Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 40, Special Issue, 2014, Pages 342–350

THE ASYMPTOTIC OF SOLUTIONS OF SINGULAR DIFFERENTIAL EQUATIONS WITH LARGE SPECTRAL PARAMETER

ELVIRA A. NAZIROVA, NURMUKHAMET F. VALEEV, AND YAUDAT T. SULTANAEV

In memory of M. G. Gasymov on his 75th birthday

Abstract. The aim of the work is the problem about essential expansion of potentials class $q(x)$ in equation

$$-y'' - q(x)y = \lambda y, \ \ 0 < x < +\infty,$$

to which Levinson's method can be applied and asymptomatic formulas can be obtained to investigate Sturm-Liouville equation with large values λ.

1. Introduction

It is well known [1, 2, 6] that for the solutions of equation:

$$-y'' - q(x)y = \lambda y, \ \ 0 < x < +\infty, \tag{1}$$

where $q(x)$ is real, positive, and such that $q(x) \to +\infty$, as $|\lambda| \to \infty$, $\lambda = \tau + i\sigma$, $\sigma = \tau^{\gamma}$, $0 < \gamma < 1$, the following asymptotic formulas uniform with respect to x are valid:

$$y_{1,2} \sim \frac{1}{\sqrt[4]{\lambda + q(x)}} \exp\{\pm i \int\limits_{x_0}^{x} \sqrt{\lambda + q(t)} dt\},$$

where $q(x)$ satisfies the following conditions:

1) $q'(x)$, $q''(x)$ do not change sign in the interval $[x_0, +\infty)$;

2) $q'(x) = o(q^{\alpha}(x))$, $0 < \alpha < 3/2$.

Levinson's method of substituting equation (1) by the system of first order equations and reducing it to L-diagonalized form was applied in [1, 2, 6], and in [5] the Liouville change was used. The former method is too cumbersome but, unlike the latter, can be applied to equations of an order higher than second.

L-diagonal systems are the systems of the form:

$$u'(x) = \Lambda(x)u(x) + \Theta(x)u(x),$$

2010 *Mathematics Subject Classification.* 34K08.

Key words and phrases. spectral theory of singular differential equations, Sturm-Liouville equation, asymptotic methods.

The work was done with partial support by RFBR (grant N12-01-00567-a, E.A.Nazirova, N.F.Valeev), partial financial support of the Ministry of Education and Science of the Republic Kazakhstan (projects 2989/GFZ MON RK, Ya.T.Sultanaev).

where $\Theta(x)$ is a summable matrix on $[x_0, \infty)$. Let us note that conditions 1) and 2) are rather hard but they provide the summability of matrix $\Theta(x)$.

In [5] a Sturm-Liouville equation with rapidly-oscillating potential was considered, but Liouville's method was applied in the research, and in paper [4] asymptotic formulas with $x \to \infty$ were obtained for fundamental system of solutions of a Sturm-Liouville equation also with rapidly-oscillating potential.

The aim of the work is the problem about essential expansion of potentials class $q(x)$ to which Levinson's method can be applied and asymptomatic formulas can be obtained to investigate Sturm-Liouville equation with large values λ.

2. Transformation of the Sturm-Liouville equation

Let us consider the following equation:

$$-y'' - q(x)y - (q_1(x) - q(x))y = \lambda y,$$

$|\lambda| \to \infty$, $\lambda \in \Gamma$, wher $\Gamma = \{\lambda \in \mathbb{C} |\, \lambda = \tau + i\sigma,\ \sigma = \tau^\gamma,\ 0 < \gamma < 1\}$.

Let us denote $u(x) = q_1(x) - q(x)$. Here $q(x)$ is the function satisfying conditions 1), 2) of part 1. Let us move from the given equation to the system bringing into consideration column vector $Y = (y, y')$:

$$Y' = A(x, \lambda)Y, \tag{2}$$

$$A(x, \lambda) = A_1(x, \lambda) + A_2(x), \qquad A_1(x, \lambda) = \begin{pmatrix} 0 & 1 \\ -q(x) - \lambda & 0 \end{pmatrix},$$

$$A_2(x) = \begin{pmatrix} 0 & 0 \\ -u(x) & 0 \end{pmatrix}.$$

Let us find eigenvalues of matrix $A_1(x, \lambda)$, denoting through I–identical matrix of size 2×2:

$$det(A_1 - \omega \cdot I) = 0, \quad \omega^2 = -(q(x) + \lambda),$$

$$\omega_1(x, \lambda) = i\sqrt{\lambda + q(x)}, \quad \omega_2(x, \lambda) = -i\sqrt{\lambda + q(x)},$$

choosing $arg(\sqrt{\lambda + q(x)}) \in [0, \pi/2]$, then $Re(i\sqrt{\lambda + q(x)}) < 0$.

Let matrix $T(x)$ reduce A_1 to diagonalized form:

$$T^{-1}(x, \lambda)A_1(x, \lambda)T(x, \lambda) = \Lambda(x, \lambda), \ \ \Lambda(x, \lambda) = diag\{\omega_1(x, \lambda), -\omega_1(x, \lambda)\}.$$

It is known that off-diagonal elements of matrix $T(x, \lambda)$ are defined within the accuracy of multiplying the elements of every column by scalar function, in such case these functions may be chosen so that the following relation is valid:

$$(T^{-1}(x, \lambda)T'(x, \lambda))_{ii} = 0, \ \ i = 1, 2.$$

Then $T(x, \lambda)$ has the form:

$$T(x, \lambda) = \delta(x, \lambda) \begin{pmatrix} 1 & 1 \\ \omega_1(x, \lambda) & -\omega_1(x, \lambda) \end{pmatrix}, \ \ \delta(x, \lambda) = (\lambda + q(x))^{-1/4},$$

$$T^{-1} = \frac{1}{2} \begin{pmatrix} 1/\delta & \delta/i \\ 1/\delta & -\delta/i \end{pmatrix}, \quad T^{-1}T' =: C(x, \lambda) = \frac{\delta'}{\delta} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let us make a change in system (2):

$$Y = T \cdot U. \tag{3}$$

Then we obtain the system:

$$U' = \Lambda U - CU + T^{-1}A_2T. \tag{4}$$

Now let us assume that matrix $G(x,\lambda)$ is such that $G\Lambda - \Lambda G = -C$,

$$G(x,\lambda) = \frac{\delta'(x,\lambda)\delta(x,\lambda)}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let us make a change in (4):

$$U = (I+G)W. \tag{5}$$

Then we come to system

$$W' = \Lambda W - (I+G)^{-1}CGW - (I+G)^{-1}G'W - (I+G)^{-1}T^{-1}A_2T(I+G)W. \tag{6}$$

Consider the functions:

$$a(x,\lambda) = \frac{\delta'(x,\lambda)\delta(x,\lambda)}{2},\ \ \Delta(x,\lambda) = 1 + a^2(x,\lambda),\ \ p(x,\lambda) = u(x)\cdot\delta^2(x,\lambda) = \frac{u(x)}{\sqrt{\lambda+q(x)}}. \tag{7}$$

Suppose, for brevity $D := (I+G)^{-1}CG + (I+G)^{-1}G'$, $\ p(x,\lambda)\cdot B := (I+G)^{-1}T^{-1}A_2T(I+G)$. It is not hard to write out the off-diagonal elements of matrixes D, B, taking into account notifications (7):

$$D(x,\lambda) = \frac{\delta'^2(x,\lambda)}{2\Delta(x,\lambda)}\begin{pmatrix} 1 & a(x,\lambda) \\ a(x,\lambda) & -1 \end{pmatrix} + \frac{a'(x,\lambda)}{\Delta(x,\lambda)}\begin{pmatrix} a(x,\lambda) & 1 \\ -1 & a(x,\lambda) \end{pmatrix}, \tag{8}$$

$$B(x,\lambda) = \frac{1}{2\Delta(x,\lambda)}\begin{pmatrix} -1+a^2(x,\lambda) & -(1+a(x,\lambda))^2 \\ (1-a(x,\lambda))^2 & 1-a^2(x,\lambda) \end{pmatrix}. \tag{9}$$

Suppose in system (6)

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = e^{\int \omega_1(t,\lambda)dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tag{10}$$

we obtain the system of equations:

$$\begin{cases} z_1' = -(d_{11} + p\cdot b_{11})z_1 - (d_{12} + p\cdot b_{12})z_2, \\ z_2' = -2\cdot\omega_1 z_2 - (d_{21} + p\cdot b_{21})z_1 - (d_{22} + p\cdot b_{22})z_2. \end{cases} \tag{11}$$

Assuming system (11) has a solution that meets the conditions:

$$\lim_{x\to\infty} z_1(x) = 1,$$

$$\lim_{x\to\infty} z_2(x) = 0,$$

and, like in proving lemma 1 in ([6, p.288]) we turn from system (11) to the system of integral equations:

$$\begin{cases} z_1 = 1 - \int_x^\infty \left[(d_{11} + p\cdot b_{11})z_1 - (d_{12} + p\cdot b_{12})z_2\right]dt, \\ z_2 = -\int_x^\infty \psi\left[(d_{21} + p\cdot b_{21})z_1 - (d_{22} + p\cdot b_{22})z_2\right]dt, \end{cases} \tag{12}$$

$$\psi(x,t,\lambda) = e^{2\int_x^t \omega_1(s,\lambda)ds}.$$

Integrating by parts the summands in the right parts of the equations that contain function $p(x,\lambda)$, after the generation we can reduce (12) to the following form:

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + P(x,\lambda)\cdot\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} +$$

$$\int\limits_x^\infty P(t,\lambda)\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} dt - \int\limits_x^\infty \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} dt, \qquad (13)$$

$$P(x,\lambda) = \int\limits_x^\infty p(t,\lambda)dt = \int\limits_x^\infty \frac{u(t)}{\sqrt{q(t)+\lambda}}dt,$$

$$\alpha_{11} = b'_{11} - b_{11}d_{11} - pb_{11}^2 - b_{12}(d_{21} + b_{21}p),$$
$$\alpha_{12} = b'_{12} - b_{11}(d_{12} + pb_{12}) - b_{12}(2\omega_1 + d_{22} + b_{22}p),$$
$$\alpha_{21} = (\psi b_{21})' - \psi b_{21}(d_{11} + pb_{11}) - \psi b_{22}(d_{21} + b_{21}p),$$
$$\alpha_{22} = (\psi b_{22})' - \psi b_{21}(d_{12} + pb_{12}) - \psi b_{22}(2\omega_1 + d_{22} + b_{22}p).$$

Let us make an operator notation of system (13) and take into consideration M – a continuity set on $[x_0,\infty)$ functions with a finite limit with $x \to +\infty$, and the set $M \times M = \widetilde{M}$. Constant x_0 will be chosen later.

Let us introduce a norm on set M:

$$\| \phi \| = \sup_{[x_0,\infty)} |\phi(x)|.$$

We will also define the norm on set $\widetilde{M}$ according to the rule:

$$\| y \| = \max\{\| y_1 \|, \| y_2 \|\}.$$

Then the operator $A[y]$ norm is:

$$\| A \| = \sup_{y\in\widetilde{M}} \frac{\| A[y] \|}{\| y \|}.$$

Note that if A – operator of multiplying by matrix, then the consistent norm will have the form:

$$\| A \| = \max_{j=1,2}\{\| a_{1j} \| + \| a_{2j} \|\}.$$

Let us introduce the operator:

$$K[z] = P(x,\lambda)\cdot Bz + \int\limits_x^\infty P(t,\lambda)\Upsilon(t,\lambda)zdt - \int\limits_x^\infty D(t,\lambda)zdt,$$

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad (\Upsilon(t,\lambda))_{ij} = \alpha_{ij}(t,\lambda).$$

Then system (13) has the form:

$$z = e_1 + K[z], \;\; (I-K)[z] = e_1, \;\; z = (I-K)^{-1}[e_1], \qquad (14)$$

where e_1 is a unit two dimensional vector.

Let us find out under what conditions besides conditions 1), 2) from part 1. operator K functions out of set $\widetilde{M} \to \widetilde{M}$ and is contracting.

If $\| K \| < 1$, for λ, such that $\lambda \in \Gamma, |\lambda| > R_0$ for some R_0 and all $x \in [x_0,\infty)$, then there exist such solution of equation (14), and such that: $z(x,\lambda) \to e_1$, with

$\lambda \in \Gamma$, $|\lambda| \to +\infty$, uniformly on x. It is obvious that the solution will also meet the condition: $z(x,\lambda) \to e_1$, $x \to +\infty$. Let us show that such evaluation for operator norm $\| K \| < 1$ is reachable. Suppose $f = K[z]$, $f = (f_1, f_2)^T$. We obtain asymptotic on $\lambda \in \Gamma$, $|\lambda| \to \infty$, evaluation of norm f. Taking into account (8), (9),

$$\|f\| \le |P(x,\lambda)\cdot(b_{11}z_1 + b_{12}z_2)| + \left|\int\limits_x^\infty P(t,\lambda)(\alpha_{11}z_1+\alpha_{12}z_2)dt\right| +$$

$$\left|\int\limits_x^\infty (d_{11}z_1 + d_{12}z_2)dt\right| \le |P(x,\lambda)|(|b_{11}||z_1| + |b_{12}|z_2|)+$$

$$\int\limits_x^\infty (|P(t,\lambda)\alpha_{11}||z_1| + |P(t,\lambda)\alpha_{12}||z_2|)dt + \int\limits_x^\infty (|d_{11}||z_1| + |d_{12}||z_2|)dt \le$$

$$|P(x,\lambda)|(|b_{11}| \parallel z_1 \parallel + |b_{12}| \parallel z_2 \parallel) + \int\limits_x^\infty (|P(t,\lambda)\alpha_{11}| \parallel z_1 \parallel + |P(t,\lambda)\alpha_{12}| \parallel z_2 \parallel)dt +$$

$$\int\limits_x^\infty (|d_{11}| \parallel z_1 \parallel + |d_{12}| \parallel z_2 \parallel)dt \le |P(x,\lambda)| \parallel B \parallel \parallel z \parallel +$$

$$\parallel z \parallel \int\limits_x^\infty (|P(t,\lambda)\alpha_{11}| + |P(t,\lambda)\alpha_{12}|)dt + \parallel z \parallel \int\limits_x^\infty (|d_{11}| + |d_{12}|)dt =$$

$$\|z\| \left(|P(x,\lambda)| \parallel B \parallel + \int\limits_{x_0}^\infty (|P(t,\lambda)\alpha_{11}| + |P(t,\lambda)\alpha_{12}|)dt + \int\limits_{x_0}^\infty (|d_{11}| + |d_{12}|)dt \right). \tag{15}$$

We get evaluations for the summands in the right part of inequality (15). Let us consider the behavior at $\lambda \in \Gamma$, $|\lambda| \to \infty$, the behavior of the functions included in the coefficients of matrixes $D(x,\lambda)$, $B(x,\lambda)$, $\Upsilon(x,\lambda)$:

$$\delta'(x) = -1/4(\lambda + q(x))^{-5/4} \cdot q'(x),$$

$$\frac{\delta'(x)}{\delta(x)} = -\frac{1}{4}\frac{q'(x)}{\lambda + q(x)} = -\frac{1}{4}\frac{q'(x)}{\tau + i\sigma + q(x)},$$

$$\delta'(x)\delta(x) = -\frac{1}{4}\frac{q'(x)}{(\lambda+q(x))^{3/2}} = -\frac{1}{4}\frac{q'(x)}{(\tau + i\sigma + q(x))^{3/2}} =$$

$$-\frac{1}{4}\frac{q'(x)}{(\tau + q(x))^{3/2}(1 + i\sigma/(\tau + q(x)))^{3/2}}.$$

Under conditions 1), 2) in part 1 and $\sigma = \tau^\gamma$, $0 < \gamma < 1$, we obtain:

$$a(x,\lambda) = \frac{\delta'(x)\delta(x)}{2} = o(1), \quad x \to \infty,$$

$$a(x,\lambda) = \frac{\delta'(x)\delta(x)}{2} = o(\tau^{-3/2}) = o(|\lambda|^{-3/2}), \quad |\lambda| \to \infty.$$

From the aforementioned and formula (9), in particular, it follows that off-diagonal elements of matrix $B(x,\lambda)$ are limited on x and λ, that is

$$B(x,\lambda) \to \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$$

which means that choosing R_0 large enough for $\lambda \in \Gamma, |\lambda| > R_0$ the following is true: $\| B \| = B_0 < 3$. Taking into account conditions 1), 2) of part 1 and formula (8), we also obtain for the functions that are a part of off-diagonal elements of matrix $D(x,\lambda)$:

$$\frac{\delta'^2(x,\lambda)}{2\Delta(x,\lambda)} = \frac{\delta'^2(x,\lambda)}{2(1+a^2(x,\lambda)} =$$

$$\frac{1}{32}\frac{q'^2(x)}{(\lambda+q(x))^{5/2}(1+a^2(x,\lambda))} = o(|\lambda|^{-5/2}).$$

Note that because of the proven properties of function $a(x,\lambda)$ the following is true:

$$\left|\frac{1}{1+a^2(x,\lambda)}\right| < 1, \quad |\lambda| > R_0. \tag{16}$$

Then taking into account (16),

$$\int_x^\infty \left|\frac{\delta'^2(t,\lambda)}{2\Delta(t,\lambda)}\right| dt = \int_x^\infty \left|\frac{\delta'^2(t,\lambda)}{2(1+a^2(t,\lambda)}\right| dt < \int_x^\infty \frac{1}{16}\frac{q'^2(t)}{(\tau+q(t))^{5/2}} dt =: I_1.$$

And in this way

$$I_1 = \int_x^\infty \frac{1}{16}\frac{q'(t)O(q(t)^\alpha)}{(\tau+q(t))^{5/2}} dt = \int_x^\infty \frac{1}{16}\frac{q'(t)O(1)q(t)^\alpha}{(\tau+q(t))^{5/2}} dt =$$

$$\int_x^\infty O(1)\frac{q'(t)(\tau+q(t))^\alpha}{(\tau+q(t))^{5/2}} dt = \int_x^\infty O(1)(\tau+q(t))^{\alpha-5/2} d(q(t)+\tau) =$$

$$O(1)(\tau+q(t))^{\alpha-3/2}|_x^\infty = O(|\lambda|^{-\varepsilon}),$$

where $\varepsilon = 3/2-\alpha > 0$ due to conditions 2) of part 1. Let us also consider the function

$$\left|\frac{a'(x,\lambda)}{\Delta(x,\lambda)}\right| = \left|\frac{a'(x,\lambda)}{1+a^2(x,\lambda)}\right| < |a'(x,\lambda)|,$$

$$|a'(x,\lambda)| < \left|\frac{1}{8}\frac{q''(x)}{(\lambda+q(x))^{3/2}}\right| + \left|\frac{3}{16}\frac{q'^2(x)}{(\lambda+q(x))^{5/2}}\right|.$$

Taking into account conditions 1), 2) of part 1, we get:

$$\int_x^\infty \left|a'(t,\lambda)\right| dt < \int_x^\infty \left|\frac{1}{8}\frac{q''(t)}{(\lambda+q(t))^{3/2}}\right| dt + \int_x^\infty \left|\frac{q'^2(t)}{(\lambda+q(t))^{5/2}}\right| dt <$$

$$C_0\left|\int_x^\infty \frac{1}{8}\frac{q''(t)}{(\tau+q(t))^{3/2}} dt\right| + C_0\int_x^\infty \frac{3}{16}\frac{q'^2(t)}{(\tau+q(t))^{5/2}} dt,$$

where C_0 is a constant.

Integrating the integral in the first summand by parts, we get the evaluation:

$$\int_x^\infty \left|a'(t,\lambda)\right| dt < \frac{1}{8}\frac{q'(x)}{(\tau+q(t))^{3/2}} + 6I_1 = O(|\lambda|^{-\varepsilon}).$$

We conclude that

$$\int_{x_0}^{\infty} (|d_{11}| + |d_{12}|)dt = O(|\lambda|^{-\varepsilon}).$$

Let us consider function $P(x, \lambda)$. We denote:

$$\int_{x}^{\infty} u(t)dt =: U(x).$$

Suppose the following condition is satisfied:

$$|U(x)| < r_1(x), \;\; r_1(x) \in L[x_0, \infty), \tag{17}$$

from which

$$|P(x, \lambda)| = \left| \int_{x}^{\infty} \frac{u(t)}{\lambda + q(t))^{1/2}} dt \right| = \left| -U(t) \,|_x^{\infty} -1/2 \int_{x}^{\infty} \frac{U(t)q'(t)}{(\lambda + q(t))^{3/2}} dt \right| \leq$$

$$|U(x)| + 1/2 \max_{x \in [x_0, \infty)} |U(x)| \left| \int_{x}^{\infty} \frac{q'(t)}{(\lambda + q(t))^{3/2}} \right| dt.$$

Let us choose x_0 large enough for $\max\limits_{x \in [x_0, \infty)} |U(x)| < \varepsilon_1$, where $\varepsilon_1 > 0$ is a fixed small enough number. Then

$$|P(x, \lambda)| < \varepsilon_1 (1 + O(|\lambda|^{-\varepsilon})),$$

$$|P(x, \lambda)| \parallel B(x, \lambda) \parallel < 3\varepsilon_1 (1 + O(|\lambda|^{-\varepsilon})).$$

Taking into account the formulas for functions $\alpha_{ij}(t, \lambda)$, from the proven evaluations we can see that

$$\alpha_{ij} = 2\omega_1 (1 + o(1)), \;\; \lambda \to \infty, \;\; \lambda \in \Gamma.$$

Let us evaluate the summands of the form:

$$\int_{x_0}^{\infty} |P(t, \lambda)\alpha_{ij}| dt = \int_{x_0}^{\infty} |P(t, \lambda) 2\omega_1 (1 + o(1))| dt.$$

As in [5] it can be showed that from condition (17) the following inequalities should be performed:

$$\int_{x_0}^{\infty} |P(t, \lambda) 2\omega_1| dt < r(x), \;\; r(x) \in L[x_0, \infty).$$

Taking x_0 large enough, we can always obtain

$$\int_{x_0}^{\infty} |P(t, \lambda)\alpha_{ij}| dt < \varepsilon_2, \;\; 0 < \varepsilon_2 < 1,$$

this way, we can always find such R_0 and x_0 so that with $\lambda \in \Gamma$, $|\lambda| > R_0$, uniformly for all $x > x_0$:

$$\|f_1\|_M \leq \varepsilon \|z\|_{\widetilde{M}}.$$

Similarly we can obtain evaluations:

$$\|f_2\|_M \le \varepsilon \|z\|_{\widetilde{M}}, \quad \|K[z]\| \le \varepsilon \|z\|, \quad \|K\| \le \varepsilon < 1.$$

In this way we showed that system (14) has a solution such that:

$$\lim_{|\lambda|\to\infty,\, \lambda\in\Gamma} z = e_1.$$

Going back with the help of inverse changes (3), (5), (10), we obtain asymptotic formulas for system (2).

So, we proved the theorem.

Theorem. *Let for functions* $q(x), q_1(x)$ *the following conditions be met*:

1) $q(x) > 0$, $q(x) \to \infty$ *with* $x \to \infty$, *and* $q'(x)$, $q''(x)$ *do not change sign in the interval* $[x_1, +\infty)$;

2) $q'(x) = o(q^{\alpha}(x))$, $0 < \alpha < 3/2$;

3) $|\int\limits_x^{\infty}(q_1(t) - q(t))dt| < r_1(x)$, $r_1(x) \in L[x_1, \infty)$, *for all* $\lambda \in \Gamma$.

Then equation $-y'' - q(x)y - (q_1(x) - q(x))y = \lambda$ *has a solution* $y(x)$, *such that, with* $|\lambda| \to \infty$, $\lambda = \tau + i\sigma$, $\sigma = \tau^{\gamma}$, $0 < \gamma < 1$, *uniformly on* x *for* $x > x_0 > x_1$,

$$y(x) = \frac{1}{\sqrt[4]{\lambda + q(x)}} \exp\{i \int\limits_{x_0}^{x} \sqrt{\lambda + q(t)} dt\}(1 + o(1)),$$

$$y'(x) = i \sqrt[4]{\lambda + q(x)} \exp\{i \int\limits_{x_0}^{x} \sqrt{\lambda + q(t)} dt\}(1 + o(1)).$$

Note. The asymptotic $|\lambda| \to \infty$, $\lambda \in \Gamma$, uniform on x formulas are supposed to be applied in the further study of function $N(\lambda)$.

References

[1] M. V. Fedoryuk, *Asymptotic methods for linear ordinary differential equations*, Nauka, Moscow, 1983 (in Russian).

[2] A. G. Kostyucenko and I. S. Sargsyan, *Distribution of eigenvalues. Selfadjoint ordinary differential operators*, Nauka, Moscow, 1979 (in Russian).

[3] L. K. Kusainova, A. Zh. Monashova, and A. A. Shkalikov, Asymptotics of eigenvalues of a second-order non-self-adjoint differential operator on the axis, *Math. Notes*, **93** (2013), no. 3-4, 629–632 (translated from *Mat. Zametki*, **93** (2013), no. 4, 630–633).

[4] N. K. Makina, E. A. Nazirova, Ya. T. Sultanaev, On the methods of study of the asymptotic behavior of solutions of singular differential equations, *Math. Notes*, **96** (2014), no. 3-4, 603–608 (translated from *Mat. Zametki*, **96** (2014), no. 4, 627–632).

[5] Kh. Kh. Murtazin and Ya. T. Sultanaev, Formulas for the distribution of eigenvalues of nonsemibounded Sturm-Liouville operators, *Math. Notes*, **28** (1980), no. 4, 733–737 (translated from *Mat. Zametki*, **28** (1980), no. 4, 545–553).

[6] M. A. Naimark, *Linear differential operators*, Nauka, Moscow, 1969 (in Russian).

[7] Ya. T. Sultanaev, Deficiency indices and the spectrum of one-dimensional singular differential operators in the degenerate case, *Dokl. Akad. Nauk SSSR*, **284** (1985), no. 3, 551–555 (in Russian).

[8] N. F. Valeev, Ya. T. Sultanaev, Spectral properties of Sturm-Liouville operators in the space of vector functions, *Math. Notes*, **65** (1999), no. 6, 781–786 (translated from *Mat. Zametki*, **65** (1999), no. 6, 932–938).

Elvira A. Nazirova
Bashkir State University, Ufa, Russia.
E-mail address: ellkid@gmail.com

Nurmukhamet F. Valeev
Institute of Mathematics with Computer Center of Russian Academy of Sciences, Ufa, Russia.
E-mail address: valeevnf@yandex.ru

Yaudat T. Sultanaev
Bashkir State Pedagogical University, Ufa, Russia.
E-mail address: sultanaevyt@gmail.com

Received: September 09, 2014; Accepted: October 02, 2014