 \left\|A^2u\right\|^2_{L_2(\mathbb{R}_+;H)} + 2Im\bar{p}Im\left(u'', A^2u\right)_{L_2(\mathbb{R}_+;H)} - Rep\,\|u'(0)\|^2_{1/2} + \\ 2Imp\bar{q}Im\left(Au', A^2u\right)_{L_2(\mathbb{R}_+;H)}. \tag{2.5}$$

Similarly we have:

$$\|F(d/dt;\beta;A)\,u\|^2_{L_2(\mathbb{R}_+;H)} = \left\|u'' + \alpha_1(\beta)\,Au' + \alpha_0(\beta)\,A^2u\right\|^2_{L_2(\mathbb{R}_+;H)} = \\ \|u''\|^2_{L_2(\mathbb{R}_+;H)} + \left(|\alpha_1(\beta)|^2 - 2Re\alpha_0(\beta)\right)\|Au'\|^2_{L_2(\mathbb{R}_+;H)} + |\alpha_0(\beta)|^2\left\|A^2u\right\|^2_{L_2(\mathbb{R}_+;H)} \\ +2Im\overline{\alpha_1(\beta)}Im\left(u'', A^2u\right)_{L_2(\mathbb{R}_+;H)} - Re\bar{\alpha}_1(\beta)\,\|u'(0)\|^2_{1/2} + \\ 2Im\alpha_1(\beta)\,\overline{\alpha_0(\beta)}Im\left(Au', A^2u\right)_{L_2(\mathbb{R}_+;H)} - Re\alpha_1(\beta)\,\|u'(0)\|^2_{1/2}. \tag{2.6}$$

Taking into account relations 2)-5) in equality (2.5), allowing for (2.6), we get the validity of equality (2.3).

Corollary 2. *For $\beta \in \left[0,\, d^{-2}\right)$ and $u \in W_2^2(\mathbb{R}_+;H;0,1)$ it holds the equality*

$$\|P_0 u\|^2_{L_2(\mathbb{R}_+;H)} - \beta\left\|Au'\right\|^2_{L_2(\mathbb{R}_+;H)} = \|F(d/dt;\beta;A)\,u\|^2_{L_2(\mathbb{R}_+;H)}. \tag{2.7}$$

Using equality (2.7), we prove the following theorem.

Theorem 3. *The norm $N_1(0,\,1) = d$, where*

$$N_1(0,\ 1) = \sup_{0 \neq u\, \in W_2^2(\mathbb{R}_+;H;0,1)} \left\|Au'\right\|_{L_2(\mathbb{R}_+;H)} \|P_0 u\|^{-1}_{L_2(\mathbb{R}_+;H)}.$$

Proof. Passing to limit as $\beta \to d^{-2}$ in equality (2.7), we get $\|P_0u\|^2_{L_2(\mathbb{R}_+;H)} \geq d^{-2}\|Au'\|^2_{L_2(\mathbb{R}_+;H)}$ for all $u \in W_2^2(\mathbb{R}_+;H;0,1)$, i.e. $N_1(0,\,1) \leq d$. Show that $N_1(0,\,1) = d$. For that, for any $\varepsilon > 0$ it suffices to construct a vector-function $u_\varepsilon(t) \in W_2^2(\mathbb{R}_+;H;0,1)$ such that

$$\mathrm{E}(u_\varepsilon) = \|P_0 u_\varepsilon\|^2_{L_2(\mathbb{R}_+;H)} - \left(d^2 + \varepsilon\right)\left\|Au'_\varepsilon\right\|^2_{L_2(\mathbb{R}_+;H)} < 0.$$

Let the vector $x \in H_4$, $\|x\| = 1$, and $g(t)$ be a scalar function from $W_2^2(\mathbb{R})$. Then by the Plancherel theorem,

$$\mathrm{E}(g(t)\,x) = \int_{-\infty}^{+\infty}\left(\left(P_0(i\zeta, A)\,P_0^*(i\zeta, A) - \left(d^2+\varepsilon\right)\zeta^2A^2\right)x, x\right)|\hat{g}(\zeta)|^2\,d\zeta = \\ \int_{-\infty}^{+\infty}\left(\Phi\ \left(i\zeta, d^2+\varepsilon, A\right)x, x\right)|\hat{g}(\zeta)|^2\,d\zeta.$$

If μ is an eigenvalue, and x is an eigenvector of the operator A, then

$$\left(\Phi\left(i\zeta_0, d^2+\varepsilon, \mu\right)x, x\right) = \Phi\left(i\zeta_0, d^2+\varepsilon, \mu\right) < 0.$$

If $\mu \in \sigma(A)$, then for any $\delta > 0$ we can find x_δ, $\|x_\delta\| = 1$ and $A^m x = \mu^m x_\delta + o(1)$, $\delta \to 0$, $m = 1, 2, \ldots$. Then for small δ we again have

$$\left(\Phi\left(i\zeta_0, d^2 + \varepsilon, A\right) x_\delta, x_\delta\right) < 0.$$

Since the function $q(\varepsilon, x) = \left(\Phi\ \left(i\zeta, d^2 + \varepsilon, A\right) x, x\right)$ is continuous with respect to the argument ζ, we can find an interval $(\tau_0(\varepsilon), \tau_1(\varepsilon))$ on which $q(\varepsilon, x) < 0$. Now we choose $g(t) \in W_2^2(\mathbb{R})$ so that its Fourier transform $\hat{g}(\zeta)$ has a support in the interval $(\tau_0(\varepsilon), \tau_1(\varepsilon))$. Then we get $\mathrm{E}(g(t)x) < 0$. Since the functional $\mathrm{E}(\cdot)$ is continuous in the space $W_2^2(\mathbb{R}; H)$, then from the theorem on density of finite vector-functions in $W_2^2(\mathbb{R}; H)$ (see [4]) it follows that there exists the function $u_N(t) \in W_2^2(\mathbb{R}; H)$ with the support $(-N, N) \subset \mathbb{R}$, and $\mathrm{E}(u_N(t)) < \varepsilon$. Assuming $u_\varepsilon(t) = u_N(t - N) \in W_2^2(\mathbb{R}_+; H; 0, 1)$ we have $\mathrm{E}(u_\varepsilon(t)) < 0$. The theorem is proved.

Since $W_2^2(\mathbb{R}_+; H; 0, 1) \subset W_2^2(\mathbb{R}_+; H; 0)$, then $N_1(0) \geq N_1(0, 1) = d$.

Theorem 4. *The norm $N_1(0) = d$ if and only if $Re(\alpha_1(\beta) - p) > 0$ for all $\beta \in \left(0, d^{-2}\right)$.*

Proof. Let $N_1(0) = d$. Then for any $\beta \in \left(0, d^{-2}\right)$ and $u \in W_2^2(\mathbb{R}_+; H; 0)$ it holds the inequality

$$\|P_0 u\|^2_{L_2(\mathbb{R}_+;H)} - \beta \left\|Au'\right\|^2_{L_2(\mathbb{R}_+;H)} \geq \left(1 - \beta N_1^2(0)\right) \|P_0 u\|^2_{L_2(\mathbb{R}_+;H)} > 0.$$

Further, from the form $F(\lambda; \beta; A)$ and equality (2.2) it follows that the Cauchy problem

$$F(d/dt; \beta; A)\, u(0) = 0,\ \ u(0) = 0,\ \ u'(0) = x$$

for any $x \in H_{1/2}$ has the solution $u(\beta, t)$. Then taking into account equality (2.3), we get $Re(\alpha_1(\beta) - p) > 0$ for $\beta \in \left(0, d^{-2}\right)$. Vice versa, if $Re(\alpha_1(\beta) - p) > 0$, then from (2.3) it follows

$$\|P_0 u\|^2_{L_2(\mathbb{R}_+;H)} - \beta \left\|Au'\right\|^2_{L_2(\mathbb{R}_+;H)} > 0.$$

Now, passing here to limit as $\beta \to d^{-2}$, we get $N_1(0) \leq d$, i.e. $N_1(0) = d$. The theorem is proved.

Hence we get

Corollary 3. *Let $Rep \leq 0$ $(Re(\omega_1 + \omega_2) \geq 0)$. Then $N_1(0) = d$.*

Now consider the case when $Re(\alpha_1(\beta) - Rep) < 0$ for some $\beta \in \left(0, d^{-2}\right)$. In this case $N_1^{-2}(0) \in \left(0, d^{-2}\right)$. Then for $\beta \in \left(0, N_1^{-2}(0)\right)$ and $u \in W_2^2(\mathbb{R}_+; H; 0)$ we have:

$$\|P_0 u\|^2_{L_2(\mathbb{R}_+;H)} - \beta \left\|Au'\right\|^2_{L_2(\mathbb{R}_+;H)} \geq \left(1 - \beta N_1^2(0)\right) \|P_0 u\|^2_{L_2(\mathbb{R}_+;H)} > 0.$$

As in the proof of theorem 4, having considered the Cauchy problem, we get $Re(\alpha_1(\beta) - p) > 0$ for $\beta \in \left(0, N_1^{-2}(0)\right)$.

By definition of $N_1(0)$ for any $\beta \in \left(N_1^{-2}(0), d^{-2}\right)$ there exists a function $u_\beta(t) \in W_2^2(\mathbb{R}_+; H; 0)$ such that

$$\|P_0 u_\beta\|^2_{L_2(\mathbb{R}_+;H)} - \beta \left\|Au'_\beta\ \right\|^2_{L_2(\mathbb{R}_+;H)} < 0.$$

Then from equality (2.3) it follows that $Re(\alpha_1(\beta) - p) < 0$. Thus, $Re\alpha_1\left(N_1^{-2}(0)\right) = Rep$. Hence it follows that for finding $N_1(0)$ we should solve the equation $Re\alpha_1(\beta) = Rep$ together with relations 1)-5). If this equation has a solution from the interval $\left(0,\, d^{-2}\right)$ equal β_0, then $N_1\left(0\right) = \beta_0^{-1/2}$. But if this equation has no solution from the interval $\left(0,\, d^{-2}\right)$, then $N_1\left(0\right) = d$. Thus, for $Rep > 0$ we solve the equation $Re\alpha_1\left(\beta\right) = Rep$. Taking into account relation 2), we get $\alpha_1\left(\beta\right) = p$. From relation 4) it follows that $Rep\left(Im\alpha_0\left(\beta\right) - Imq\right) = Im\left(Re\alpha_0\left(\beta\right) - Req\right)$. Then, taking into account relation 3), we get $Re\alpha_0\left(\beta\right) = Req + \frac{\beta}{2}$ and $Im\alpha_0\left(\beta\right) = \frac{\beta}{2}\frac{Imp}{Rep} + Imq$. Finally, from 5) for β we get the equation

$$\left(Req + \frac{\beta}{2}\right)^2 + \left(Imq + \frac{\beta}{2}\frac{Imp}{Rep}\right)^2 = \left|q\right|^2.$$

Taking into account $\beta \neq 0$, we get

$$\beta_0 = \frac{-4Rep \cdot Rep\bar{q}}{\left|p\right|^2}. \tag{2.8}$$

Thus, the following theorem is proved

Theorem 5. *The norm $N_1\left(0\right)$ is determined as follows:*

$$N_1\left(0\right) = \begin{cases} d & for \quad Rep \le 0, \\ d & for \quad \beta_0 \notin \left(0, d^{-2}\right), \quad Rep > 0, \\ \beta_0^{-1/2} & for \quad \beta_0 \in \left(0, d^{-2}\right), \quad Rep > 0, \end{cases}$$

where β_0 is determined from equality (2.8).

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