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PROPERTIES OF THE SPECTRUM AND UNIQUENESS OF RECONSTRUCTION OF STURM-LIOUVILLE OPERATOR WITH A SPECTRAL PARAMETER IN THE BOUNDARY CONDITION

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In memory of M. G. Gasymov on his 75th birthday

Abstract. In this paper we consider the boundary value problem generated on a finite interval by the Sturm-Liouville equation and nonseparated boundary conditions, one of which contains the spectral parameter. We study the main properties of the eigenvalues of the boundary value problem, provide a uniqueness theorem and construct an algorithm for solving the inverse problem by spectral data.

1. Introduction

One of the rapidly developing sections of modern mathematics is the spectral theory of differential operators subject of study of which is the boundary value problems of the mathematical physics. Investigation of a spectrum, expansion of the given function in eigenfunctions of a differential operator, the solution of inverse problems of spectral analysis belong to basic issues of this theory. Interest to spectral theory of differential operators (of Schrödinger, Dirac and others) especially increased at the last ten years in connection with the development of quantum mechanics.

Inverse problems of spectral analysis is to determine the operators on their known spectral data, which may include one, two, and more number of spectra, the spectral function, spectra and the normalizing numbers, the Weyl function, etc. Theory of inverse problems plays a great role in various sections of mathematics and has a number of applications in natural science and engineering.

Depending on the choice of the spectral data the inverse problems of spectral analysis differ by their statements. The inverse problems that have a unique solution are of great interest. In such problems relatively small collection of spectral data (one or two spectra) more natural from physical point of view is used. Therefore they may be interesting for solving many applied problems of the modern mathematics.

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At present there is extensive literature (see [2], [13], [15], [18], [23], [24], [26], [29], [30], [34]) on theory of inverse problems. The more exhaustive results in the field of inverse spectral problems are known for the Sturm-Liouville differential equation

$$-y'' + q(x)\, y = \lambda^2 y. \tag{1.1}$$

The swedish mathematician G. Borg was the first who took steps in investigation of reconstruction of the Sturm-Liouville operator by spectral information [1]. He proved that two spectra of Sturm-Liouville operators with one common boundary condition uniquely define the function $q(x)$. N. Levinson [17] suggested another method for proving G. Borg's result. The transformation operators played a great role in various fields of spectral theory of operators. V. A. Marchenko [22] (see also [23], [24]) applied the transformation operators to the investigation of inverse problems and asymptotic behavior of the spectral function of singular Sturm-Liouville operator. In these papers the uniqueness of the Sturm-Liouville operator with the given spectral function and the uniqueness of the solution of a number of the inverse spectral problems were proved. The transformation operators tool was also used by I. M. Gelfand and B. M. Levitan in their fundamental work [5], where necessary and sufficient conditions and reconstruction method of the Sturm-Liouville operator by spectral function were obtained. The paper by M. G. Krein [16] was devoted to effective construction of the classic Sturm-Liouville operator by two spectra. The complete solution of the renewal problem of the Sturm-Liouville operator on the segment by two spectra is in M. G. Gasymov and B. M. Levitan's work [4].

Periodical problems for differential and difference equations play an important role in many physical and engineering applications. The paper [31] was the first work devoted to the inverse periodic problem for equation (1.1) in which the method using the Lyapunov function was applied. In [30] the uniqueness theorem on reconstruction of the periodic problem by the method of mapping of solutions spaces was proved. The total characteristics of the spectrum of periodic and antiperiodic boundary value problems that was based on parametrization of a class of real entire functions with the help of special conformal mappings of an upper half-plane onto an upper half-plane with vertical sections was obtained in [23] by another method. By the direct method (in which the results of the paper [23], the Gelfand-Levitan-Marchenko equation and the trace formulas are not used) in [14], the inverse problem was solved by the length of the gaps for the Hill operator. Here real-analytic isomorphism between the Hilbert spaces and bilateral estimations of the potential in terms of spectral data are widely used.

The paper [28] was devoted to characteristics of spectral data of similar boundary value problems (i.e. problems whose characteristic functions differ by a constant). In this paper the methods and results of the paper [23] are generalized and developed for the case of non-separated boundary conditions (i.e. when boundary forms contain combination of values of functions on the end of the section). The problem of reconstruction of a class of similar boundary value problems by spectral data was completely studied in [33] by another method. The classes of non-similar (self-adjoint) boundary value problems were considered in the papers of M. G. Gasymov, I. M. Guseinov, and I. M. Nabiev [3], I.M. Guseinov and

I. M. Nabiev [10], [11]. The properties of spectral data were studied, uniqueness theorems, necessary and sufficient conditions for solvability of the inverse problems were proved. The characteristics of the spectrum of Sturm-Liouville operators in the case of regular, irregular and degenerate boundary conditions were given in the papers [19]-[21], in the case of boundary conditions containing a spectral parameter in [6]. Inverse problems on a whole axis, semi-axis and section for Sturm-Liouville operators with a spectral parameter in the discontinuity condition were solved in [8], [9], [12]. Note that in the general case of boundary conditions, a uniqueness problem of the reconstruction of a differential equation and boundary conditions was studied in the papers [30], [32], [35] by different methods. One can find information on similar results on the solution of inverse problems (in different statements) for a quadratic bundle of Sturm-Liouville operators in [27] (see also [26]).

In the present paper we consider a boundary value problem generated on the segment $[0,\pi]$ by the Sturm-Liouville equation (1.1) and nonseparated boundary conditions

$$y(0)+\omega\, y(\pi)=0, \bar{\omega}\, y'(0)+(\alpha\lambda+\beta)\, y(\pi)+y'(\pi)=0, \tag{1.2}$$

where the coefficient of equation (1.1) $q(x)$ is a real function belonging to $L_2[0,\pi]$, λ is a spectral parameter $\omega\neq\pm i$ is a pure imaginary number $\alpha\neq 0$, β are real numbers. We will denote this problem by L_β. Basic properties of the eigenvalues of the problem L_β was studied, a uniqueness theorem was given, an algorithm for solving the inverse problem of reconstruction of boundary value problems L_{β_1} and L_{β_2} $(\beta_1\neq\beta_2)$ by spectral data was composed.

By $W_2^n[0,\pi]$ we denote the Sobolev space of complex-valued functions on $[0,\pi]$ which has $n-1$ absolutely continuous derivative and nth derivative of which is square integrable on $[0,\pi]$. For brevity, in the sequel, we will say that the condition (T) holds, if for any function $y\,(x)\in W_2^2\,[0,\,\pi]$, $y\,(x)\not\equiv 0$ satisfying the condition (1.2) the inequality

$$\beta\,|y(\pi)|^2+\int_0^\pi\left(|y'(x)|^2+q(x)\,|y(x)|^2\right)dx>0 \tag{1.3}$$

holds. Note that this inequality is necessarily satisfied if $\beta\geq 0,\ \ q\,(x)>0$.

2. Properties of eigenvalues of the problem L_β

In this section we assume that the condition (T) holds. Denote

$$M=\int_0^\pi|y\,(x)|^2\,dx,\ N=\alpha\,|y\,(\pi)|^2\,, P=\beta\,|y\,(\pi)|^2+$$

$$\int_0^\pi\left(|y'\,(x)|^2+q\,(x)\,|y\,(x)|^2\right)dx. \tag{2.1}$$

Lemma 1. *The eigenvalues of boundary value problem L_β are real, nonzero and simple.*

Proof. Let λ be an eigenvalue of the problem L_β and $y(x)$ be a corresponding eigenfunction. Multiplying both sides of the equality (1.1) by $\overline{y(x)}$ and integrating the result from 0 to π with respect to x, we obtain

$$-\int_0^\pi y''(x)\overline{y(x)}dx + \int_0^\pi q(x)\,|y(x)\,|^2\,dx = \lambda^2\int_0^\pi |y(x)|^2\,dx.$$

Applying the integration by part we arrive at

$$y'(0)\overline{y(0)} - y'(\pi)\overline{y(\pi)} + \int_0^\pi \left(|y'(x)|^2 + q(x)\,|y(x)|^2\right)dx = \lambda^2\int_0^\pi |y(x)|^2\,dx. \tag{2.2}$$

From the boundary conditions (1.2) we obtain

$$y'(0)\overline{y(0)} - y'(\pi)\overline{y(\pi)} = -y'(0)\overline{\omega y(\pi)} + \left[\overline{\omega}y'(0) + (2\alpha\lambda+\beta)\,y(\pi)\right]\overline{y(\pi)} =$$
$$(\alpha\lambda+\beta)\,|y(\pi)|^2\,.$$

Taking into account (2.1) and (2.2), we find that

$$M\lambda^2 - N\lambda - P = 0,$$

and hence

$$\lambda = \frac{N \pm \sqrt{N^2+4MP}}{2M}. \tag{2.3}$$

By the inequality (1.3) we have$P > 0$. Since $M > 0$, (2.3) implies that , λ is real and nonzero. Since the number ω is nonreal, by the Theorem 2.1 of [25] we obtain that eigenvalues of the problem L_β are simple. The lemma is proved.

Lemma 2. *If $y(x)$ is an eigenfunction of the problem L_β corresponding to the eigenvalue λ, then*

$$2\lambda M - N \neq 0. \tag{2.4}$$

Besides it, the relation

$$\operatorname{sign}(2\lambda M - N) = \operatorname{sign}\lambda. \tag{2.5}$$

holds.

Proof. Since $M > 0$, $P > 0$, it follows from (2.3) that

$$2\lambda M - N = \pm\sqrt{N^2+4MP} \neq 0.$$

It also follows from (2.3) that, if $\lambda > 0$ then under the root it must be the $+$ sign , and $-$ sign if $\lambda < 0$. Therefore, the sign of the left hand side of (2.4) coincides whit the sign of λ, i.. (2.5) holds. The lemma is proved.

We denote by $c(x,\lambda)$ and $s(x,\lambda)$ solutions of the equation (1.1) satisfying the initial conditions $c(0,\lambda)$=$s'(0,\lambda) = 1$, $c'(0,\lambda)$ =$s(0,\lambda) = 0$. Then it is easy to see that the characteristic function of the boundary-value problem L_β is

$$\Delta(\lambda) = |\omega|^2\,c(\pi,\lambda) + (\alpha\lambda+\beta)\;s(\pi,\lambda) + s'(\pi,\lambda)\,. \tag{2.6}$$

The zeros of this functions are the eigenvalues of the problem L_β.

Lemma 3. *The inequality $\Delta(0) > 0$ holds.*

The proof of this lemma is analogous to the proof of Lemma 1.4 of [25] (see also [7]).

Theorem 1. *The following asymtotic formulas hold (as $|k| \to \infty$) for the eigenvalues μ_k ($k = \pm1, \pm2, ...$) of the boundary value problem:*

$$\mu_k = k + a + \frac{B}{k} + \frac{\tau_k}{k}, \tag{2.7}$$

where

$$a=-\frac{1}{\pi}\mathrm{arctg}\frac{b}{\alpha}\ ,\ \ B=\frac{1}{\pi}\left(A+\frac{\beta\, b}{\alpha^2+b^2}\right), \tag{2.8}$$

$$b=1+|\omega|^2,\ \ A=\frac{1}{2}\int_0^\pi q(x)\,dx,\ \ \sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty}|\tau_k|^2<\infty.$$

Proof. According to (2.6) the characteristic equation of the problem L_β is of the form $|\omega|^2 c(\pi,\lambda)+(\alpha\lambda+\beta)\, s(\pi,\lambda)+s'(\pi,\lambda)=0$. It follows from the representations of the functions $c(\pi,\lambda)$, $s(\pi,\lambda)$ and $s'(\pi,\lambda)$ (see [23, p. 18]) that

$$b\cos\lambda\pi+\alpha\sin\lambda\pi+(Ab+\beta)\frac{\sin\lambda\pi}{\lambda}-\alpha A\frac{\cos\lambda\pi}{\lambda}+\frac{f(\lambda)}{\lambda}=0, \tag{2.9}$$

where $f(\lambda)$ is an entire function of an exponential type not exceeding π, square summable on real line. By Rouche's theorem the roots μ_k $(k=\pm1,\pm2,...)$ of this equation satisfies the asymptotic formula (as $|k|\to\infty$)

$$\mu_k=k+a+\varepsilon_k, \varepsilon_k=O\left(\frac{1}{k}\right). \tag{2.10}$$

Putting (2.10) into (2.9) and using asymptotic equalities

$$\cos\mu_k\pi=(-1)^k\cos(a+\varepsilon_k)\pi=(-1)^k(\cos a\pi-\pi\varepsilon_k\sin a\pi)+O\left(\frac{1}{k^2}\right),$$

$$\sin\mu_k\pi=(-1)^k\sin(a+\varepsilon_k)\pi=(-1)^k(\sin a\pi+\pi\varepsilon_k\cos a\pi)+O\left(\frac{1}{k^2}\right),$$

$$\frac{\sin\mu_k\pi}{\mu_k}=\frac{(-1)^k\sin a\pi}{k}+O\left(\frac{1}{k^2}\right),\ \ \frac{\cos\mu_k\pi}{\mu_k}=\frac{(-1)^k\cos a\pi}{k}+O\left(\frac{1}{k^2}\right),$$

$$\frac{f(\mu_k)}{\mu_k}=\frac{f(k+a)}{k}+O\left(\frac{1}{k^2}\right)$$

(in the establishment of the last formula we use Lemma 1.4.3 of [23]), we obtain the asymptotic

$$\varepsilon_k=\frac{B}{k}+\frac{\tau_k}{k},$$

substitution of which into (2.10) implies the desired formula (2.7). The theorem is proved.

Theorem 2. The eigenvalues $\mu_k^{(1)}$ *and* $\mu_k^{(2)}$ *$(k=\pm1,\pm2,...)$ of the boundary value problems L_{β_1} and L_{β_2} $(\beta_1<\beta_2)$ alternate (interlace) in the following sense:*

$$...<\mu_{-2}^{(2)}<\mu_{-2}^{(1)}<\mu_{-1}^{(2)}<\mu_{-1}^{(1)}<0<\mu_1^{(1)}<\mu_1^{(2)}<\mu_2^{(1)}<\mu_2^{(2)}<.... \tag{2.11}$$

Proof. According to (2.6)

$$\Delta_j(\lambda)=|\omega|^2 c(\pi,\lambda)+(\alpha\lambda+\beta_j)\ s(\pi,\lambda)+s'(\pi,\lambda). \tag{2.12}$$

is a characteristic function of L_{β_j} $(j=1,\ 2)$. Consider a solution of the equation (1.1) which is of the form

$$z(x,\lambda)=[1+\omega A(\pi,\lambda)]\ s(x,\lambda)-\omega s(\pi,\lambda)\ c(x,\lambda). \tag{2.13}$$

Differentiating the equality

$$z''(x,\lambda) + \left[\lambda^2 - q(x)\right] z(x,\lambda) = 0\,, \tag{2.14}$$

with respect toλ and then taking the complex conjugation in the obtained equality, for real λ we obtain

$$\overline{\dot{z}''(x,\lambda)} + 2\lambda\,\overline{z(x,\lambda)} + \left[\lambda^2 - q(x)\right]\,\overline{\dot{z}(x,\lambda)} = 0.$$

Then, multiplying this equality by $z(x,\lambda)$, and the relation (2.14) by $\overline{\dot{z}(x,\lambda)}$, subtracting the second result from the first one and then integrating on$[0,\pi]$ with respect to x, and using (2.13) we find

$$2\lambda\int_0^\pi |z(x,\lambda)|^2\,dx - \alpha\,|z(\pi,\lambda)|^2 = \overline{\dot{z}(\pi,\lambda)}\,z'(\pi,\lambda) -$$

$$z(\pi,\lambda)\,\overline{\dot{z}'(\pi,\lambda)} - \overline{\dot{z}(0,\lambda)}\,z'(0,\lambda) + z(0,\lambda)\,\overline{\dot{z}'(0,\lambda)} =$$

$$\dot{s}(\pi,\lambda)\,\left[|\omega|^2\,c(\pi,\lambda) + s'(\pi,\lambda) + \alpha\lambda s(\pi,\lambda)\right] -$$

$$s(\pi,\lambda)\,\left[|\omega|^2\,\dot{c}(\pi,\lambda) + \dot{s}(\pi,\lambda) + \alpha\lambda\,\dot{s}(\pi,\lambda) + \alpha\,s(\pi,\lambda)\right].$$

From here, using the equalities

$$s(\pi,\lambda) = \frac{\Delta_2(\lambda) - \Delta_1(\lambda)}{\beta_2 - \beta_1}, \tag{2.15}$$

$$|\omega|^2\,c(\pi,\lambda) + s'(\pi,\lambda) + \alpha\lambda s(\pi,\lambda) = \frac{\beta_2\Delta_1(\lambda) - \beta_1\Delta_2(\lambda)}{\beta_2 - \beta_1}, \tag{2.16}$$

obtained from (2.12), we find that

$$\Delta_1(\lambda)\,\dot{\Delta}_2(\lambda) - \dot{\Delta}_1(\lambda)\,\Delta_2(\lambda) = (\beta_2 - \beta_1)\left\{2\lambda\int_0^\pi |z(x,\lambda)|^2\,dx - \alpha\,|z(\pi,\lambda)|^2\right\}. \tag{2.17}$$

It is evident that $z\left(x,\mu_k^{(j)}\right)$ is an eigenfunction of the problem L_{β_j}. Then it follows from (2.17) and Lemma 2 that functions $\Delta_1(\lambda)$ and $\Delta_2(\lambda)$ have only simple zeroes and they do not have common zeros. Putting $\lambda = \mu_k^{(2)}$ $(k = \pm1, \pm2, ...)$ in (2.17) and taking into account that $\Delta_2\left(\mu_k^{(2)}\right) = 0$ we obtain

$$\Delta_1\left(\mu_k^{(2)}\right)\dot{\Delta}_2\left(\mu_k^{(2)}\right) = (\beta_2 - \beta_1)\left\{2\mu_k^{(2)}\int_0^\pi \left|z\left(x,\mu_k^{(2)}\right)\right|^2 dx - \alpha\left|z\left(\pi,\mu_k^{(2)}\right)\right|^2\right\}.$$

Since$\beta_2 - \beta_1 > 0$, according to Lemma 2, the last equality implies that

$$\text{sign}\left\{\Delta_1\left(\mu_k^{(2)}\right)\dot{\Delta}_2\left(\mu_k^{(2)}\right)\right\} = \text{sign}\mu_k^{(2)}.$$

Hence,

$$\text{sign}\,\Delta_1\left(\mu_k^{(2)}\right) = \text{sign}\,\dot{\Delta}_2\left(\mu_k^{(2)}\right) \text{ if } \mu_k^{(2)} > 0,$$
$$\text{sign}\,\Delta_1\left(\mu_k^{(2)}\right) = -\text{sign}\,\dot{\Delta}_2\left(\mu_k^{(2)}\right) \text{ if } \mu_k^{(2)} < 0.$$

Using lemma 3 and arguing as in the proof of Theorem 4.1 of [25], we find from last relations that the function $\Delta_1(\lambda)$ has two zeros of different sign in the interval $\left(\mu_{-1}^{(2)}, \mu_1^{(2)}\right)$ and exactly one zero in each of the intervals

$$..., \left(\mu_{-3}^{(2)}, \mu_{-2}^{(2)}\right), \left(\mu_{-2}^{(2)}, \mu_{-1}^{(2)}\right), \left(\mu_1^{(2)}, \mu_2^{(2)}\right), \left(\mu_2^{(2)}, \mu_3^{(2)}\right), ...,$$

i.e. (2.11) holds. The theorem is proved.

3. Uniqueness of reconstruction of the problems L_{β_1} and L_{β_2}

We denote by λ_k $(k = \pm 1, \pm 2, ...)$ zeros of the function $s(\pi, \lambda)$, squares of which are the eigenvalues of the boundary-value problem generated by the equation (1.1) and boundary conditions $y(0) = y(\pi) = 0$. The following uniqueness theorem holds.

Theorem 3. *The boundary-value problems L_{β_1} and L_{β_2} are uniquely determined by their spectra, a number ω and the sequence of signs*

$$\sigma_k = sign\left[|\omega| - \left|s'(\lambda_k, \pi)\right|\right].$$

We give an algorithm of reconstruction of the boundary value problems L_{β_1} and L_{β_2} (at the same time we give a short proof of Theorem 3).

Algorithm. The sequences $\left\{\mu_k^{(1)}\right\}$, $\left\{\mu_k^{(2)}\right\}$, $\{\sigma_k\}$ and a number ω - the spectral data of the problems L_{β_1}, L_{β_2} are given.

1) Construct the function $\Delta_j(\lambda)$ in the form of an infinite product by using the sequence $\left\{\mu_k^{(j)}\right\}$.

2) Determine the parameter α by the formula$\alpha = -b\text{ctg}\pi\, a$ (see (11)), where $b = 1 + |\omega|^2$, $a = \lim\limits_{k\to\infty} \left(\mu_k^{(j)} - k\right)$ (see (10)).

3) Calculate the difference $\beta_2 - \beta_1 = \frac{\pi}{b}\left(\alpha^2 + b^2\right)(B_2 - B_1)$, where $B_2 - B_1 = \lim\limits_{k\to\infty} k\left(\mu_k^{(j)} - k - a\right)$ (according to (10)).

4) Construct the function $s(\pi, \lambda)$ by (2.15) and find the zeros λ_k of this function.

5) Define the parameters β_j by the formula

$$\beta_j = \frac{\pi}{b}\left(\alpha^2 + b^2\right)\lim_{k\to\infty} k\left(\mu_k^{(j)} - \lambda_k - a\right).$$

6) Reconstruct the function $u_+(\lambda) = |\omega|^2 c(\pi, \lambda) + s'(\pi, \lambda) + \alpha\lambda s(\pi, \lambda)$ by (2.16).

7) Find the value of the function $u_-(\lambda) = |\omega|^2 c(\pi, \lambda) - s'(\pi, \lambda) + \alpha\lambda s(\pi, \lambda)$ at the points λ_k as follows: $u_-(\lambda_k) = (-1)^k \sigma_k \sqrt{u_+^2(\lambda_k) - 4|\omega|^2}$.

8) Using $s(\pi, \lambda)$ and $u_\pm(\lambda_k)$ we reconstruct the function $g(\lambda) = \frac{1-|\omega|^2}{4|\omega|^2}u_+(\lambda) + \frac{1+|\omega|^2}{4|\omega|^2}u_-(\lambda) = \frac{1}{2}\left[c(\pi, \lambda -)\, s'(\pi, \lambda)\right]$ by the interpolation formula

$$g(\lambda) = s(\pi, \lambda) \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{g(\lambda_k)}{(\lambda - \lambda_k)\,\dot{s}(\pi, \lambda_k)}.$$

9) Define $u_-(\lambda)$ from 8).

10) Define the characteristic function $s'(\pi, \lambda)$ of the boundary-value problem, generated by the equation (1.1) and boundary conditions $y(0) = y'(\pi) = 0$ by the formula $s'(\pi, \lambda) = \frac{1}{2}[u_+(\lambda) - u_-(\lambda)]$.

11) Determine the coefficient $q(x)$ from the sequences of zeros of the functions $s(\pi, \lambda)$ and $s'(\pi, \lambda)$ by the known procedure (see, e.g., [23]).

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