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CONSTRUCTION OF A KERNEL OF THE TRANSFORMATION OPERATOR FOR A FOURTH ORDER DIFFERENTIAL BUNDLE WITH MULTIPLE CHARACTERISTICS

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In memory of M. G. Gasymov on his 75*th birthday*

Abstract. In the paper for a fourth order differential operator with a triple characteristic root i the transformation operator, taking the solution of the considered equation to the solution of the equation containing only principal terms, is constructed. The integral equations for a kernel are found, the existence of the solutions of these equations is proved, and the partial differential equations of fourth order satisfied by the kernels with certain conditions of Goursat type are received.

1. Introduction

Let's consider on the interval $0 \leq x < \infty$ a differential equation

$$l(x, \frac{d}{dx}, \lambda)Y = (\frac{d}{dx} + i\lambda)^3(\frac{d}{dx} - i\lambda)Y + r(x)\frac{dY}{dx} + (\lambda p(x) + q(x))Y = 0, \quad (1.1)$$

where λ is a spectral parameter, $r(x), p(x), q(x)$ are defined and continuous on $[0, \infty)$, respectively have continuous derivatives to 3, 4, 5 order inclusively, the integrals

$$\begin{array}{l} \int_0^\infty x^4 \left|r^{(s)}(x)\right| dx < \infty, \; s = \overline{0,3}, \; \int_0^\infty x^4 \left|p^{(s)}(x)\right| dx < \infty, \; s = \overline{0,5}, \\ \int_0^\infty x^4 \left|q^{(s)}(x)\right| dx < \infty, \; s = \overline{0,4}, \end{array} \quad (1.2)$$

converge.

In the paper [1] it is proved the existence of the kernels of a transformation operator taking the solution of the equation

$$(\frac{d}{dx} + i\lambda)^3(\frac{d}{dx} - i\lambda)Y = 0 \quad (1.3)$$

to the solution of equation (1.1) with restrictions of conditions at the point $x = 0$ and $x = \infty$. However, the conclusion provided there gives not enough information

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on the kernels $K_j^{\pm}(x,t)$ of the transformation operator for the solutions

$$Y_j(x,\lambda) = x^{j-1}e^{i\lambda x} + \int_x^{\infty} K_j^{\pm}(x,t)e^{i\lambda t}dt,\ j=\overline{1,3},\ Jm\lambda \geq 0,$$

$$Y_4(x,\lambda) = e^{-i\lambda x} + \int_x^{\infty} K_4(x,t)e^{-i\lambda t}dt,\ Jm\lambda \leq 0, \tag{1.4}$$

which satisfy certain conditions at infinity. Informations found in [1] are insufficient also for the solution of the inverse problem – definition of the differential operator with respect to a scattering matrix, directly connected with construction of integral equations of Marchenko type and its solution. Similar problems were studied in [2, 3, 4, 8, 9] for differential bundles of fourth order and any any even order on finite and infinite intervals, in case of two various characteristic roots $\pm i$ of the corresponding principal characteristic polynomial.

More detailed plausible researches of kernels of transformation operators in case of the triple characteristic roots, which is a key tool for solving inverse problems, are of interest. From the point of view of appendices of these considerations let's notice that as it is noted in [10], the solution of problems of stability of plates made of a plastic material is very difficult because of need to solve the corresponding ordinary differential equation in the complete form (members containing Y'' and Y' appear there).

In the sense of direct spectral problems in the last decade certain results were obtained for a problem of a plate with such properties. Statements of the inverse spectral problems and their solutions haven't been investigated up to now.

The present paper can be considered as continuation of the paper [1]. However, here for reducing the calculations volume we will assume that among the coefficients of the equation (1.1) of the paper [1], some coefficients are zero and the coefficients are specified under weaker conditions. Also let's pay attention that for receiving a formula of expansion in eigenfunctions the essential role will be played by the coefficients on decreasing degrees of a spectral parameter at the exponent function in the representation of Green's function, so that it is necessary to consider not only the principal terms, but also other members to the maximum multiplicity.

2. Main results

Integral equation (9) of the paper [1], is equivalent to differential equation (1.1) with the boundary conditions

$$\lim_{x\to\infty} \frac{1}{x^j} F_j(x,\lambda)e^{i\lambda x} = 1, j=\overline{0,2},\ \lim_{x\to\infty} \frac{1}{x^j} F_j(x,\lambda)e^{-i\lambda x} = 1, \tag{2.1}$$

for any real λ, and its solution has the form:

$$\begin{array}{l} F_j^{\pm}(x,\lambda) = x^{j-1}e^{-i\lambda x} + \int_x^{\infty} K_j^{\pm}(x,t)e^{-i\lambda t}dt,\ j=0,1,2, \\ F_3^{\pm}(x,\lambda) = e^{i\lambda x} + \int_x^{\infty} K_3^{\pm}(x,t)e^{i\lambda t}dt,\ \pm Jm \geq 0. \end{array} \tag{2.2}$$

By substituting in formula (9) of the paper [1] instead of $F_j^{\pm}(x,\lambda)$ its representation (2.2), we will get

$$\begin{aligned}
&\textstyle\int_x^\infty K_j^{\pm}(x,t)e^{-i\lambda t}dt = \\
&\textstyle\int_x^\infty G_1(x,t,\lambda)r(t)t^j e^{-i\lambda t}dt - \int_x^\infty G(x,t,\lambda)(t)q_0(t)t^j e^{-i\lambda t}dt \\
&\textstyle-\lambda\int_x^\infty G(x,t,\lambda)(t)p(t)t^j e^{-i\lambda t}dt + \int_x^\infty G_1(x,t,\lambda)r(t)\left\{\int_t^\infty K_j^{\pm}(t,\xi)e^{\pm i\lambda\xi}d\xi\right\}dt \\
&\textstyle-\int_x^\infty G(x,t,\lambda)q_0(t)\left\{\int_t^\infty K_j^{\pm}(t,\xi)e^{-i\lambda\xi}d\xi\right\}dt \\
&\textstyle-\lambda\int_x^\infty G(x,t,\lambda)p(t)\left\{\int_t^\infty K_j^{\pm}(t,\xi)e^{-i\lambda\xi}d\xi\right\}dt,
\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}
&G(x,\xi,\lambda) = -\tfrac{1}{8i\lambda}\left[\left\{(x-\xi)^2+\tfrac{(x-\xi)}{\lambda}+\tfrac{1}{2\lambda^2}\right\}e^{-i\lambda(x-\xi)}-\tfrac{1}{2\lambda^2}e^{i\lambda(x-\xi)}\right], \\
&q_0(x) = q(x)-r'(x),\, G_1(x,\xi,\lambda) = \tfrac{\partial G(x,\xi,\lambda)}{\partial\xi}.
\end{aligned}$$

Let's transform the right part of formula (2.3) so that it was in the form of Fourier transformation of some function. As the function

$$\Phi(a,\lambda,\xi) = \left\{-\frac{1}{8}\left[\frac{a^2}{\lambda}+\frac{a}{\lambda^2}+\frac{1}{2\lambda^3}\right]e^{-ia\lambda}+\frac{1}{16\lambda^3}e^{ia\lambda}\right\}e^{i\lambda\xi},\ \text{where } a = s-x, \tag{2.4}$$

satisfies the conditions of the Paley-Wiener theorem [5], there exists a function such that

$$\left\{-\frac{1}{8}\left[\frac{a^2}{\lambda}+\frac{a}{\lambda^2}+\frac{1}{2\lambda^3}\right]e^{-ia\lambda}+\frac{1}{16\lambda^3}e^{ia\lambda}\right\}e^{i\lambda\xi} = \int_{\xi-a}^{\xi+a}\mathrm{P}(s)e^{i\lambda s}ds. \tag{2.5}$$

Hence

$$\mathrm{P}(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{-\frac{1}{8}\left[\frac{a^2}{\lambda}+\frac{a}{\lambda^2}+\frac{1}{2\lambda^3}\right]e^{-i\lambda a}+\frac{1}{16\lambda^3}e^{i\lambda a}\right\}e^{-i\lambda s}d\lambda. \tag{2.6}$$

It is possible to apply here a residual method [7] to the calculation of integrals in the sense of principal value. Depending on a sign of exponents indicator we choose an integration contour in the upper half-plane a symmetric with respect to real axis along the boundary of area $\pm Jm\lambda > 0, |\lambda| < R, |\lambda| > \rho$, and then passing to the limit as $R\to\infty$ and $\rho\to 0$, we get

$$\begin{aligned}
&\mathrm{P}(s) = \tfrac{1}{2\pi}\left\{\tfrac{\pi i}{8}\operatorname*{Res}_{\lambda=0}\left[\tfrac{a^2}{\lambda}+\tfrac{a}{\lambda^2}+\tfrac{1}{2\lambda^3}\right]e^{-i\lambda(a+s)}+\tfrac{\pi i}{16}\operatorname*{Res}_{\lambda=0}\left[\tfrac{1}{16\lambda^3}e^{i\lambda(a-s)}\right]\right\} = \\
&= \tfrac{1}{2\pi}\left\{\tfrac{\pi i}{8}(a^2-i(a+s)-\tfrac{1}{4}(a+s)^2)+\tfrac{\pi i}{32}(a-s)^2\right\} = \tfrac{i}{32}\left[a^2-2i(a+s)-s^2\right].
\end{aligned} \tag{2.7}$$

The direct calculation shows that

$$G(x,s,\lambda)e^{\pm i\lambda\xi} = \frac{1}{16}\int_{\xi-(s-x)}^{\xi+(s-x)}\left[(s-x)^2-2i(x-s+t-\xi)-(t-\xi)^2\right]e^{\pm i\lambda t}dt, \tag{2.8}$$

$$G_1(x,s,\lambda)e^{\pm i\lambda\xi} = \frac{1}{8}\int_{\xi-(s-x)}^{\xi+(s-x)}(s-x+i)e^{\pm i\lambda t}dt. \tag{2.9}$$

Then

$$\begin{gathered}\int_x^\infty G(x,s,\lambda)q_0(s)s^j e^{\pm is\lambda}ds = \\ \int_x^\infty s^j q_0(s)\left\{\int_x^{2s-x}\frac{\left[(s-x)^2-2i(s-x)-2i(t-s)-(t-s)^2\right]}{16}e^{\pm i\lambda t}dt\right\} = \\ \int_x^\infty s^j q_0(s)\left\{\int_x^{2s-x}\frac{\left[(s-x)^2+2i(x-t)-(t-s)^2\right]}{16}e^{\pm i\lambda t}dt\right\}ds.\end{gathered} \tag{2.10}$$

Whence after changing the order of integration, we find

$$\begin{gathered}\int_x^\infty G(x,s,\lambda)q_0(s)s^j e^{\pm i\lambda s}ds = \\ \int_x^\infty e^{\pm i\lambda s}\left\{\int_{\frac{x+t}{2}}^\infty s^j q_0(s)\frac{(s-x)^2-2i(x-t)-(t-s)^2}{16}ds\right\}dt.\end{gathered} \tag{2.11}$$

Again by using formula (2.8), we will receive:

$$\begin{gathered}\int_x^\infty G(x,s,\lambda)q_0(s)\left\{\int_s^\infty K^\pm(s,\xi)e^{\pm i\lambda\xi}d\xi\right\}ds = \\ \int_x^\infty q_0(s)\left\{\int_s^\infty K^\pm(s,\xi)d\xi\int_{\xi-(s-x)}^{\xi+(s-x)}\frac{(s-x)^2-2i(x-u)-(u-\xi)^2}{16}e^{\pm i\lambda u}du\right\}ds.\end{gathered} \tag{2.12}$$

Continuing the function $K_j^\pm(s,\xi)$ by zero at $\xi < s$, we find that for all $s \geq x$

$$\begin{gathered}\int_s^\infty K_j^\pm(s,\xi)\left\{\int_{\xi-(s-x)}^{\xi+(s-x)}\frac{(s-x)^2-2i(x-u)-(u-\xi)^2}{16}e^{\pm i\lambda u}du\right\}d\xi = \\ \int_{-\infty}^\infty K_j^\pm(s,\xi)\int_{\xi-(s-x)}^{\xi+(s-x)}\frac{(s-x)^2-2i(x-u)-(u-\xi)^2}{16}e^{\pm i\lambda u}du = \\ \int_{-\infty}^\infty e^{\pm i\lambda t}dt\int_{t-(s-x)}^{t+(s-x)}K_j^\pm(s,\xi)\frac{(s-x)^2-2i(x-t)-(t-\xi)^2}{16}d\xi = \\ \int_x^\infty e^{\pm i\lambda t}dt\int_{t-(s-x)}^{t+(s-x)}K_j^\pm(s,\xi)\frac{(s-x)^2-2i(x-t)-(t-\xi)^2}{16}d\xi.\end{gathered} \tag{2.13}$$

As for $t < x$ ($\xi \leq t+(s-x)$ and $t < x \Rightarrow \xi < s$)

$$\int_{t-(s-x)}^{t+(s-x)}K_j^\pm(s,\xi)\frac{(x-s)^2-2i(x-t)-(t-\xi)^2}{16}d\xi = 0, \tag{2.14}$$

then

$$\begin{gathered}\int_x^\infty G(x,s,\lambda)q_0(s)\left\{\int_s^\infty K_j^\pm(s,\xi)e^{\pm i\lambda\xi}d\xi\right\}ds = \\ \int_x^\infty e^{\pm i\lambda t}dt\left\{\int_x^\infty q_0(s)ds\int_{t-(s-x)}^{t+(s-x)}K_j^\pm(s,\xi)\frac{(s-x)^2-2i(x-t)-(t-\xi)^2}{16}d\xi\right\}.\end{gathered} \tag{2.15}$$

Considering (2.10) in a similar way we receive that

$$\begin{gathered}\int_x^\infty G_1(x,s,\lambda)r(s)F_j^\pm(s,\lambda)ds = \frac{1}{8}\int_x^\infty e^{\pm i\lambda t}dt\times \\ \left\{\int_{\frac{x+t}{2}}^\infty r(s)s^j(x-s+i)ds+\int_x^\infty r(s)(x-s+i)ds\int_{t+x-s}^{t-x+s}K_j^\pm(s,\xi)d\xi\right\}\end{gathered} \tag{2.16}$$

and

$$\begin{gathered}\int_x^\infty G(x,s,\lambda)p(s)F_j^\pm(s,\lambda)ds = \int_x^\infty e^{\pm i\lambda t}\left\{\int_{\frac{x+t}{2}}^\infty \xi^j p(\xi)\frac{(\xi-x)^2-2i(x-t)-(t-\xi)^2}{16}d\xi\right. \\ \left.+\int_x^\infty p(s)ds\int_{t-(s-x)}^{t+(s-x)}K_j^\pm(s,\xi)\frac{(s-x)^2-2i(x-t)-(t-\xi)^2}{16}d\xi\right\}.\end{gathered} \tag{2.17}$$

By partial integration of (2.17), we get

$$\begin{gathered}\int_x^\infty G(x,s,\lambda)p(s)F_j^\pm(s,\lambda)ds = \pm\frac{1}{\lambda i}\int_x^\infty e^{\pm i\lambda t}dt\times \\ \left\{\int_{\frac{x+t}{2}}^\infty \xi^j p(\xi)\frac{[t-\xi+i]}{8}d\xi+\int_0^{\frac{x+t}{2}}p(s)ds\int_{t-(s-x)}^{t+(s-x)}K_j^\pm(s,\xi)(t-\xi+i)d\xi+\right. \\ \left.\int_{\frac{x+t}{2}}^\infty p(s)\int_s^{t+(s-x)}K_j^\pm(s,\xi)(t-\xi+i)d\xi\right\}.\end{gathered} \tag{2.18}$$

From formulas (2.10), (2.16)-(2.18) it follows that equality (2.3) is fulfilled if $K_j^{\pm}(x,t)$satisfies the equation

$$\begin{aligned}
&K_j^{\pm}(x,t) = \\
&\pm\tfrac{1}{8i}\int_{\frac{x+t}{2}}^{\infty} s^j p(s)(s-t-i)ds \pm \tfrac{1}{8i}\int_x^{\infty} p(s)ds\int_{t-(s-x)}^{t+(s-x)} K_j^{\pm}(s,\xi)(t-\xi+i)d\xi \\
&-\tfrac{1}{16}\int_{\frac{x+t}{2}}^{\infty} s^j q_0(s)[(s-x)^2-2i(x-t)-(t-s)^2]ds \\
&-\tfrac{1}{16}\int_x^{\infty} q_0(s)ds\int_{t-(s-x)}^{t+(s-x)} K_j^{\pm}(s,\xi)[(s-x)^2-2i(x-t)-(t-\xi)^2]d\xi \\
&+\tfrac{1}{8}\int_{\frac{x+t}{2}}^{\infty} r(s)s^j(x-s+i)ds + \tfrac{1}{8}\int_x^{\infty} r(s)(x-s+i)ds\int_{t-(s-x)}^{t+(s-x)} K_j^{\pm}(s,\xi)d\xi
\end{aligned} \tag{2.19}$$

and condition $K_j^{\pm}(x,t)=0$ for $t<x$.

Now let's use the condition $K_j^{\pm}(x,t)=0$ at $t<x$. If $t-(s-x)>s$, i.e. $s<\frac{1}{2}(x+t)$, then $\xi>s$ and, therefore $K_j^{\pm}(x,t)\neq 0$. If $t-(s-x)<s$, i.e. $s>\frac{1}{2}(x+t)$, then ξ can be less than s and for these values ξ, $K(s,\xi)=0$. Therefore the integral equation (2.16) takes the of form:

$$\begin{aligned}
&K_j^{\pm}(x,t) = \pm\tfrac{1}{8i}\int_{\frac{x+t}{2}}^{\infty} s^j p(s)(s-t+i)ds- \\
&\tfrac{1}{16}\int_{\frac{x+t}{2}}^{\infty} s^j\left[q(s)-r'(s)\right]\left[(s-x)^2-2i(x-t)-(t-s)^2\right]ds+ \\
&\tfrac{1}{8}\int_{\frac{x+t}{2}}^{\infty} r(s)s^j(x-s+i)ds \pm \tfrac{1}{8i}\int_x^{\frac{x+t}{2}} p(s)ds\int_{t-(s-x)}^{t+(s-x)} K_j^{\pm}(s,\xi)(t-\xi+i)d\xi\pm \\
&\tfrac{1}{8i}\int_{\frac{x+t}{2}}^{\infty} p(s)ds\int_{t-(s-x)}^{t+(s-x)} K_j^{\pm}(s,\xi)(s+t-i)d\xi+ \\
&\tfrac{1}{8}\int_x^{\frac{x+t}{2}} r(s)(s-x+i)ds\int_{t-(s-x)}^{t+(s-x)} K_j^{\pm}(s,\xi)d\xi+ \\
&\tfrac{1}{8}\int_{\frac{x+t}{2}}^{\infty} r(s)(s-x+i)ds\int_s^{t+(s-x)} K_j^{\pm}(s,\xi)d\xi- \\
&\tfrac{1}{16}\int_x^{\frac{x+t}{2}}\left[q(s)-r(s)\right]ds\int_{t-(s-x)}^{t+(s-x)} K_j^{\pm}(s,\xi)\left[(s-x)^2-2i(x-t)-(\xi-t)^2\right]d\xi- \\
&\tfrac{1}{16}\int_x^{\frac{x+t}{2}}\left[q(s)-r'(s)\right]ds\int_s^{t+(s-x)} K_j^{\pm}(s,\xi)\left[(s-x)^2-2i(x-t)-(\xi-t)^2\right]d\xi.
\end{aligned} \tag{2.20}$$

The equation (2.20) can be solved by the of successive approximation method. To the end we assume

$$\begin{aligned}
&K_{j0}^{\pm}(x,t) = \pm\tfrac{1}{8i}\int_{\frac{x+t}{2}}^{\infty} s^j p(s)(s-t+i)ds- \\
&\tfrac{1}{16}\int_{\frac{x+t}{2}}^{\infty} s^j\left[q(s)-r'(s)\right]\left[(s-x)^2-2i(x-t)-(t-s)^2\right]ds+ \\
&\tfrac{1}{8}\int_{\frac{x+t}{2}}^{\infty} r(s)s^j(x-s+i)ds,
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
&K_{jn}^{\pm}(x,t) = \pm\tfrac{1}{8i}\int_x^{\frac{x+t}{2}} r(s)ds\int_{t-(s-x)}^{t+(s-x)} K_{jn-1}^{\pm}(s,t)(t-\xi+i)d\xi\pm \\
&\tfrac{1}{8i}\int_{\frac{x+t}{2}}^{\infty} p(s)ds\int_s^{t+(s-x)} K_{jn-1}^{\pm}(s,\xi)(s-t+i)d\xi+ \\
&\tfrac{1}{8}\int_x^{\frac{x+t}{2}} r(s)(s-x+i)ds\int_{t-(s-x)}^{t+(s-x)} K_{jn-1}^{\pm}(s,\xi)d\xi+ \\
&\tfrac{1}{8}\int_{\frac{x+t}{2}}^{\infty} r(s)(s-x+i)ds\int_s^{t+(s-x)} K_{jn-1}^{\pm}(s,\xi)d\xi- \\
&\tfrac{1}{16}\int_x^{\frac{x+t}{2}}\left[q(s)-r'(s)\right]ds\int_{t-(s-x)}^{t+(s-x)} K_{jn-1}^{\pm}(s,\xi)\left[(s-x)^2-2i(x-t)-(\xi-t)^2\right]d\xi- \\
&\tfrac{1}{16}\int_x^{\frac{x+t}{2}}\left[q(s)-r'(s)\right]ds\int_s^{t+(s-x)} K_{jn-1}^{\pm}(s,\xi)\left[(s-x)^2-2i(x-t)-(\xi-t)^2\right]d\xi, \\
&n=1,2,\ldots.
\end{aligned}$$

The following estimates hold:

$$\left|K_{j0}^{\pm}(x,t)\right| \leq \frac{1}{8}\int_{\frac{x+t}{2}}^{\infty} s^{j+1}\left\{|p(s)|+|r(s)|+1+s\left[|r'(s)|+|q(s)|\right]\right\}ds = \frac{1}{8}\sigma_j\left(\frac{x+t}{2}\right), \tag{2.22}$$

where

$$\sigma_j\left(\frac{x+t}{2}\right) = \int_{\frac{x+t}{2}}^{\infty} s^{j+1}\left\{|p(s)|+|r(s)|+1+s\left[\left|r'(s)\right|+|q(s)|\right]\right\}ds.$$

It is easy to find the value

$$K_{jn}^{\pm}(x,t) \leq \frac{1}{8}\sigma_j\left(\frac{x+t}{2}\right)\left(\frac{1}{2}\right)^n \frac{\tau^n(x)}{n!},\ j=\overline{0,1,2},\ n=1,2,3,..., \tag{2.23}$$

where

$$\tau(x) = \int_x^{\infty} s^2\left\{1+|p(s)|+|r(s)|+s\left[|q(s)|+\left|r'(s)\right|+1\right]\right\}ds.$$

From this estimation and from assumptions (2.17) it follows that the series

$$\sum_{n=0}^{\infty} K_{jn}^{\pm}(x,t),\ \ j=0,1,2,$$

converges absolutely and uniformly in the area $0 \leq x < t < \infty$ and for its sum $K_j^{\pm}(x,t)$ the following equality is valid

$$\left|K_j^{\pm}(x,t)\right| \leq \frac{1}{8}\sigma_j\left(\frac{x+t}{2}\right)e^{\tau(x)},\ \ j=0,1,2. \tag{2.24}$$

Thus we proved the following theorem.

Theorem 2.1. *Let Ω_j, $j=0,1,2$, be a set of the functions $A_j^{\pm}(x,t)$ defined at $t \geq x \geq 0$ and having continuous derivatives to the 3rd order on both variables such that $\left|\frac{\partial^{\alpha+\beta}}{\partial x^{\alpha}\partial t^{\beta}}A_j^{\pm}(x,t)\right| \leq \sigma_j^0\left(\frac{x+t}{2}\right), \alpha+\beta \leq 3, j=0,1,2,$ where $\sigma_j^0(x), j=0,1,2,$ are non-increasing positive functions defined for all $x \in [0,\infty)$ and integrable on this interval, the functions $r(x)$, $p(x)$, $q(x)$ satisfy conditions* (1.2)*. Then the solution of equation* (1.1) *satisfying conditions* (2.1) *is presented in the form* (2.2) *if and if only $K_j^{\pm}(x,t)$, $j=0,1,2,$ in the class Ω_j is the solution of the integral equation* (2.20).

Considering that the area of integration of double integrals in the right part of (2.20) consists of two parts in one of which $K_j^{\pm}(s,\xi)=0$, i.e. double integrals are actually are taken in area where $K_j^{\pm}(s,\xi) \neq 0$, with the help of replacement $\xi+s=2\alpha, \xi-s=2\beta$ from (2.20) we will receive:

$$\begin{aligned}
&K_j^{\pm}(x,t) = \pm\tfrac{1}{8i}\int_{\frac{x+t}{2}}^{\infty}\xi^j P(\xi)(\xi-t+i)d\xi - \\
&\tfrac{1}{16}\int_{\frac{x+t}{2}}^{\infty}\xi^j\left[q(\xi)-r'(\xi)\right]\left[(\xi-x)^2-2i(x-t)-(t-\xi)^2\right]d\xi + \\
&\tfrac{1}{8}\int_{\frac{x+t}{2}}^{\infty}\xi^j r(\xi)(x-\xi+i)d\xi \pm \\
&\tfrac{1}{4i}\int_{\frac{x+t}{2}}^{\infty}d\alpha\int_0^{\frac{t-x}{2}} p(\alpha-\beta)(t-\alpha-\beta+i)K_j^{\pm}(\alpha-\beta,\alpha+\beta)d\beta + \\
&\tfrac{1}{4}\int_{\frac{x+t}{2}}^{\infty}d\alpha\int_0^{\frac{t-x}{2}} r(\alpha-\beta)(\alpha-\beta-x+i)K_j^{\pm}(\alpha-\beta,\alpha+\beta)d\beta - \\
&\tfrac{1}{8}\int_{\frac{x+t}{2}}^{\infty}d\alpha\int_0^{\frac{t-x}{2}}\left[q(\alpha-\beta)-r'(\alpha-\beta)\right]K_j^{\pm}(\alpha-\beta,\alpha+\beta)\times \\
&\left[(\alpha-\beta-x)^2-2i(x-t)-(\alpha+\beta-t)^2\right]d\beta.
\end{aligned} \tag{2.25}$$

For finding the derivatives $K_j^{\pm}(x,t)$ it is convenient to pass to a new function $H_j^{\pm}(\alpha,\beta) = K_j^{\pm}(\alpha-\beta,\alpha+\beta)$. Assuming $x+t=2u$, $t-x=2v$, we can rewrite equation (2.25) as:

$$\begin{array}{l} H_j^{\pm}(u,v) = \pm\frac{1}{8i}\int_u^{\infty}\xi^j p(\xi)(\xi-u+v)d\xi + \frac{1}{8}\int_u^{\infty}\xi^j r(\xi)(uv-\xi+i)d\xi- \\ \frac{1}{16}\int_u^{\infty}\xi^j\left[q(\xi)-r'(\xi)\right]\left[(\xi-u+v)^2+4iv-(u+v-\xi)^2 d\xi\right]\pm \\ \frac{1}{4i}\int_u^{\infty}d\alpha\int_0^v p(\alpha-\beta)H_j^{\pm}(\alpha,\beta)(u+v-\alpha-\beta+i)d\beta+ \\ \frac{1}{4}\int_u^{\infty}d\alpha\int_0^v r(\alpha-\beta)(\alpha-\beta-u+v+i)H_j^{\pm}(\alpha,\beta)d\beta- \\ \frac{1}{8}\int_0^{\infty}d\alpha\int_0^v\left[q(\alpha-\beta)-r'(\alpha-\beta)\right]H_j^{\pm}(\alpha,\beta)\times \\ \left[(\alpha-\beta-u+v)^2+4iv-(\alpha+\beta-u-v)^2\right]d\beta. \end{array} \tag{2.26}$$

It is easy to receive the following theorem by using the same methodology as in [3], by taking into account $x=u-v$, $t=u+v$, $H_j^{\pm}(u,v)=K_j^{\pm}(x,t)$ and by applying the methods of the paper [6] using direct calculations of derivatives.

Theorem 2.2. *If functions* $p(x), r(x)$ and $q(x)$ *satisfy assumptions* (1.2), *then* $K_j^{\pm}(x,t)$ *has all derivatives to fourth order inclusively, and the integral equation* (2.20) *is equivalent to the problem*

$$\begin{array}{l} l\left(x,\frac{\partial}{\partial x},\pm i\frac{\partial}{\partial t}\right)K_j^{\pm}(x,t)=0, \\ \lim\limits_{x+t\to\infty}\frac{\partial^{\alpha+\beta}}{\partial x^{\alpha}\partial t^{\beta}}K_j^{\pm}(x,t)=0,\ \alpha+\beta\le 4, \\ \int_x^{\infty}\left|K_j^{\pm}(x,t)\right|^2 dx<\infty. \end{array} \tag{2.27}$$

The functions $K_j^{\pm}(x,t)$ *and their derivatives satisfy Goursat type conditions on the characteristic* $t=x$.

Analyticity of the function $F_j^{+}(x,\lambda)$, $j=\overline{0,3}$, in open upper half-plane of the spectral parameter λ directly follows from representation (9) of the paper [1] and estimates (2.24) and these functions are continuous up to a real axis. For these functions the following estimates hold:

$$\begin{array}{l} \left|F_j^{\pm}(x,\lambda)\right| = \left|x^j e^{i\lambda x}+\int_x^{\infty}K_j^{\pm}(x,t)e^{i\lambda t}dt\right|\le x^j e^{-xJm\lambda}+ \\ \int_x^{\infty}\left|K_j^{\pm}(x,t)\right|e^{-tJm\lambda}dt = e^{-xJm\lambda}\left[x^j+\int_x^{\infty}\left|K_j^{+}(x,t)\right|e^{-(t-x)Jm\lambda}dt\right]\le \\ e^{-xJm\lambda}\left[x^j+\int_x^{\infty}\left|K_j^{+}(x,t)\right|dt\right]. \end{array}$$

Hence, using (2.24), we receive:

$$\left|F_j^{\pm}(x,\lambda)\right|\le e^{-xJm\lambda}\left[x^j+\frac{1}{4}e^{\tau(x)}\int_x^{\infty}\sigma_j\left(\frac{x+t}{2}\right)dt\right]=Q_j(x)e^{-xJm\lambda}, \tag{2.28}$$

where

$$Q_j(x)=x^j+\frac{1}{4}e^{\tau(x)}\int_x^{\infty}\sigma_j(s)ds,\ j=0,1,2.$$

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