Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 40, Special Issue, 2014, Pages 359–374

ON THE SPECTRUM OF THE PENCIL OF SECOND ORDER DIFFERENTIAL OPERATORS WITH PERIODIC COEFFICIENTS ON THE SEMI-AXIS

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In memory of M. G. Gasymov on his 75th birthday

Abstract. In this paper, the spectrum and resolvent of the operator L_λ generated by the differential expression $\ell_\lambda(y) = y'' + p(x)y' + (\lambda^2 + i\lambda p(x) + q(x))y$ and the initial condition $y(0) = 0$ is investigated in the space $L_2(0, +\infty)$. Here the coefficients $p(x)$, $q(x)$ are periodic functions whose Fourier series are absolutely convergent and Fourier exponents are positive. It is shown that continuous spectrum of the operator L_λ consists of the interval $(-\infty, +\infty)$. Moreover, at most a countable set of spectral singularities can exists over the continuous spectrum and at most a countable set of eigenvalues can be located outside of the interval $(-\infty, +\infty)$. Eigenvalues and spectral singularities with sufficiently large modulus are simple and lie near the points $\lambda = \pm\frac{n}{2}$, $n \in \mathbb{N}$.

1. Introduction

In this study, the spectrum and resolvent of the maximal differential operator L_λ generated by the linear differential expression

$$\ell_\lambda(y) = y'' + p(x)y' + (\lambda^2 + \lambda i p(x) + q(x))y$$

and the boundary condition $y(0) = 0$ have been investigated in the space $L_2(0, +\infty)$. Here λ is a complex parameter,

$$p(x) = \sum_{n=1}^{\infty} p_n e^{inx},\ q(x) = \sum_{n=1}^{\infty} q_n e^{inx} \tag{1}$$

with complex coefficients p_n, q_n for which

$$\sum_{n=1}^{\infty} n \mid p_n \mid< +\infty,\ \sum_{n=1}^{\infty} \mid q_n \mid< +\infty. \tag{2}$$

2010 *Mathematics Subject Classification.* 34L05, 47E05.
Key words and phrases. Floquet solutions, spectrum, resolvent, spectral singularity, eigenvalue.

The domain of the operator L_λ is

$$\begin{aligned} D(L_\lambda) &= \{y(x)|y(x), y'(x) \in AC([0,R]) \text{ for all } R>0, \\ &\quad y(0)=0,\ y(x), \ell_\lambda(y) \in L_2(0,+\infty)\}. \end{aligned}$$

Let Q be the class of periodic functions $q(x) = \sum_{n=1}^{\infty} q_n e^{inx}$ with $\|q(x)\| = \sum_{n=1}^{\infty} |q_n| < +\infty$. Then Q is a complex normed space and $p(x)$, $q(x)$, $p'(x) \in Q$. It is clear that if at least one of the functions $p(x)$ and $q(x)$ is not zero, then the operator L_λ is nonselfadjoint for each $\lambda \in \mathbb{C}$.

In the study [4], the Floquet solutions of equation $\ell_\lambda(y) = 0$ in the case $p(x) \equiv 0$ have been constructed and using these solutions direct and inverse spectral problems have been investigated for the operator $L = -\frac{d^2}{dx^2} + q(x)$ in the space $L_2(\mathbb{R})$. Later using some different methods, the inverse problem for the operator $L = -\frac{d^2}{dx^2} + q(x)$ with periodic potential $q(x) \in L_2(0, 2\pi)$ was investigated in [13], the spectrum and resolvent operator was studied in [14]. Some results of [4] were generalized for the $2n$ order linear differential operators with almost periodic coefficients in [5], [7]. The spectrum and resolvent of a pencil of high order differential operators with periodic and almost periodic coefficients were investigated in [8] and [12], [9] respectively. The inverse problem for a pencil of $2n$ order differential operators with periodic coefficients from the class Q was studied in [3]. The pencil of the second order differential operators with periodic coefficients has been investigated in [2], [10]. Afterwords, the spectrum and resolvent for the pencil of the second order differential operators with almost periodic coefficients under more general conditions on the coefficients was investigated in [11].

In the present study the operator L_λ is investigated in the space $L_2(0, +\infty)$. It is proved that the continuous spectrum of the operator pencil L_λ consists of the interval $(-\infty, +\infty)$. There may be at most a countable set of spectral singularities on the continuous spectrum. Moreover, there may be a countable set $\sigma_p(L_\lambda)$ of eigenvalues outside the interval $(-\infty, +\infty)$. Singular values $\lambda_n^{\pm}$ (eigenvalues or spectral singularities) with sufficiently large modulus are simple, lie in the neighborhood of points $\pm\frac{n}{2}$, $n \in \mathbb{N}$, and satisfy the asymptotic formula

$$\lambda_n^{\pm} = \pm\frac{n}{2} + O(\frac{1}{n}),\ n \to \infty.$$

2. Floquet solutions of the equation $\ell_\lambda(y) = 0$

The system of the linear independent solutions of an equation of type $\ell_\lambda(y) = 0$ with almost periodic coefficient was investigated in [11]. According to Theorem 1 in the study [11], we can formulate the following theorem related with the equation

$$y'' + p(x)y' + [\lambda^2 + i\lambda p(x) + q(x)]y = 0, \quad -\infty < x < +\infty. \tag{3}$$

Theorem 1. *If the functions $p(x)$ and $q(x)$ satisfy the conditions (1) and (2), then for $\forall\lambda \neq \pm\frac{n}{2}$, $n \in \mathbb{N}$, the differential equation (3) has the solutions*

$$f_1(x,\lambda)=e^{i\lambda x}\left(1+\sum_{n=1}^{\infty}U_n^{(1)}(\lambda)e^{inx}\right),\ f_2(x,\lambda)=e^{-i\lambda x}\left(1+\sum_{n=1}^{\infty}U_n^{(2)}(\lambda)e^{inx}\right), \tag{4}$$

where the series

$$\sum_{n=1}^{\infty}\left|U_n^{(s)}(\lambda)\right|n^2,\ s=1,2,$$

is uniform convergent in each compact set $S\subseteq\mathbb{C}$ *which doesn't contain the numbers* $\lambda=-\frac{n}{2}$, $n\in N$, *in case* $s=1$ *and* $\lambda=\frac{n}{2}$, $n\in\mathbb{N}$, *in case* $s=2$. *Here* $U_n^{(1)}(\lambda)=U_{0n}^{(1)}+\sum_{k=1}^{n}\frac{U_{kn}^{(1)}}{k+2\lambda}$, $U_n^{(2)}(\lambda)=U_{0n}^{(2)}+\sum_{k=1}^{n}\frac{U_{kn}^{(2)}}{k-2\lambda}$, $n\in\mathbb{N}$.

The solutions $f_1(x,\lambda)$ and $f_2(x,\lambda)$ can be used for the investigation of the structure of the spectrum and the kernel of the resolvent operator, but they are not sufficient for studying the asymptotics of the singular values of the operator L_λ. For this reason it is convenient to use the Floquet solutions of the form

$$\begin{cases} f_1(x,\lambda)=e^{i\lambda x}\left(1+\sum\limits_{n=1}^{\infty}U_{0n}^{(1)}e^{inx}+\sum_{k=1}^{\infty}\frac{1}{k+2\lambda}\sum\limits_{n=k}^{\infty}U_{kn}^{(1)}e^{inx}\right), \\ f_2(x,\lambda)=e^{-i\lambda x}\left(1+\sum\limits_{n=1}^{\infty}U_{0n}^{(2)}e^{inx}+\sum_{k=1}^{\infty}\frac{1}{k-2\lambda}\sum\limits_{n=k}^{\infty}U_{kn}^{(2)}e^{inx}\right) \end{cases} \tag{5}$$

with conditions $\sum\limits_{n=1}^{\infty}n^2\left|U_{0n}^{(s)}\right|<+\infty$, $\sum\limits_{k=1}^{\infty}\frac{1}{k}\sum\limits_{n=k}^{\infty}n^2\left|U_{kn}^{(s)}\right|<+\infty$, $s=1,2$. It is clear that these representations of the solutions are a modified form of formulas (4).

The special solutions of type (5) are used in [2], [10] under various conditions on the coefficients of the considered equations. We use the following theorem about existence of the Floquet solutions of the equation (3).

Theorem 2. *If* $p(x)$, $q(x)$, $p'(x)\in Q$, *then for each* $\lambda\neq-\frac{n}{2}$, $n\in\mathbb{N}$, *the differential equation (3) has solutions as*

$$f(x,\lambda)=e^{i\lambda x}\left(1+\sum_{n=1}^{\infty}u_ne^{inx}+\sum_{k=1}^{\infty}\frac{1}{k+2\lambda}\sum_{n=k}^{\infty}u_{kn}e^{inx}\right), \tag{6}$$

where the sequence $\{u_n\},\{u_{kn}\}$ *of complex numbers uniquely determined from the system of equations*

$$-n^2u_n-n\sum_{k=1}^{n}u_{kn}+q_n+\sum_{m=1}^{n-1}(imp_{n-m}+q_{n-m})u_m+\sum_{m=1}^{n-1}ip_{n-m}\sum_{k=1}^{m}u_{km}=0, \tag{7.1}$$

$$-nu_n+ip_n+\sum_{m=1}^{n-1}ip_{n-m}u_m=0,\ n\in\mathbb{N}, \tag{7.2}$$

$$-n(n-k)u_{kn}+\sum_{m=k}^{n-1}[i(m-k)p_{n-m}+q_{n-m}]u_{km}=0,\ k,n\in\mathbb{N},\ n\geq k+1 \tag{7.3}$$

and the series

$$\sum_{n=1}^{\infty}n^2\left|u_n\right|<+\infty,\ \sum_{k=1}^{\infty}\frac{1}{k}\sum_{n=k}^{\infty}n^2\left|u_{kn}\right|<+\infty \tag{8}$$

converge.

Remark. In what follows we suppose the sum $\sum\limits_{m=1}^{n-1} a_m$ to be equal to zero for $n = 1$.

Proof. If we assume the existence of the solution of equation (3) of the form (6), according to (7) we can find the derivatives of $f(x,\lambda)$ with respect to x as follows

$$f'(x,\lambda) = e^{i\lambda x}\left(i\lambda + \sum_{n=1}^{\infty} i(\lambda+n)\left(u_n + \sum_{k=1}^{n}\frac{u_{kn}}{k+2\lambda}\right)e^{inx}\right),$$

$$f''(x,\lambda) = e^{i\lambda x}\left(-\lambda^2 - \sum_{n=1}^{\infty}(\lambda+n)^2\left(u_n + \sum_{k=1}^{n}\frac{u_{kn}}{k+2\lambda}\right)e^{inx}\right).$$

If we substitute these expressions in the equation (3) and divide both sides by $e^{i\lambda x}$, according to uniqueness properties of the Fourier series we have the following system of equations for the sequences $\{u_n\}$, $\{u_{kn}\}$:

$$-\left(u_n + \sum_{k=1}^{n}\frac{u_{kn}}{k+2\lambda}\right)n(n+2\lambda) + 2i\lambda p_n + q_n +$$

$$\sum_{s+m=n}\left[ip_s(2\lambda+m)+q_s\right]\left(u_m + \sum_{k=1}^{m}\frac{u_{km}}{k+2\lambda}\right) = 0,\ n\in\mathbb{N}.$$

From this system we get the system of equations

$$\begin{cases} -n^2u_n - n\sum\limits_{k=1}^{n}u_{kn} + q_n + \sum\limits_{s+m=n}(imp_s+q_s)u_m + \sum\limits_{s+m=n}ip_s\sum\limits_{k=1}^{m}u_{km} = 0, \\ -nu_n + ip_n + \sum\limits_{s+m=n}ip_su_m = 0,\ n\in\mathbb{N}, \\ -n(n-k)u_{kn} + \sum\limits_{\substack{s+m=n\\ m\geq k}}[i(m-k)p_s+q_s]u_{km} = 0, \\ k,n\in\mathbb{N},\ n\geq k+1, \end{cases}$$

to determine $\{u_n\}$, $\{u_{kn}\}$. The last system of equations can be rewritten as (7.1)-(7.3). On the contrary if $\{u_n\}$, $\{u_{kn}\}$ satisfy the system of equations (7.1)-(7.3) and series (8) converge, then the function $f(x,\lambda)$ determined by (6) is a solution of (3). Therefore to prove the theorem it is sufficient to show the solvability of the system (7.1)-(7.3) and convergent of series (8). It is easy to see that the system of equations (7.1)-(7.3) has a unique solution. Indeed, from the equation (7.2) the sequence $\{u_n\}$ is determined by the recurrent manner uniquely. Furthermore by the known sequence $\{u_n\}$ from equations (7.1), (7.3) the sequence $\{u_{kn}\}$ also is determined by the recurrent manner uniquely.

Now let us show that for the solution $\{u_n\}$, $\{u_{kn}\}$ of the system (7.1)-(7.3) the series $\sum\limits_{n=1}^{\infty}|u_n|\,n^2$ and $\sum\limits_{k=1}^{\infty}\frac{1}{k}\sum\limits_{n=k}^{\infty}|u_{kn}|\,n^2$ converge, therefore the function $f(x,\lambda)$ is a solution of equation (3) for $\forall\lambda\in\mathbb{C}$, $\lambda\neq-\frac{n}{2}$, $n\in\mathbb{N}$. For this reason from equation (7.2) we have $n^2|u_n| \leq n|p_n| + n\sum\limits_{m=1}^{n-1}|p_{n-m}|\,|u_m|$ and by summing for $n=1,2,...,j$ it is found that

$$\sum_{n=1}^{j}n^2|u_n| \leq \sum_{n=1}^{j}n|p_n| + \sum_{n=2}^{j}\sum_{m=1}^{n-1}n|p_{n-m}|\,|u_m| \leq \sum_{n=1}^{j}n|p_n| +$$

$$\sum_{n=2}^{j}\sum_{m=1}^{n-1}(n-m)\left|p_{n-m}\right|m\left|u_m\right|\frac{n}{(n-m)m}\leq$$

$$\sum_{n=1}^{j}n\left|p_n\right|+2\sum_{n=1}^{j-1}n\left|p_n\right|\sum_{m=1}^{j-1}m\left|u_m\right|\leq\left\|p'(x)\right\|+2\left\|p'(x)\right\|\sum_{n=1}^{j-1}n\left|u_n\right|$$

or

$$\sum_{n=1}^{j}n^2\left|u_n\right|\leq\left\|p'(x)\right\|+2\left\|p'(x)\right\|\sum_{n=1}^{j-1}n\left|u_n\right|,\forall j\in\mathbb{N}.$$

Here we use the following lemma.

Lemma. *Let* $0<a_1<a_2<...a_n<...$, $\lim\limits_{n\to\infty}a_n=+\infty$ *and* $b_n\geq 0$, $n\in\mathbb{N}$. *If there exist* $n_0\in\mathbb{N}$ *and constants* $C_0>0$, $C_1>0$ *such that the inequality*

$$\sum_{k=1}^{n}a_kb_k\leq C_0+C_1\sum_{k=1}^{n-1}b_k$$

holds for any $n\geq n_0$, *then the series* $\sum\limits_{k=1}^{\infty}a_kb_k$ *converges.*

By this Lemma if $a_n=n$, $b_n=n\left|u_n\right|$, $C_0=\|p'(x)\|$, $C_1=2\left\|p'(x)\right\|$, then the series $\sum\limits_{n=1}^{\infty}n^2\left|u_n\right|$ converges.

Now let us show that if the series $\sum\limits_{n=1}^{\infty}n^2\left|u_n\right|$ converges, then for the sequence $\{u_{kn}\}$ obtained from the system (7.1)-(7.3) the series $\sum\limits_{n=1}^{\infty}\frac{1}{k}\sum\limits_{n=k}^{\infty}\left|u_{kn}\right|n^2$ also converges.

Since the series $\sum\limits_{n=1}^{\infty}\left|u_n\right|n^2$ converges, by setting $U_n=\sum_{k=1}^{n}u_{kn}$ from the equation (7.1) we have

$$n\sum_{k=1}^{n}u_{kn}=-n^2u_n+q_n+\sum_{m=1}^{n-1}(imp_{n-m}+q_{n-m})u_m+\sum_{m=1}^{n-1}ip_{n-m}\sum_{k=1}^{m}u_{km},$$

$$nU_n=-n^2u_n+q_n+\sum_{m=1}^{n-1}(imp_{n-m}+q_{n-m})u_m+\sum_{m=1}^{n-1}ip_{n-m}U_m$$

which implies

$$n\left|U_n\right|\leq n^2\left|u_n\right|+\left|q_n\right|+\sum_{m=1}^{n-1}\left(m\left|p_{n-m}\right|+\left|q_{n-m}\right|\right)\left|u_m\right|+\sum_{m=1}^{n-1}\left|p_{n-m}\right|\left|U_m\right|\leq$$

$$n^2\left|u_n\right|+\left|q_n\right|+\sum_{m=1}^{n-1}\left(\left|p_{n-m}\right|+\left|q_{n-m}\right|\right)m\left|u_m\right|+\sum_{m=1}^{n-1}\left|p_{n-m}\right|\left|U_m\right|.$$

By summing with respect to n, we obtain

$$\sum_{n=1}^{j}n\left|U_n\right|\leq\sum_{n=1}^{j}(n^2\left|u_n\right|+\left|q_n\right|)+\sum_{n=2}^{j}\sum_{m=1}^{n-1}\left(\left|p_{n-m}\right|+\left|q_{n-m}\right|\right)m\left|u_m\right|+$$

$$\sum_{n=2}^{j}\sum_{m=1}^{n-1}|p_{n-m}|\,|U_m| \leq \sum_{n=1}^{j}(n^2|u_n|+|q_n|)+$$
$$\sum_{n=1}^{j-1}(|p_n|+|q_n|)\sum_{m=1}^{j-1}m\,|u_m|+\sum_{n=1}^{j-1}|p_n|\sum_{m=1}^{j-1}|U_m|$$

or

$$\sum_{n=1}^{j}n\,|U_n| \leq \sum_{n=1}^{\infty}(n^2|u_n|+|q_n|)+\sum_{n=1}^{\infty}(|p_n|+|q_n|)\sum_{m=1}^{\infty}m\,|u_m|+$$
$$\sum_{n=1}^{\infty}|p_n|\sum_{m=1}^{j-1}|U_m|,\ j\in\mathbb{N}.$$

If we set

$$C_0'=\sum_{n=1}^{\infty}(n^2|u_n|+|q_n|)+(\|p(x)\|+\|q(x)\|)\sum_{m=1}^{\infty}m\,|u_m|,\ C_1'=\|p(x)\|,$$

then it is obtained

$$\sum_{n=1}^{j}n\,|U_n| \leq C_0'+C_1'\sum_{n=1}^{j-1}|U_n|,\ \forall j\in\mathbb{N}.$$

Then according to the Lemma the series $\sum\limits_{n=1}^{\infty}n\,|U_n|=\sum\limits_{n=1}^{\infty}n\left|\sum\limits_{k=1}^{n}u_{kn}\right|$ converges. On other hand, from the equation $\sum\limits_{k=1}^{n}u_{kn}=U_n$ we have

$$u_{nn}=U_n-\sum_{k=1}^{n-1}u_{kn}\text{ and }|u_{nn}|\leq|U_n|+\sum_{k=1}^{n-1}|u_{kn}|,\ n\in\mathbb{N}.$$

Considering this in the equation (7.3), we get

$$n(n-k)\,|u_{kn}|\leq\sum_{m=k}^{n-1}((m-k)\,|p_{n-m}|+|q_{n-m}|)\,|u_{km}|\Rightarrow$$
$$n\sum_{k=1}^{n-1}(n-k)\,|u_{kn}|\leq\sum_{k=1}^{n-1}\sum_{m=k}^{n-1}((m-k)\,|p_{n-m}|+|q_{n-m}|)\,|u_{km}|=$$
$$\sum_{m=1}^{n-1}\sum_{k=1}^{m}((m-k)\,|p_{n-m}|+|q_{n-m}|)\,|u_{km}|=$$
$$\sum_{m=1}^{n-1}\sum_{k=1}^{m-1}((m-k)\,|p_{n-m}|+|q_{n-m}|)\,|u_{km}|+\sum_{m=1}^{n-1}|q_{n-m}|\,|u_{mm}|\leq$$
$$\sum_{m=1}^{n-1}\sum_{k=1}^{m-1}(m-k)\,(|p_{n-m}|+|q_{n-m}|)\,|u_{km}|+\sum_{m=1}^{n-1}|q_{n-m}|\left(|U_m|+\sum_{k=1}^{m-1}|u_{km}|\right)$$

or

$$n\sum_{k=1}^{n-1}(n-k)\,|u_{kn}|\leq\sum_{m=1}^{n-1}(|p_{n-m}|+2\,|q_{n-m}|)\sum_{k=1}^{m-1}(m-k)\,|u_{km}|+\sum_{m=1}^{n-1}|q_{n-m}|\,|U_m|).$$

If we set $V_1 = 0$, $V_n = \sum\limits_{k=1}^{n-1} (n-k)\,|u_{kn}|$ by summing the last inequality term by term, we have

$$\sum_{n=2}^{j} nV_n \le \sum_{n=2}^{j}\sum_{m=1}^{n-1}(|p_{n-m}| + 2\,|q_{n-m}|)V_m + \sum_{n=2}^{j}\sum_{m=1}^{n-1}|q_{n-m}|\,|U_m| \le$$

$$\sum_{n=1}^{j-1}(|p_n| + 2\,|q_n|)\sum_{m=2}^{j-1} V_m + \sum_{n=1}^{j-1}|q_n|\sum_{m=1}^{j-1}|U_m| \le$$

$$\sum_{n=1}^{\infty}|q_n|\sum_{m=1}^{\infty}|U_m| + \sum_{n=1}^{\infty}(|p_n| + 2\,|q_n|)\sum_{m=1}^{j-1} V_m,\ \forall j \ge 2.$$

As a result, we prove the inequality

$$\sum_{n=1}^{j} nV_n \le A_0 + A_1\sum_{n=1}^{j-1} V_n,\ \forall j \ge 2,$$

where $A_0 = \|q(x)\| \sum\limits_{m=1}^{\infty} |U_m|$, $A_1 = \|p(x)\| + 2\,\|q(x)\|$. From here according to the Lemma we get that the series $\sum\limits_{n=2}^{\infty} nV_n$ converges, consequently the series $\sum\limits_{n=2}^{\infty}\sum\limits_{k=1}^{n-1} n(n-k)\,|u_{kn}|$ also converges. Therefore because of the inequality $\sum\limits_{n=2}^{\infty}\sum\limits_{k=1}^{n-1} n\,|u_{kn}| < \sum\limits_{n=2}^{\infty}\sum\limits_{k=1}^{n-1} n(n-k)\,|u_{kn}|$ the series $\sum\limits_{n=2}^{\infty}\sum\limits_{k=1}^{n-1} n\,|u_{kn}|$ also converges. On the other hand, taking into our account the inequality

$$|u_{nn}| \le |U_n| + \sum_{k=1}^{n-1}|u_{kn}|,\ n \in \mathbb{N},$$

we have

$$|u_{nn}| \le |U_n| + \sum_{k=1}^{n-1}|u_{kn}| \Rightarrow n\,|u_{nn}| \le n\,|U_n| + n\sum_{k=1}^{n-1}|u_{kn}| \Rightarrow$$

$$\sum_{n=1}^{\infty} n\,|u_{nn}| \le \sum_{n=1}^{\infty} n\,|U_n| + \sum_{n=2}^{\infty} n\sum_{k=1}^{n-1}|u_{kn}|,$$

consequently the series $\sum\limits_{n=1}^{\infty} n\,|u_{nn}|$ and $\sum\limits_{n=1}^{\infty} n\sum\limits_{k=1}^{n}|u_{kn}|$ also converge. Now show that the series

$$\sum_{n=2}^{\infty} n^2\sum_{k=1}^{n-1}\frac{|u_{kn}|}{k}$$

converges. For this reason from equation (7.3) we can write

$$n^2 u_{kn} = knu_{kn} + \sum_{m=k}^{n-1}[i(m-k)p_{n-m} + q_{n-m}]u_{km} \Rightarrow n^2\,|u_{kn}| \le$$

$$kn\left|u_{kn}\right|+\sum_{m=k}^{n-1}\left[(m-k)\left|p_{n-m}\right|+\left|q_{n-m}\right|\right]\left|u_{km}\right|\Rightarrow\sum_{k=1}^{n-1}n^2\frac{\left|u_{kn}\right|}{k}\leq$$

$$n\sum_{k=1}^{n-1}\left|u_{kn}\right|+\sum_{k=1}^{n-1}\sum_{m=k}^{n-1}\left[(m-k)\left|p_{n-m}\right|+\left|q_{n-m}\right|\right]\frac{\left|u_{km}\right|}{k}\leq n\sum_{k=1}^{n-1}\left|u_{kn}\right|+$$

$$\sum_{m=1}^{n-1}\sum_{k=1}^{m-1}m\left|p_{n-m}\right|\frac{\left|u_{km}\right|}{k}+\sum_{m=1}^{n-1}\sum_{k=1}^{m}\left|q_{n-m}\right|\left|u_{km}\right|\Rightarrow\sum_{k=1}^{n-1}n^2\frac{\left|u_{kn}\right|}{k}\leq$$

$$n\sum_{k=1}^{n-1}\left|u_{kn}\right|+\sum_{m=1}^{n-1}\left|p_{n-m}\right|\sum_{k=1}^{m-1}m\frac{\left|u_{km}\right|}{k}+\sum_{m=1}^{n-1}\left|q_{n-m}\right|\sum_{k=1}^{m}\left|u_{km}\right|\Rightarrow$$

$$\sum_{n=2}^{j}\sum_{k=1}^{n-1}n^2\frac{\left|u_{kn}\right|}{k}\leq\sum_{n=2}^{j}n\sum_{k=1}^{n-1}\left|u_{kn}\right|+\sum_{n=2}^{j}\sum_{m=1}^{n-1}\left|p_{n-m}\right|\sum_{k=1}^{m-1}m\frac{\left|u_{km}\right|}{k}+$$

$$\sum_{n=2}^{j}\sum_{m=1}^{n-1}\left|q_{n-m}\right|\sum_{k=1}^{m}\left|u_{km}\right|\leq\sum_{n=2}^{j}n\sum_{k=1}^{n-1}\left|u_{kn}\right|+\sum_{n=1}^{j-1}\left|p_n\right|\sum_{m=2}^{j-1}\sum_{k=1}^{m-1}m\frac{\left|u_{km}\right|}{k}+$$

$$\sum_{n=1}^{j-1}\left|q_n\right|\sum_{m=1}^{j-1}\sum_{k=1}^{m}\left|u_{km}\right|\Rightarrow\sum_{n=2}^{j}\sum_{k=1}^{n-1}n^2\frac{\left|u_{kn}\right|}{k}\leq\sum_{n=2}^{\infty}n\sum_{k=1}^{n-1}\left|u_{kn}\right|+$$

$$\sum_{n=1}^{\infty}\left|p_n\right|\sum_{m=2}^{j-1}\sum_{k=1}^{m-1}m\frac{\left|u_{km}\right|}{k}+\sum_{n=1}^{\infty}\left|q_n\right|\sum_{m=1}^{\infty}\sum_{k=1}^{m}\left|u_{km}\right|$$

or

$$\sum_{n=2}^{j}\sum_{k=1}^{n-1}n^2\frac{\left|u_{kn}\right|}{k}\leq\sum_{n=2}^{\infty}n\sum_{k=1}^{n-1}\left|u_{kn}\right|+$$

$$\|q(x)\|\sum_{m=1}^{\infty}\sum_{k=1}^{m}\left|u_{km}\right|+\|p(x)\|\sum_{m=2}^{j-1}\sum_{k=1}^{m-1}\frac{m\left|u_{km}\right|}{k},$$

$$\sum_{n=2}^{j}n\sum_{k=1}^{n-1}n\frac{\left|u_{kn}\right|}{k}\leq B_0+B_1\sum_{n=2}^{j-1}\sum_{k=1}^{n-1}\frac{n\left|u_{kn}\right|}{k},\ j\geq 2$$

where $B_0=\sum\limits_{n=2}^{\infty}n\sum\limits_{k=1}^{n-1}\left|u_{kn}\right|+\|q(x)\|\sum\limits_{m=1}^{\infty}\sum\limits_{k=1}^{m}\left|u_{km}\right|$ and $B_1=\|p(x)\|$. From here according to the Lemma the series $\sum\limits_{n=2}^{\infty}\sum\limits_{k=1}^{n-1}n^2\frac{\left|u_{kn}\right|}{k}$ converges. Hence, from the convergence of the series $\sum\limits_{n=2}^{\infty}\sum\limits_{k=1}^{n-1}n^2\frac{\left|u_{kn}\right|}{k}$ and $\sum\limits_{n=1}^{\infty}n\left|u_{nn}\right|$, by the equality

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}n^2\frac{\left|u_{kn}\right|}{k}=\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}n^2\frac{\left|u_{kn}\right|}{k}+\sum_{n=1}^{\infty}n\left|u_{nn}\right|,$$

we obtain that the series

$$\sum_{k=1}^{\infty}\frac{1}{k}\sum_{n=k}^{\infty}\left|u_{kn}\right|n^2$$

also converges. Therefore, for the solution $\{u_n\}$, $\{u_{kn}\}$ of the system (7.1)-(7.3) the series $\sum\limits_{n=1}^{\infty}|u_n|\,n^2$ and $\sum\limits_{k=1}^{\infty}\frac{1}{k}\sum\limits_{n=k}^{\infty}n^2\,|u_{kn}|$ converge. Then the function $f(x,\lambda)$ is a solution of equation (3). The proof is completed.

In order to find the second solution of the equation (3) which is linearly independent with the solution $f(x,\lambda)$ we put $\mu=-\lambda$ in equation (3). Then the equation (3) is written as

$$y''+p(x)y'+[\mu^2-i\mu p(x)+q(x)]y=0,\quad -\infty<x<+\infty. \tag{9}$$

The substitution $y(x)=e^{-\int p(x)dx}z(x)$ (where $\int p(x)dx\in Q$) in the equation (9) after some simplifications gives

$$z''-p(x)z'+[\mu^2-i\mu p(x)+q(x)-p'(x)]z=0,\quad -\infty<x<+\infty. \tag{10}$$

According to the above proved, the equation (10) for each $\mu\neq-\frac{n}{2}$, $n\in\mathbb{N}$, has a solution $z(x,\mu)$ in the form of

$$z(x,\mu)=e^{i\mu x}\left(1+\sum_{n=1}^{\infty}\widetilde{u}_n e^{inx}+\sum_{k=1}^{\infty}\frac{1}{k+2\mu}\sum_{n=k}^{\infty}\widetilde{u}_{kn}e^{inx}\right).$$

Therefore, for each $\lambda\neq\frac{n}{2}$, $n\in\mathbb{N}$, the function

$$\widetilde{f}(x,\lambda)=z(x,-\lambda)e^{-\int p(x)dx}=$$

$$e^{-i\lambda x-\int p(x)dx}\left(1+\sum_{n=1}^{\infty}\widetilde{u}_n e^{inx}+\sum_{k=1}^{\infty}\frac{1}{k-2\lambda}\sum_{n=k}^{\infty}\widetilde{u}_{kn}e^{inx}\right),$$

where the series $\sum\limits_{n=1}^{\infty}|\widetilde{u}_n|\,n^2$ and $\sum\limits_{k=1}^{\infty}\frac{1}{k}\sum\limits_{n=k}^{\infty}n^2\,|\widetilde{u}_{kn}|$ converge, is the second solution of equation (3). Since $\int p(x)dx\in Q$, $p'(x)\in Q$, from the Wiener and Levy's Theorem (see [1], p. 34) it is easy to obtain the existence of the periodic function $q_0(x)$ such that $e^{-\int p(x)dx}=1+q_0(x)$, $q_0'(x)$, $q_0''(x)\in Q$. Consequently, we can write the representation

$$\widetilde{f}(x,\lambda)=e^{-i\lambda x}\left(1+\sum_{n=1}^{\infty}v_n e^{inx}+\sum_{k=1}^{\infty}\frac{1}{k-2\lambda}\sum_{n=k}^{\infty}v_{kn}e^{inx}\right)$$

for which the series $\sum\limits_{n=1}^{\infty}|v_n|\,n^2$ and $\sum\limits_{n=1}^{\infty}\frac{1}{k}\sum\limits_{n=k}^{\infty}n^2\,|v_{kn}|$ converge.

In what follows we use the representations (5) for the solutions $f(x,\lambda)$, $\widetilde{f}(x,\lambda)$.

Corollary 1. *If the functions $p(x)$, $p'(x)$ and $q(x)$ belong to the class Q, then for $\forall\lambda\neq\pm\frac{n}{2}$, $n\in\mathbb{N}$, the equation (3) has the Flouqet solutions*

$$\begin{aligned} f_1(x,\lambda) &= e^{i\lambda x}\left(1+\sum_{n=1}^{\infty}U_{0n}^{(1)}e^{inx}+\sum_{k=1}^{\infty}\frac{1}{k+2\lambda}\sum_{n=k}^{\infty}U_{kn}^{(1)}e^{inx}\right),\\ f_2(x,\lambda) &= e^{-i\lambda x}\left(1+\sum_{n=1}^{\infty}U_{0n}^{(2)}e^{inx}+\sum_{k=1}^{\infty}\frac{1}{k-2\lambda}\sum_{n=k}^{\infty}U_{kn}^{(2)}e^{inx}\right), \end{aligned}$$

in $\mathbb{R}$ for which the series of type (8) converge.

Corollary 2. *For $\forall x\in\mathbb{R}$ the functions $f_j(x,\lambda)$, $j=1,2$, and their derivatives $f_j'(x,\lambda)$, $f_j''(x,\lambda)$ with respect to x are meromorphic functions with respect to λ*

and they may have only simple poles $\lambda = (-1)^j n/2$, $n \in \mathbb{N}$, *and they are also continuous functions of* (x, λ) *for all* $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$, $\lambda \neq (-1)^j n/2$, $n \in \mathbb{N}$.

Note that if $p(x)$, $q(x)$ are distinct from zero, then the functions $f_j(x, \lambda)$ have at most one pole. Namely, if $U_{nn}^{(j)} \neq 0$, then $\lambda = (-1)^j n/2$, $j = 1, 2$, is the pole of the function $f_j(x, \lambda)$.

Wronskian of the functions $f_1(x, \lambda)$ and $f_2(x, \lambda)$ is found as $W[f_1, f_2](x, \lambda) = -2i\lambda e^{-\int p(x)dx}$, where $\int p(x)dx = \sum\limits_{n=1}^{\infty} \frac{p_n}{in} e^{inx}$ for each $\lambda \neq \mp\frac{n}{2}$, $n \in \mathbb{N}$ (see [4], p. 200). Therefore, for each $\lambda \neq 0, \pm\frac{n}{2}$, $n \in \mathbb{N}$, the functions $f_1(x, \lambda)$ and $f_2(x, \lambda)$ are linearly independent in $\mathbb{R}$.

The linearly independent solutions of equation (3) for $\lambda = 0$ or $\lambda = \mp\frac{n}{2}$, $n \in \mathbb{N}$ have been constructed in [11]. These solutions are the functions $\widetilde{f}_{1n}(x) = e^{-\frac{n}{2}x}(\psi_{1n}(x) + x\phi_{1n}(x))$ and $f_2(x, -\frac{n}{2})$ when $\lambda = -\frac{n}{2}$, $f_1(x, \frac{n}{2})$ and $\widetilde{f}_{2n}(x) = e^{i\frac{n}{2}x}(\psi_{2n}(x) + x\phi_{2n}(x))$ when $\lambda = \frac{n}{2}$, and $\widetilde{f}_1(x) = \psi_0(x)$, $\tilde{f}_2(x) = x\psi_0(x) + \phi_0(x)$ when $\lambda = 0$. Here the functions $\psi_{1n}(x)$, $\phi_{1n}(x)$, $\psi_{2n}(x)$, $\phi_{2n}(x)$, $\psi_0(x)$, $\phi_0(x)$ belong to the class Q.

3. The spectrum and resolvent of the operator L_λ

Theorem 3. *The operator* L_λ *has not real eigenvalues.*

Proof. Let's show that the equation $L_\lambda y = 0$ has only trivial solution which belongs to $L_2(0, +\infty)$ for $\forall\lambda \in \mathbb{R}$. In case $\lambda \neq 0, \pm\frac{n}{2}$, $n \in \mathbb{N}$ this follows from the properties of solutions $f_1(x, \lambda)$ and $f_2(x, \lambda)$. Really, if $y(x) = c_1 f_1(x, \lambda) + c_2 f_2(x, \lambda)$ the solution of the equation $l_\lambda(y) = 0$ belonging to $L_2(0, +\infty)$ and satisfying the condition $y(0) = 0$, then $y(x)$ is almost a periodic function and necessarily $c_1 = 0$, $c_2 = 0$, $y(x) \equiv 0$. If linearly independent solutions of (3) according to $\lambda = \pm\frac{n}{2}$, $n \in \mathbb{N}$ or $\lambda = 0$ are taken instead of $f_1(x, \lambda)$ and $f_2(x, \lambda)$, then similar result is also valid. Hence $\mathbb{R} \cap \sigma_p(L_\lambda) = \varnothing$. The theorem is proved.

Theorem 4. *The operator* L_λ *has at most a countable set of eigenvalues in* $\mathbb{C}\backslash\mathbb{R}$.

Proof. It is easy to see that, $f_1(x, \lambda) \in L_2(0, +\infty)$, $f_2(x, \lambda) \notin L_2(0, +\infty)$ for $Im\lambda > 0$ and $f_1(x, \lambda) \notin L_2(0, +\infty)$, $f_2(x, \lambda) \in L_2(0, +\infty)$ for $Im\lambda < 0$. These relations imply

$$y(x, \lambda) = c_1 f_1(x, \lambda) + c_2 f_2(x, \lambda) \in L_2(0, +\infty),\ y(0, \lambda) = 0,\ \lambda \in \mathbb{C}\backslash\mathbb{R},$$

if and only if the eigenvalue equation

$$f_1(0, \lambda) = 0 \text{ when } \operatorname{Im}\lambda > 0 \text{ or } f_2(0, \lambda) = 0 \text{ when } \operatorname{Im}\lambda < 0$$

are satisfied.

Because of $f_1(x, \lambda)$ and $f_2(x, \lambda)$ are holomorphic functions in the upper and lower half planes respectively, these equations have at most a countable set of roots in $\mathbb{C}\backslash\mathbb{R}$. The theorem is proved.

Theorem 3 and Theorem 4 imply that $\sigma_p(L_\lambda) \subseteq \mathbb{C}\backslash\mathbb{R}$.

Theorem 5. *The residual spectrum of the operator* L_λ *is an empty set, i.e.* $\sigma_r(L_\lambda) = \varnothing$.

Proof. Since the operator L_λ is one to one for every $\lambda \in \mathbb{C}\backslash\sigma_p(L_\lambda)$ then $\lambda \in \sigma_r(L_\lambda)$ if and only if when the range $R(L_\lambda)$ is not dense in $L_2(0, +\infty)$. It means the equation $L_\lambda^*(z) = 0$ has a nontrivial solution $z(x)$, in other words, $\overline{\lambda} \in \sigma_p(L_\lambda^*)$ or there is a nontrivial solution $\overline{z(x)}$ of the conjugate equation

$$z''(x) - p(x)z'(x) + [\lambda^2 + i\lambda p(x) + q(x) - p'(x)]z(x) = 0, \ 0 < x < +\infty, \quad (11)$$

satisfying conditions $\overline{z}(0) = 0, \ z(x) \in L_2(0, +\infty)$. Since equation (11) is reduced to equation of type (3) by putting $\mu = -\lambda$, then according to Theorem 3 the equation (11) for $\lambda \in \mathbb{R}$ has not a nontrivial solution $z(x) \in L_2(0, +\infty)$, i.e. $\lambda \notin \sigma_p(L_\lambda^*)$ or $\sigma_r(L_\lambda) \cap \mathbb{R} = \varnothing$.

In general, if $\lambda \in \mathbb{C}$, $\lambda \neq \pm\frac{n}{2}$, $n \in \mathbb{N}$, then according to Corollary 1 the equation (11) has the solutions in $\mathbb{R}$ as

$$\varphi_1(x,\lambda) = e^{-i\lambda x}\left(1 + \sum_{n=1}^{\infty} V_{0n}^{(1)} e^{inx} + \sum_{k=1}^{\infty} \frac{1}{k - 2\lambda} \sum_{n=k}^{\infty} V_{kn}^{(1)} e^{inx}\right)$$

and

$$\varphi_2(x,\lambda) = e^{i\lambda x}\left(1 + \sum_{n=1}^{\infty} V_{0n}^{(2)} e^{inx} + \sum_{k=1}^{\infty} \frac{1}{k + 2\lambda} \sum_{n=k}^{\infty} V_{kn}^{(2)} e^{inx}\right)$$

in the real line. Clearly the functions

$$z_1(x,\lambda) = -\frac{2i\lambda f_2(x,\lambda)}{W[f_1, f_2](x,\lambda)}, \ z_2(x,\lambda) = -\frac{2i\lambda f_1(x,\lambda)}{W[f_1, f_2](x,\lambda)} \quad (12)$$

are linear independent solutions of equation (11). It is easy to see that,

$$\varphi_1(x,\lambda) = -\frac{2i\lambda f_2(x,\lambda)}{W[f_1, f_2](x,\lambda)}, \ \varphi_2(x,\lambda) = -\frac{2i\lambda f_1(x,\lambda)}{W[f_1, f_2](x,\lambda)} \quad (13)$$

From here we immediately have $\varphi_1(x,\lambda) \notin L_2(0,+\infty)$, $\varphi_2(x,\lambda) \in L_2(0,+\infty)$ for $\operatorname{Im}\lambda > 0$ and $\varphi_1(x,\lambda) \in L_2(0,+\infty)$, $\varphi_2(x,\lambda) \notin L_2(0,+\infty)$ for $\operatorname{Im}\lambda < 0$. Consequently, the solution $z(x,\lambda) = c_1\varphi_1(x,\lambda) + c_2\varphi_2(x,\lambda)$ of the equation (9) with conditions $z(0,\lambda) = 0$, $z(x,\lambda) \in L_2(0,+\infty)$ only exists for values of the parameter λ satisfying the equation $\varphi_2(0,\lambda) = 0$ when $\operatorname{Im}\lambda > 0$ or equation $\varphi_1(0,\lambda) = 0$ when $\operatorname{Im}\lambda < 0$.

According to equation (12) this is equivalent to relations

$$f_1(0,\lambda) = 0 \text{ when } \operatorname{Im}\lambda > 0 \text{ or } f_2(0,\lambda) = 0 \text{ when } \operatorname{Im}\lambda < 0,$$

that is $\lambda \in \sigma_p(L_\lambda)$. Hence $\overline{\lambda} \in \sigma_p(L_\lambda^*)\backslash\mathbb{R}$ and $\lambda \in \sigma_p(L_\lambda)\backslash\mathbb{R}$ are equivalent. From here we have that if $\lambda \notin \sigma_p(L_\lambda) \cup \mathbb{R}$, then $\overline{\lambda} \notin \sigma_p(L_\lambda^*)$, i.e. $\lambda \notin \sigma_r(L_\lambda)$. On the other hand if $\lambda \in \mathbb{R}$ then $\lambda \notin \sigma_r(L_\lambda)$, and consequently we get $(\mathbb{C}\backslash\sigma_p(L_\lambda)) \cap \sigma_r(L_\lambda) = \varnothing$ i.e. $\sigma_r(L_\lambda) = \varnothing$. The theorem is proved.

According to Theorem 5, for each $\lambda \in \mathbb{C}\backslash\sigma_p(L_\lambda)$ the inverse operator L_λ^{-1} is defined in a dense set of the space $L_2(0,+\infty)$. Show that for each $\lambda \in \mathbb{C}\backslash\left(\sigma_p(L_\lambda) \cup \mathbb{R}\right)$ the operator L_λ^{-1} is bounded on $L_2(0,+\infty)$. For this reason let us investigate the solution $y(x,\lambda) \in L_2(0,+\infty)$ of

$$y'' + p(x))y' + (\lambda^2 + i\lambda p(x) + q(x))y = f(x) \quad (13)$$

satisfying the condition $y(0) = 0$, where $f(x) \in L_2(0,+\infty)$. If we apply the Lagrange method by using Floquet solutions of equation (3), we find the solution of (13) as

$$y(x,\lambda) = \int_0^{+\infty} G(x,t,\lambda) f(t) dt,$$

where

$$G(x,t,\lambda)=\frac{1}{f_1(0,\lambda)W[f_1,f_2](t,\lambda)}\begin{cases} f_1(x,\lambda)f_0(t,\lambda), & 0\le t<x, \\ f_0(t,\lambda)f_1(t,\lambda), & t\ge x, \end{cases}$$

if $\operatorname{Im}\lambda>0,\ f_1(0,\lambda)\neq 0$,

$$G(x,t,\lambda)=\frac{1}{f_2(0,\lambda)W[f_1,f_2](t,\lambda)}\begin{cases} f_2(x,\lambda)f_0(t,\lambda), & 0\le t<x, \\ f_0(x,\lambda)f_2(t,\lambda), & t\ge x, \end{cases}$$

if $\operatorname{Im}\lambda<0,\ f_2(0,\lambda)\neq 0$, where

$$f_0(x,\lambda)=f_2(0,\lambda)f_1(x,\lambda)-f_1(0,\lambda)f_2(x,\lambda)$$

is the solution of equation (3) in $(-\infty,+\infty)$ with initial condition $f_0(0,\lambda)=0$, $f_0'(0,\lambda)=-W[f_1,f_2](0,\lambda)=2i\lambda w_0,\ w_0=e^{-\sum\limits_{n=1}^{\infty}\frac{p_n}{in}}$. It is easy to see that the function

$$\widehat{\varphi}(x,\lambda)=\frac{f_0(x,\lambda)}{W[f_1,f_2](x,\lambda)}=\frac{f_2(0,\lambda)\varphi_2(x,\lambda)-f_1(0,\lambda)\varphi_1(x,\lambda)}{-2i\lambda}$$

is the solution of equation (11) with the conditions $\widehat{\varphi}(0,\lambda)=0,\ \widehat{\varphi}'(0,\lambda)=-1$ and the function

$$\widehat{f}(x,\lambda)=\frac{f_0(x,\lambda)}{-2i\lambda}=\frac{f_2(0,\lambda)f_1(x,\lambda)-f_1(0,\lambda)f_2(x,\lambda)}{-2i\lambda}$$

is the solution of equation (3) with the conditions $\widehat{f}(0,\lambda)=0,\ \widehat{f}'(0,\lambda)=-w_0$. Therefore these solutions are holomorphic functions of λ in $\mathbb{C}$. Using these expressions we can write

$$G(x,t,\lambda)=\frac{1}{f_1(0,\lambda)}\begin{cases} f_1(x,\lambda)\widehat{\varphi}(t,\lambda), & 0\le t<x, \\ \widehat{f}(x,\lambda)\varphi_2(t,\lambda), & t\ge x, \end{cases}$$

if $\operatorname{Im}\lambda>0,\ f_1(0,\lambda)\neq 0$,

$$G(x,t,\lambda)=\frac{1}{f_2(0,\lambda)}\begin{cases} f_2(x,\lambda)\widehat{\varphi}(t,\lambda), & 0\le t<x, \\ \widehat{f}(x,\lambda)\varphi_1(t,\lambda), & t\ge x, \end{cases}$$

if $\operatorname{Im}\lambda<0,\ f_2(0,\lambda)\neq 0$.

From the explicit expression of functions $f_i(x,\lambda)$ and $\varphi_i(x,\lambda)$ it follows that for $\forall\lambda\in\mathbb{C}\backslash\sigma_p(L_\lambda),\ \operatorname{Im}\lambda\neq 0$

$$|G(x,t,\lambda)|\le C(\lambda)e^{-\tau|x-t|}, \tag{14}$$

where $C(\lambda)>0,\ \tau=|\operatorname{Im}\lambda|,\ \forall x,t\in(0,+\infty)$. By considering the explicit form of the function $G(x,t,\lambda)$ and the formula (14) it can be proved by the standard method (see [6], p. 302-304) that for each $f(x)\in L_2(0,+\infty)$ the function

$$y(x,\lambda)=\int_0^{+\infty}G(x,t,\lambda)f(t)dt,$$

belong to $L_2(0,+\infty)$ and satisfy condition $y(0,\lambda)=0$. Further the integral operator $L_\lambda^{-1}:L_2(\mathbb{R}^+)\to L_2(\mathbb{R}^+)$ defined by

$$(R_\lambda f)(x)=\left(L_\lambda^{-1}f\right)(x)=\int_0^{+\infty}G(x,t,\lambda)f(t)dt$$

is bounded for $\forall\lambda \in \mathbb{C}\setminus(\sigma_p(L_\lambda)\cup\mathbb{R})$ and $\|R_\lambda\| \le \frac{2|C(\lambda)|}{\tau}$. It means that $\lambda \in \rho(L_\lambda)$. On other hand for $Im\lambda = 0$ the operator L_λ^{-1} is unbounded which means $\lambda \in \sigma_c(L_\lambda)$.

It is clear that root λ of the equations $f_1(0,\lambda) = 0$, $\operatorname{Im}\lambda \ge 0$ or $f_2(0,\lambda) = 0$, $\operatorname{Im}\lambda \le 0$ may be a pole of the kernel $G(x,t,\lambda)$. If $\operatorname{Im}\lambda \ne 0$ then these poles are eigenvalues of the operator L_λ. If $\operatorname{Im}\lambda = 0$, because of L_λ hasn't got $\lambda \in \mathbb{R}$ as an eigenvalue, then the kernel $G(x,t,\lambda)$ of the resolvent operator has poles at these points which are called spectral singularities (in the sense of [6], p. 306) of the operator L_λ.

Thus the following theorem is true.

Theorem 6. *L_λ has a continuous spectrum $\sigma_c(L_\lambda) \in \mathbb{R}$ and the resolvent set $\rho(L_\lambda) = \mathbb{C}\setminus(\mathbb{R}\cup\sigma_p(L_\lambda))$. The resolvent operator L_λ^{-1} is an integral operator in $L_2(0,+\infty)$ with the kernel $G(x,t,\lambda)$ of Carleman type for $\lambda \in \rho(L_\lambda)$.*

4. The asymptotic formulas for singular values of the operator L_λ

In this section we specify the location of the singular values of the operator L_λ on the λ complex plane and show that the singular values with sufficiently large modulus are located close to the points $\lambda = \pm\frac{n}{2}$, $n \in \mathbb{N}$. Note that as the singular values of the operator L_λ we mean the eigenvalues and spectral singularities of the operator L_λ. For this reason we show that the singular values are located on the strip $\{\lambda \in \mathbb{C} : \operatorname{Im}\lambda < \alpha\}$ for some $\alpha > 0$. Let us prove this fact for the case $\operatorname{Im}\lambda \ge 0$, $f_1(0,\lambda) = 0$, in the other case it is proved in the similar way.

First we show that there exists $\alpha > 0$ such that the equation $f_1(0,\lambda) = 0$ has not a root outside the set $E_\alpha = \{\lambda | \lambda \in \mathbb{C},\ |\operatorname{Im}\lambda| < \alpha,\ \operatorname{Re}\lambda < \alpha\}$. Because of $|k + 2\lambda| > 2\alpha$ for each $\lambda \in \mathbb{C}\setminus E_\alpha$, $k \in \mathbb{N}$, we have that the inequality

$$|f_1(0,\lambda)| = \left|1 + \sum_{n=1}^{\infty} U_{0n}^{(1)} + \sum_{k=1}^{\infty} \frac{1}{k+2\lambda} \sum_{n=k}^{\infty} U_{kn}^{(1)}\right| \ge$$

$$|U_0| - \left|\sum_{k=1}^{\infty} \frac{1}{k+2\lambda} \sum_{n=k}^{\infty} U_{kn}^{(1)}\right| \ge |U_0| -$$

$$\sum_{k=1}^{\infty} \frac{1}{|k+2\lambda|} \sum_{n=k}^{\infty} \left|U_{kn}^{(1)}\right| \ge |U_0| - \frac{1}{2\alpha} \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} \left|U_{kn}^{(1)}\right| \ge$$

$$|U_0| - \frac{1}{2\alpha} \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} \left|U_{kn}^{(1)}\right| > 0$$

is satisfied when $\alpha > \frac{\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\left|U_{kn}^{(1)}\right|}{2|U_0|}$ (here, according to equation (7.2) $1 + \sum_{n=1}^{\infty} U_{0n}^{(1)} e^{inx}$ $= e^{-\int p(x)dx}$, $U_0 = 1 + \sum_{n=1}^{\infty} U_{0n}^{(1)} \ne 0$). For some α satisfying this condition the equation $f_1(0,\lambda) = 0$ has not any root outside the set E_α. On the other hand, the function $f_1(0,\lambda)$ is holomorphic at every interior point $\lambda \ne -\frac{k}{2}$ $(k \in \mathbb{N})$ of the set E_α and on the boundary of this set. Then the equation $f_1(0,\lambda) = 0$ may have at most a countable set of roots in the set E_α and all these roots may have the unique limit point at infinity. Show that the roots of the equation $f_1(0,\lambda)=0$ with sufficiently large modulus are located close to the points $\lambda=-\frac{n}{2}$, $n\in\mathbb{N}$. For this, taking into our account the absolute convergence of the series $\sum\limits_{k=1}^{\infty}\sum\limits_{n=k}^{\infty} n\left|U_{kn}^{(1)}\right|$, we can choose the smallest number $k_0\in\mathbb{N}$, $k_0>2$ such that $\sum_{k=k_0}^{\infty}\sum\limits_{n=k}^{\infty} n\left|U_{kn}^{(1)}\right|<\frac{|U_0|}{2}$. Further, because of $\lim\limits_{\lambda\to\infty}\sum\limits_{k=1}^{k_0-1}\frac{1}{k+2\lambda}\sum\limits_{n=k}^{\infty}U_{kn}^{(1)}=0$ we can take the smallest number $r_0\in\mathbb{N}$, $r_0>k_0$ such that the inequality

$$\left|\sum_{n=1}^{k_0-1}\frac{1}{n+2\lambda}\sum_{k=n}^{\infty}U_{nk}^{(1)}\right|<\frac{|U_0|}{2}$$

is satisfied for all $|\operatorname{Im}\lambda|<\alpha$, $\operatorname{Re}\lambda\le-\frac{r_0}{2}+\frac{1}{4}$. Thus for $\lambda\in E_\alpha$, $\operatorname{Re}\lambda\le-\frac{r_0}{2}+\frac{1}{4}$, $|2\lambda+k|\ge\frac{1}{k}$, $k\ge r_0$ we have $|2\lambda+k|\ge\frac{1}{k}$, $k\ge k_0$ and

$$|f_1(0,\lambda)|=|U_0|-\left|\sum_{k=1}^{\infty}\frac{1}{k+2\lambda}\sum_{n=k}^{\infty}U_{kn}^{(1)}\right|\ge$$

$$|U_0|-\left|\sum_{k=1}^{k_0-1}\frac{1}{k+2\lambda}\sum_{n=k}^{\infty}U_{kn}^{(1)}\right|-\sum_{k=k_0}^{\infty}\frac{1}{|k+2\lambda|}\sum_{n=k}^{\infty}\left|U_{kn}^{(1)}\right|\ge$$

$$|U_0|-\frac{|U_0|}{2}-\sum_{k=k_0}^{\infty}k\sum_{n=k}^{\infty}\left|U_{kn}^{(1)}\right|=\frac{|U_0|}{2}-\sum_{k=k_0}^{\infty}\sum_{n=k}^{\infty}n\left|U_{kn}^{(1)}\right|>0.$$

Consequently, the roots of the equation $f_1(0,\lambda)=0$ satisfying the conditions $\lambda\in E_\alpha$, $\operatorname{Re}\lambda\le-\frac{r_0}{2}+\frac{1}{4}$ only can be located in the neighborhoods of the points $\lambda=-\frac{k}{2}$, $k\ge r_0$ with radius $\delta_k=\frac{1}{2k}$. On the other hand, the equation $f_1(0,\lambda)=0$ may have a finite number of roots satisfying the conditions $\lambda\in E_\alpha$, $\operatorname{Re}\lambda>-\frac{r_0}{2}+\frac{1}{4}$. Show that if the point $\lambda=-\frac{m}{2}$, $m\ge r_0$ is a pole of the function $f_1(0,\lambda)$, then there is a unique simple root in the neighborhood $\left|\lambda+\frac{m}{2}\right|<\frac{1}{2m}$. Indeed, the equations $f_1(0,\lambda)=0$ and $(m+2\lambda)f_1(0,\lambda)=0$ have the same roots in the closed disk $\left|\lambda+\frac{m}{2}\right|\le\frac{1}{2m}$. Further if we put $g(\lambda)=(m+2\lambda)U_0$, $h(\lambda)=(m+2\lambda)\sum\limits_{m=1}^{\infty}\frac{1}{k+2\lambda}\sum\limits_{n=m}^{\infty}U_{mn}^{(1)}$, then on the circle $\left|\lambda+\frac{m}{2}\right|=\frac{1}{2m}$ we have $\left|\lambda+\frac{k}{2}\right|\ge\frac{1}{2k}$ for each $k\ge k_0$ and

$$|g(\lambda)|-|h(\lambda)|=|m+2\lambda|\,|U_0|-|m+2\lambda|\left|\sum_{k=1}^{\infty}\frac{1}{k+2\lambda}\sum_{n=m}^{\infty}U_{mn}^{(1)}\right|=$$

$$\frac{1}{m}\left(|U_0|-\left|\sum_{k=1}^{\infty}\frac{1}{k+2\lambda}\sum_{n=m}^{\infty}U_{mn}^{(1)}\right|\right)>0\text{ id est. }|g(\lambda)|>|h(\lambda)|.$$

Therefore, by the Roushé theorem, the functions $(m+2\lambda)f_1(0,\lambda)=g(\lambda)+h(\lambda)$ and $g(\lambda)$ have the same number zeros in the disk $\left|\lambda+\frac{m}{2}\right|\le\frac{1}{2m}$. Since the function $g(\lambda)$ has the unique simple zero $\lambda=-\frac{m}{2}$ in this disk, we have that the function $(m+2\lambda)f_1(0,\lambda)$ also has the unique simple zero λ_m^- in this disk. It is obvious that, if $\lambda=-\frac{m}{2}$ is not a pole of $f_1(0,\lambda)$, then the equation $f_1(0,\lambda)=0$ has not any root in the disk $\left|\lambda+\frac{m}{2}\right|\le\frac{1}{2m}$. Consequently, the equation $f_1(0,\lambda)=0$

may have a unique simple root in the $\delta = \frac{1}{2k}$ -neighborhood of the point $\lambda = -\frac{k}{2}$ for all $k \geq r_0$. By the similar way we can show that the equation $f_2(0, \lambda) = 0$ has the unique simple root λ_k^+ in the $\delta = \frac{1}{2k}$ -neighborhood of the point $\lambda = \frac{k}{2}$ for all $k \geq r_1$ and some $r_1 \in \mathbb{N}$. Outside these neighborhoods the equation $f_2(0, \lambda) = 0$ may have only a finite set of roots. If we set $m_0 = \max\{r_0, r_1\}$, we have

$$\lambda_k^- = -\frac{k}{2} + O(\frac{1}{k}),\ \lambda_k^+ = \frac{k}{2} + O(\frac{1}{k}),\ k \geq m_0.$$

Here $\lambda_k^\pm$ are the simple eigenvalues of operator L_λ if $Im\lambda_k^- > 0$ or $Im\lambda_k^+ < 0$ and , then $\lambda_k^\pm$ are the simple spectral singularities of operator L_λ if $Im\lambda_k^- = 0$ or $Im\lambda_k^+ = 0$. So the following theorem is true.

Theorem 7. *The operator L_λ may have at most a countable set of spectral singularities on the continuous spectrum and at most a countable set of eigenvalues outside the real axis. Singular values $\{\lambda_n^\pm\}$ (eigenvalues or spectral singularities) with sufficiently large modulus are simple, lie in the neighborhood of points $\pm\frac{n}{2}$, $n \in \mathbb{N}$, and the asymptotic formulas*

$$\lambda_k^\pm = \pm\frac{k}{2} + O(\frac{1}{k}),\ k \to \infty,$$

are satisfied.

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Received: September 19, 2014; Accepted: October 30, 2014