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TOPOLOGICAL PROPERTIES OF THE SPACE OF COEFFICIENTS GENERATED BY THE NON-DEGENERATE SYSTEM IN METRIC SPACES

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In memory of M. G. Gasymov on his 75*th birthday*

Abstract. In this paper we consider linear metric spaces possessing certain property. The notion of a non-degenerate system is introduced. It is proved that such systems have a complete metric space of coefficients with a canonical basis. The basicity criterion is given in terms of the coefficient operator.

1. Introduction

The space of coefficients is the notion of theory of basis. It is known that the arbitrary basis in Banach space has a Banach space of coefficients that is isomorphic to initial one (see [7, 8, 3, 6]). Each non-degenerate system (will be determined later) in Banach space generates an appropriate Banach space of coefficients with a canonical basis (see [3, 6, 2, 1]). In this connection, the space of coefficients plays an important part in studying approximate properties of systems. It has very significant applications in different fields of natural science as solids and molecules, multiple birth of particles, aviation, medicine and biology, data compression and etc (see [4, 5] and references in it). All these applications are closely related to the theory of wavelet analysis. Recently, there is a great interest to this direction and a lot of monographs (see [4]) have been devoted to it. It is known that many topological spaces are unnormed. Therefore, study of these or other properties of the space of coefficients in topological and, in particular, in metric spaces is of great scientific interest.

The present paper is devoted to the study of topological properties of the space of coefficients, generated by the non-degenerate system in metric spaces. The paper is organized as follows. The main notion and properties that will be used in further presentations are given in Section 2. Section 3 is devoted to obtaining the main results. It is proved that an arbitrary non-degenerate system in the metric space in which the metric possesses a certain property, generates a complete space of coefficients with a canonical basis. The basicity criterion of the systems in such metric spaces is given in terms of the coefficient operator. The brief summary of obtained results is given in Section 4.

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2. Main denotations and notions

Accept the following standard denotation. N is a set of all positive integers ordered in a usual way, K is a field of scalars (K or R are real numbers, $\mathbb{C}$ are complex numbers). $\exists!$ means "exists" and "is unique", $\Rightarrow$ means "follows".

Recall some notion from the theory of bases. Let $(X\,;\,\rho)$ be some linear metric space over the field K. Denote the linear span of the set $M \subset X$ by $L\,[M]$, its closure by $\bar{M}$.

Definition 2.1. The system $\{x_n\}_{n\in N} \subset X$ is called complete in X if $\overline{L\left[\{x_n\}_{n\in N}\right]} \equiv X$.

Definition 2.2. The system $\{x_n\}_{n\in N} \subset X$ is called minimal in X if $x_k \notin \overline{L\left[\{x_n\}_{n\neq k}\right]}$, $\forall k \in N$.

Definition 2.3. The system $\{x_n\}_{n\in N} \subset X$ is called ω-linear independent in X if $\sum_{n=1}^{\infty} \lambda_n x_n = 0$ in X yields that $\lambda_n = 0\,,\ \forall n \in N$.

Definition 2.4. The system $\{x_n\}_{n\in N} \subset X$ is called a basis in X if for $\forall x \in X$, $\exists!\ \{\lambda_n\}_{n\in N} \subset K$:$x = \sum_{n=1}^{\infty} \lambda_n x_n$.

We'll use the following notion as well.

Definition 2.5. The system $\{x_n\}_{n\in N} \subset X$ is called non-degenerate if $x_n \neq 0$, $\forall n \in N$.

We'll suppose that the linear metric space $(X\,;\,\rho)$ possesses the following properties:

α) The linear operations of addition and multiplication by the scalar in $(X\,;\,\rho)$ are continuous in X, i.e. from $\lambda_n \to \lambda$, $n \to \infty$, in $\mathbb{C}$ and from $x_n \to x$, $y_n \to y$, $n \to \infty$ in X it follows that $\lambda_n x_n \to \lambda x$, $x_n + y_n \to x + y$, $n \to \infty$ in X;

β) Let τ_ρ be a topology in X, generated by the metric ρ. We'll assume that the boundedness of the set in $X$with respect to the topology τ_ρ and the metric ρ are equivalent, i.e. these notion in the spaces$(X\,;\,\tau_\rho)$ and $(X\,;\,\rho)$ are identical.

3. The space of coefficients

Let $(X\,;\,\rho)$ be some linear, metric complete space possessing the properties α), β) and $\{x_n\}_{n\in N} \subset X$ be some non-degenerate system.

Assume

$$\mathscr{K}_{\bar{x}} \equiv \left\{\{\lambda_n\}_{n\in N} \subset K: \quad \text{the series } \sum_{n=1}^{\infty} \lambda_n x_n \ \text{ converges in } X\right\}.$$

It is obvious that with respect to ordinary operations of component-wise addition and multiplication by a scalar, $\mathscr{K}_{\bar{x}}$ turns into a linear space. Let $\bar{\lambda}, \bar{\mu} \in \mathscr{K}_{\bar{x}}$: $\bar{\lambda} \equiv \{\lambda_n\}_{n\in N}$, $\bar{\mu} \equiv \{\mu_n\}_{n\in N}$. Assume

$$\rho_{\mathscr{K}_{\bar{x}}}\left(\bar{\lambda}; \bar{\mu}\right) = \sup_m \rho\left(\sum_{n=1}^{m} \lambda_n x_n;\ \sum_{n=1}^{m} \mu_n x_n\right).$$

Show that $\rho_{\mathscr{K}_{\bar{x}}}(\cdot\,;\,\cdot)$ is a metric in $\mathscr{K}_{\bar{x}}$.

1) It is clear that $\rho_{\mathscr{K}_{\bar{x}}}\left(\bar{\lambda};\bar{\mu}\right)\geq 0$, $\forall\bar{\lambda},\bar{\mu}\in\mathscr{K}_{\bar{x}}$. Let $\rho_{\mathscr{K}_{\bar{x}}}\left(\bar{\lambda};\bar{\mu}\right)=0\Rightarrow\rho\left(\lambda_1x_1;\mu_1x_1\right)=0\Rightarrow\lambda_1x_1=\mu_1x\Rightarrow\lambda_1=\mu_1$, since the system $\{x_n\}_{n\in N}$ is non-degenerate. From $\rho\left(\lambda_1x_1+\lambda_2x_2;\lambda_1x_1+\mu_2x_2\right)=0$ we get $\lambda_2=\mu_2$. Continuing this process, we get $\lambda_k\neq\mu_k$, $\forall k\in N\Rightarrow\bar{\lambda}=\bar{\mu}$.

2) Let $\bar{\nu}\equiv\{\nu_n\}_{n\in N}\in\mathscr{K}_{\bar{x}}$. We have

$$\rho_{\mathscr{K}_{\bar{x}}}\left(\bar{\lambda};\bar{\mu}\right)=\sup_m\rho\left(\sum_{n=1}^m\lambda_nx_n;\sum_{n=1}^m\mu_nx_n\right)\leq$$

$$\sup_m\left[\rho\left(\sum_{n=1}^m\lambda_nx_n;\sum_{n=1}^m\nu_nx_n\right)+\rho\left(\sum_{n=1}^m\nu_nx_n;\sum_{n=1}^m\mu_nx_n\right)\right]\leq$$

$$\rho\left(\bar{\lambda};\bar{\nu}\right)+\rho\left(\bar{\nu};\bar{\mu}\right).$$

Consequently, $\rho_{\mathscr{K}_{\bar{x}}}\left(\cdot\,;\,\cdot\right)$ is a metric in $\mathscr{K}_{\bar{x}}$. Show that $\left(\mathscr{K}_{\bar{x}};\rho_{\mathscr{K}_{\bar{x}}}\right)$ is complete. Let $\left\{\bar{\lambda}_n\right\}_{n\in N}\subset\mathscr{K}_{\bar{x}}$ be some fundamental sequence, where $\bar{\lambda}_n=\left\{\lambda_k^{(n)}\right\}_{k\in N}$.

On establishing the completeness we'll need the following

Lemma 3.1. *Let $x\in X$, $x\neq 0$, $\{\lambda_n\}_{n\in N}\subset\mathbb{C}$ and $\lambda_nx\to 0$, $n\to\infty$. Then $\lambda_n\to 0$, $n\to\infty$.*

Proof. Indeed, let $\lim\limits_{n\to\infty}\lambda_n=0$ do not hold. Assume that $\{\lambda_n\}_{n\in N}$ has a bounded subsequence $\{\lambda_{n_k}\}_{k\in N}$. Then from it we can isolate a convergent sequence and not loosing generality, we'll assume that $\lambda_{n_k}\to\lambda_0$, $k\to\infty$. We have $\lambda_{n_k}x\to\lambda_0\,x$, $k\to\infty$, and consequently $\lambda_0=0$. Thus, an arbitrary bounded subsequence of $\{\lambda_n\}_{n\in N}$ converges to zero. It follows from the accepted assumption that $\{\lambda_n\}_{n\in N}$ contains an unbounded subsequence, and let $\lambda_{n_k}\to\infty$, $k\to\infty$. Consequently, $\frac{1}{\lambda_{n_k}}\to 0$, $k\to\infty$. As a result, $\frac{1}{\lambda_{n_k}}\lambda_{n_k}x=x\neq 0$, $\forall k\in N$. On the other hand, by the conditions of the lemma $\lim\limits_{k\to\infty}\left(\frac{1}{\lambda_{n_k}}\lambda_{n_k}x\right)=\lim\limits_{k\to\infty}\frac{1}{\lambda_{n_k}}\lim\limits_{k\to\infty}\left(\lambda_{n_k}x\right)=0$. The obtained contradiction proves the lemma. $\square$

So, $\rho_{\mathscr{K}_{\bar{x}}}\left(\bar{\lambda}_n\,;\,\bar{\lambda}_m\right)\to 0$, $n,m\to\infty$. It is easy to see that

$$\rho\left(\lambda_k^{(n)}x_k;\,\lambda_k^{(m)}x_k\right)\leq 2\rho_{\mathscr{K}_{\bar{x}}}\left(\bar{\lambda}_n\,;\,\bar{\lambda}_m\right)\,,\ \forall k\in N. \tag{3.1}$$

Further, we'll suppose that ρ is invariant with respect to the shift, i.e.. $\rho\left(x;y\right)=\rho\left(x-y;0\right)$,$\forall x,y\in X$. it follows directly from the inequality (3.1) that the sequence $\left\{\lambda_k^{(n)}\right\}_{n\in N}$ is fundamental for each fixed $k\in N$. Let $\lambda_k^{(n)}\to\lambda_k$, as $n\to\infty$. Denote $\bar{\lambda}\equiv\{\lambda_n\}_{n\in N}$. Show that $\rho_{\mathscr{K}_{\bar{x}}}\left(\bar{\lambda}_n\,;\,\bar{\lambda}\right)\to 0$, $n\to\infty$. Take $\forall\varepsilon>0$. It is clear that $\exists n_0:\rho_{\mathscr{K}_{\bar{x}}}\left(\bar{\lambda}_n\,;\,\bar{\lambda}_{n+p}\right)<\varepsilon$, $\forall n\geq n_0$, $\forall p\in N$. Consequently

$$\sup_m\rho\left(\sum_{k=1}^m\left(\lambda_k^{(n)}-\lambda_k^{(n+p)}\right)\,x_k;\,0\right)<\varepsilon,\ \forall n\geq n_0,\ \forall p\in N.$$

Hence, it follows

$$\rho\left(\sum_{k=1}^m\left(\lambda_k^{(n)}-\lambda_k^{(n+p)}\right)\,x_k;\,0\right)<\varepsilon,\ \forall n\geq n_0,\ \forall p,m\in N.$$

Passing to limit as $p \to \infty$ we get

$$\rho\left(\sum_{k=1}^{m}\left(\lambda_k^{(n)}-\lambda_k\right)x_k;\,0\right)\le\varepsilon,\ \forall n\ge n_0,\ \forall m\in N. \tag{3.2}$$

We have

$$\rho\left(\sum_{k=r}^{r+p}\left(\lambda_k^{(n)}-\lambda_k\right)x_k;\,0\right)\le\rho\left(\sum_{k=1}^{r+p}\left(\lambda_k^{(n)}-\lambda_k\right);\,0\right)+$$

$$\rho\left(\sum_{k=1}^{r-1}\left(\lambda_k^{(n)}-\lambda_k\right);\,0\right)\le 2\varepsilon,\ \forall n\ge n_0,\ \forall r,p\in N.$$

It follows from $\bar{\lambda}_n \in \mathscr{K}_{\bar{x}}$ that $\exists m_0^{(n)}$:

$$\rho\left(\sum_{k=m}^{m+p}\lambda_k^{(n)}x_k;\,0\right)<\varepsilon,\ \forall m\ge m_0^{(n)},\ \forall p\in N.$$

Consequently, for a fixed $n\ge n_0$ we have

$$\rho\left(\sum_{k=m}^{m+p}\lambda_k x_k;\,0\right)=\rho\left(\sum_{k=m}^{m+p}\left(\lambda_k-\lambda_k^{(n)}\right)x_k+\sum_{k=m}^{m+p}\lambda_k^{(n)}x_k;0\right)\le$$

$$\rho\left(\sum_{k=m}^{m+p}\left(\lambda_k-\lambda_k^{(n)}\right)x_k;0\right)+\rho\left(\sum_{k=m}^{m+p}\lambda_k^{(n)}x_k;0\right)<2\varepsilon,\ \forall m\ge m_0^{(n)},\ \forall p\in N.$$

From the arbitrariness of $\varepsilon>0$ it follows that the series $\sum_{k=1}^{\infty}\lambda_k x_k$ converges in X. Thus, $\bar{\lambda}\in\mathscr{K}_{\bar{x}}$ and it follows from relation (3.2) that $\lim\limits_{n\to\infty}\rho_{\mathscr{K}_{\bar{x}}}\left(\bar{\lambda}_n\,;\,\bar{\lambda}\right)=0$. As a result, we get that $(\mathscr{K}_{\bar{x}};\rho_{\mathscr{K}_{\bar{x}}})$ is a complete metric space. It is obvious that the metric $\rho_{\mathscr{K}_{\bar{x}}}$ is invariant with respect to the shift. Show that linear operations in $\mathscr{K}_{\bar{x}}$ are continuous. Let $\mu\to\mu_0$ in $\mathbb{C}$ and $\bar{\lambda}\equiv\{\lambda_n\}_{n\in N}\in\mathscr{K}_{\bar{x}}$. We have

$$\rho_{\mathscr{K}_{\bar{x}}}\left(\mu\bar{\lambda}\,;\,\mu_0\bar{\lambda}\right)=\rho_{\mathscr{K}_{\bar{x}}}\left((\mu-\mu_0)\,\bar{\lambda};0\right)=\sup_m\rho\left((\mu-\mu_0)\,S_m;0\right), \tag{3.3}$$

where $S_m=\sum_{n=1}^{m}\lambda_n x_n,\ \forall m\in N$. Since the sequence $\{S_n\}_{n\in N}$ converges in X, it is clear that it is bounded in $(X;\rho)$, and also in $(X;\tau_\rho)$. Take $\forall\varepsilon>0$. Then $\exists\delta>0,\,\forall t,\,|t|<\delta:\ \rho\,(tS_m;0)<\varepsilon,\ \forall m\in N$. From (3.3) we get

$$\rho_{\mathscr{K}_{\bar{x}}}\left(\mu\bar{\lambda}\,;\,\mu_0\bar{\lambda}\right)\le\varepsilon,\ \text{for}\ |\mu-\mu_0|<\delta.$$

This means that $\mu\bar{\lambda}\to\mu_0\bar{\lambda}$ in $\mathscr{K}_{\bar{x}}$. Let $\left\{\bar{\lambda}_n\right\}_{n\in N};\,\{\bar{\mu}_n\}_{n\in N}\subset\mathscr{K}_{\bar{x}}$ and $\bar{\lambda}_n\to\bar{\lambda}$, $\bar{\mu}_n\to\bar{\mu}$, as $n\to\infty$.

It follows directly from the inequality of triangle and invariance of ρ with respect to the shift that

$$\rho_{\mathscr{K}_{\bar{x}}}\left(\bar{\lambda}_n+\bar{\mu}_n;\,\bar{\lambda}+\bar{\mu}\right)\le\rho_{\mathscr{K}_{\bar{x}}}\left(\bar{\lambda}_n-\bar{\lambda};\,0\right)+\rho_{\mathscr{K}_{\bar{x}}}\left(\bar{\mu}_n-\bar{\mu};\,0\right)\to 0,\ \text{as}\ n\to\infty.$$

Consequently, the linear operations are continuous in $\mathscr{K}_{\bar{x}}$.

Consider the operator $T:\mathscr{K}_{\bar{x}}\to X$, determined by the expression

$$T\bar{\lambda}=\sum_{n=1}^{\infty}\lambda_n x_n,\bar{\lambda}\equiv\{\lambda_n\}_{n\in N}\in\mathscr{K}_{\bar{x}}.$$

Let $\bar{\lambda}_n \to \bar{\lambda}$, $n \to \infty$, in $\mathscr{K}_{\bar{x}}$, where $\bar{\lambda}_n \equiv \left\{ \lambda_k^{(n)} \right\}_{k \in N} \in \mathscr{K}_{\bar{x}}$. We have

$$\rho\left(T\bar{\lambda}_n;\, T\bar{\lambda}\right) = \rho\left(\sum_{k=1}^{\infty}\left(\lambda_k^{(n)} - \lambda_k\right)x_k;\, 0\right) \leq$$

$$\sup_m \rho\left(\sum_{k=1}^{\infty}\left(\lambda_k^{(n)} - \lambda_k\right)x_k;\, 0\right) \to 0,\ n \to \infty.$$

Hence it follows directly that T is a continuous operator. Let $\bar{\lambda} \in KerT$, i.e. $T\bar{\lambda} = 0 \Rightarrow \sum_{n=1}^{\infty} \lambda_n x_n = 0$, where $\bar{\lambda} \equiv \{\lambda_n\}_{n \in N} \in \mathscr{K}_{\bar{x}}$. It is clear that if the system $\{x_n\}_{n \in N}$ is ω-linear independent, then $\lambda_n = 0$, $\forall n \in N$, and as result $KerT = \{0\}$. In this case there exists an inverse operator $T^{-1} : ImT \to \mathscr{K}_{\bar{x}}$. If in addition ImT is closed in X, then T^{-1} is also continuous.

Denote by $\{e_n\}_{n \in N} \subset \mathscr{K}_{\bar{x}}$ a canonical system, where $e_n = \{\delta_{nk}\}_{k \in N}$, δ_{nk} is Kronecker's symbol. Show that $\{e_n\}_{n \in N}$ forms a basis for $\mathscr{K}_{\bar{x}}$. Take $\forall \bar{\lambda} \equiv \{\lambda_n\}_{n \in N} \in \mathscr{K}_{\bar{x}}$. Show that the series $\sum_{n=1}^{\infty} \lambda_n e_n$converges in $\mathscr{K}_{\bar{x}}$. Indeed, since the series $\sum_{n=1}^{\infty} \lambda_n x_n$ converges in X, then for $\forall \varepsilon > 0$, $\exists m_0 \in N$:

$$\rho\left(\sum_{m_0}^{m_0+p} \lambda_n x_n;\, 0\right) < \varepsilon,\ \forall m \geq m_0,\ \forall p \in N.$$

Thus

$$\rho_{\mathscr{K}_{\bar{x}}}\left(\sum_{m_0}^{m_0+p} \lambda_n e_n; 0\right) = \rho_{\mathscr{K}_{\bar{x}}}\left(\{...; 0; \lambda_{m_0}; ...; \lambda_{m_0+p}; 0; ...\}; 0\right) =$$

$$\sup_{m_0 \leq r \leq m_0+p} \rho\left(\sum_{n=m_0}^{r} \lambda_n x_n;\, 0\right) \leq \varepsilon,\ \forall m \geq m_0,\ \forall p \in N.$$

Hence it follow that the series $\sum_{n=1}^{\infty} \lambda_n e_n$ converges in $\mathscr{K}_{\bar{x}}$. We have

$$\rho_{\mathscr{K}_{\bar{x}}}\left(\bar{\lambda} - \sum_{n=1}^{m} \lambda_n e_n; 0\right) = \rho_{\mathscr{K}_{\bar{x}}}\left(\{...; 0; \lambda_{m+1}; ...\}; 0\right) =$$

$$\sup_{r \geq m+1} \rho\left(\sum_{n=m+1}^{r} \lambda_n x_n; 0\right) \leq \varepsilon,\ \forall m \geq m_0.$$

Consequently, $\lim\limits_{m\to\infty} \sum_{n=1}^{m} \lambda_n e_n = \bar{\lambda}$ in $\mathscr{K}_{\bar{x}}$, i.e. $\bar{\lambda} = \sum_{n=1}^{\infty} \lambda_n e_n$. Consider the linear functionals $e_n^*\left(\bar{\lambda}\right) = \lambda_n$, $\forall n \in N$. Show that they are continuous. Let $\bar{\lambda}_n \to \bar{\lambda}$, $n \to \infty$, in $\mathscr{K}_{\bar{x}}$, where $\bar{\lambda}_n \equiv \left\{\lambda_k^{(n)}\right\}_{k \in N} \subset \mathscr{K}_{\bar{x}}$. As it was established $\lambda_k^{(n)} \to \lambda_k$, $n \to \infty$. Consequently, $e_k^*\left(\bar{\lambda}_n\right) = \lambda_k^{(n)} \to \lambda_k = e_k^*\left(\bar{\lambda}\right)$, as $n \to \infty$, and so, e_k^* is continuous, $\forall k \in N$. It is easy to see that $e_n^*\left(e_k\right) = \delta_{nk}$, $\forall n, k \in N$. As a result we get that $\{e_n^*\}_{n \in N}$ is a biorthogonal system to $\{e_n\}_{n \in N}$. This proves the basicity of the system $\{e_n\}_{n \in N}$ in $\mathscr{K}_{\bar{x}}$. So the following theorem is valid.

Theorem 3.1. *Let $(X; \rho)$ be a metric space possessing the properties α), β), ρ be invariant with respect to the shift, and $\{x_n\}_{n \in N} \subset X$ be a non-degenerate system. Then the appropriate space $(\mathscr{K}_{\bar{x}}; \rho_{\mathscr{K}_{\bar{x}}})$ is complete, the canonical system $\{e_n\}_{n \in N}$*

forms a basis for it. Furthermore, the metric $\rho_{\mathscr{K}_{\bar{x}}}$ is invariant with respect to the shift and linear operations are continuous in $\mathscr{K}_{\bar{x}}$.

Assume that the system $\{x_n\}_{n\in N}$ is ω-linear independent and the range of values of the operator T is closed, i.e. $ImT = \overline{ImT}$. In this case, as it follows from the previous arguments and the Banach theorem, the spaces $\mathscr{K}_{\bar{x}}$ and ImT are isomorphic. Furthermore, T is an isomorphism between them. It is easy to see that $Te_n = x_n, \ \forall n \in N$. Then it is clear that the system $\{x_n\}_{n\in N}$ forms a basis for ImT. In case of its completeness in X it also forms a basis for it. T is said to be a coefficient operator. It is easy to see that the inverse is also true, i.e. if the operator T brings about an isomorphism between $\mathscr{K}_{\bar{x}}$ and ImT , then $\{x_n\}_{n\in N}$ forms a basis for ImT. In case of its completeness in$X$, it forms a basis for it.

Thus the following theorem is valid.

Theorem 3.2. *Let $(X;\rho)$ be a metric space possessing the properties α), β), ρ be invariant with respect to the shift, and $\{x_n\}_{n\in N} \subset X$ be a non-degenerate complete system in X. Then it forms a basis for it only if the appropriate coefficient operator $T : \mathscr{K}_{\bar{x}} \to X$ is an isomorphism in $L(\mathscr{K}_{\bar{x}}; X)$.*

4. Conclusion

Summarizing the all obtained results, we find the following conclusion:

1) in each linear metric space satisfying the properties α) and β) an arbitrary non-degenerate system generates the corresponding linear metric space of coefficients also satisfying the properties α) and β);

2) regardless of the fact that this system is complete or minimal, the space of coefficients has a canonical basis;

3) this system generates a corresponding coefficient operator, which acts from $K_{\bar{X}}$ to X and it forms a basis only when this operator is an isomorphism between $K_{\bar{X}}$ and X.

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