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APPLICATION OF THE FINITE INTEGRAL TRANSFORMATION METHOD TO SOLVING MIXED PROBLEMS FOR HYPERBOLIC EQUATIONS WITH IRREGULAR BOUNDARY CONDITIOS

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In memory of M. G. Gasymov on his 75th birthday

Abstract. In the paper we study mixed problems with nonlocal and irregular boundary conditions for junction of hyperbolic systems of different orders with first kind discontinuous coefficients. Under definite conditions, using the finite integral transformation method, we get analytic (integral) representation of the solution of the problem under consideration.

1. Introduction

The symbolic calculus was a convenient but mathematically not substantiated device for solving mixed problems. Its popularization, in great extent was promoted by electrical-engineer. O. Heaviside to successfully used the symbolic calculus in electrician calculations. But Heaviside did not care of grounding the applied methods and in a number of cases came to false results. One of the methods for solving mixed problems for partial differential equations is the integral transformations method that was successfully used by Cauchy, Laplace, A.N. Tikhonov, V.A. Il'in, M.L. Rasulov and others.

The function of a complex variable λ determined by the relation

$$\tilde{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt \tag{0.1}$$

is said to be the transform of the function $f(t)$ by (Laplace).

Note that the Heviside method, as it became clear after the papers of Carson consists of going from the function $f(t)$ to the function

$$F(\lambda) = \lambda \int_0^\infty e^{-\lambda t} f(t)dt. \tag{0.2}$$

M.L. Rasulov showed that the Laplace integral transformation (0.1) (consequently the Heaviside transformation (0.2)) is a weak device in solving dynamic problems at non zero initial conditions.

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In this paper we use the finite integral transformation method [5, 6]. Application of this method to solving mixed problems is reduced to the following stages:

1) From the sought-for function $u(x,t)$ we pass to the function (”transform” $u(x,t)$) of a complex variable:

$$\tilde{u}(x,t,\lambda) = \int_0^t \omega(\tau) \exp\left(-\lambda \int_0^\tau \omega(\eta) d\eta\right) u(x,\tau)\, d\tau,$$

or

$$\tilde{u}(x,t,\lambda) = \int_t^T \omega(\tau) \exp\left(-\lambda \int_0^\tau \omega(\eta) d\eta\right) u(x,\tau)\, d\tau,$$

where T is some positive number.

2) On the transform $\tilde{u}(x,t,\lambda)$ we perform the operations corresponding to the given operations on $u(x,t)$ and get ”an operational equation” (a parametric problem).

3) In the λ - plane we find ”suitable” domain Ω and on it solve the ”operational equation” with respect to $\tilde{u}(x,t,\lambda)$:

$$\tilde{u}(x,t,\lambda) = I(x,t,\lambda; u(x,t)) + \Phi(x,t,\lambda).$$

4) In the domain Ω we find a ”suitable” smooth line $\mathcal{L}$ and from the found transform $\tilde{u}(x,t,\lambda)$ we pass to the pre-image $u(x,t)$:

$$u(x,t) = \alpha \int_{\mathrm{L}} \exp\left[\lambda \int_0^t \omega(\tau) d\tau\right] \{\tilde{u}(x,t,\lambda) - I(x,t,\lambda; u(x,t))\}\, d\lambda.$$

From the above stated follows

Problem 1. For solving the given mixed problem find ”suitable” domain Ω and line $\mathcal{L}$.

While solving mixed problems for parabolic equations with ”regular” boundary conditions[1], problem 1 was solved positively [3-11]. And also in the case when condition (2) doesn’t contain integral summands of the sought-for function in [12], at restrictions of ”regularity” of boundary conditions for problem (1)-(3) (stated for hyperbolic equations) problem 1 was solved positively. In the case of irregular ”boundary” conditions for hyperbolic equations the solution of problem 1 possesses specific peculiarites. For example, while solving mixed problems (4)-(6) for hyperbolic equations with irregular conditions (5), problem 1 may be solved in the classic way:

$$\Omega = \{\lambda : Re\lambda \geq \alpha\}, \quad \mathcal{L} = \{\lambda : Re\lambda = a\}, \tag{0.3}$$

where α and a $(a \geq \alpha)$ are some positive numbers, i.e. in this case Ω is a half-plane, $\mathcal{L}$ is the Laplace straightline. And while solving mixed problems (11)-(13) for hyperbolic equations with irregular boundary conditions (12) it is impossible to solve problem 1 in the similar way.

In the case of solving a wide class of mixed problems for hyperbolic equations with irregular boundary conditions, we suggest to choose the domain Ω and the line $\mathcal{L}$ as follows (see fig. 1).

[1]If the boundary conditions for parabolic equations are regular in the sense of Birkhoff-Tamarkin-Naimark-Rasulov, they are ”regular” by our definition [6], but the inverse statement is not true.

2. Problem statement

Find the solution of the hyperbolic system

$$\frac{\partial^{n_i}}{\partial t^{n_i}} u_i(x,t) - \sum_{s=1}^{n_i-1} \sum_{j=0}^{n_i-s} B_{i,j,s}(x) \frac{\partial^{j+s}}{\partial x^j \partial t^s} u_i(x,t) -$$

$$- \sum_{j=0}^{n_i} A_{i,j}(x) \frac{\partial^j}{\partial x^j} u_i(x,t) = f_i(x,t),$$

$$x \in (a_i, b_i), \quad t \in (0,T), \; i = 1,2,...,n, \tag{1}$$

under the conditions

$$\sum_{i=1}^{n} \sum_{j=0}^{\chi(i,k)} \sum_{m=0}^{S(i,j,k)} \left\{ \alpha_{j,m}^{(i,k)} \frac{\partial^{m+j}}{\partial x^m \partial t^j} u_i(x,t) /_{x=a_i} + \int_{a_i}^{b_i} \gamma_{j,m}^{(i,k)}(x) \frac{\partial^{m+j}}{\partial x^m \partial t^j} u_i(x,t)\, dx + \right.$$

$$\left. + \beta_{j,m}^{(i,k)} \frac{\partial^{m+j}}{\partial x^m \partial t^j} u_i(x,t) /_{x=b_i} \right\} = \varphi_k(t), \; t \in (0,T), \; k = \overline{1,N}, \tag{2}$$

and initial conditions

$$\frac{\partial^k}{\partial t^k} u_i(x,t) /_{t=0} = D_{i,k}(x), \; x \in (a_i, b_i), \; k = \overline{0, n_i - 1}, \; \overline{i = 1,n}, \tag{3}$$

where $B_{i,j,s}$, $A_{i,j}$ are the square matrices of order r_i, $\alpha_{j,m}^{(i,k)}$, $\beta_{j,m}^{(i,k)}$,$\gamma_{j,m}^{(i,k)}(x)$ are the vectors of the row of dimension r_i; $\varphi_k(t)$ is a scalar function $D_{i,k}$, f_i, u_i are the columns of dimension r_i; $S(i,j,k)$ and $\chi(i,k)$ are non-negative integers less or equal to $n_i - 1$ and n_i respectively; $N \equiv \sum\limits_{\nu=1}^{n} d_\nu$, $d_\nu = n_\nu r_\nu$; r_i, n_i, n are natural numbers; a_i, b_i $(a_i < b_i)$ are finite numbers ; T $(0 < T \leq \infty)$ is some number.

In (1)-(3) $u_1, u_2, ..., u_n$ is the sought-for solution, and the remaining ones are considered to be known.

At the same time, note that the problem of vibrations of a end fastened string, at some points of which there are concentrated masses, are reduced to the problems of the form (1)-(3) (see [1], p. 147).

In the present paper we'll consider the case when conditions (3) are irregular. For simplicity of notation we consider the following model problems 2 and 3.

Problem 2. Find the solution of the hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + F(x,t), \; 0 < x < 1, \; 0 < t < \infty, \tag{4}$$

satisfying the nonlocal conditions

$$U_i(u) \equiv \int_0^1 K_i(x) u(x,t) dx = \varphi_i(t), \; 0 < t < \infty, \; i = 1,2, \tag{5}$$

and initial conditions

$$\left. \frac{\partial^k u}{\partial t^k} \right|_{t=0} = f_k(x), \; 0 < x < 1, \; k = 0,1, \tag{6}$$

where $u \equiv u(x,t)$ is the sought-for positive solution, and the remaining ones are the known functions.

At the same time note that conditions (5) are irregular.

1^0. Let the functions $F(x,t),\ f_k(x),\ \ (k=0,1),\ \varphi_i(t)\,,\ (i=1,2)$ be continuous for $0 \le x \le 1,\ 0 \le t < \infty$.

2^0. Let $K_i(x) \in C^2([0,1])\,,\ i=1,2$ and $A \equiv K_1(0)K_2(1) - K_1(1)K_2(0) \neq 0$.

Applying the finite integral transformation

$$\tilde{\varphi}(t,\lambda) = \int_0^t e^{-\lambda\tau}\varphi(\tau)\, d\,\tau, \tag{7}$$

to problem 2, we get the parametric problem

$$y'' - \lambda^2 y = \psi(x)\,,\quad x \in (0,1), \tag{8}$$

$$U_i(y) = \gamma_i\,,\quad i=1,2. \tag{9}$$

Let [2] $G(x,\xi,\lambda)$ be the Green function of problem (8)-(9); $\Delta(\lambda)$ be the denominator of the Green function; $\delta(x,\lambda,\gamma_1,\gamma_2)$ be the solution of the homogeneous equation corresponding to (8), satisfying nonhomogeneous conditions (9);

$$y_1(x,\lambda) = e^{-\lambda x};\ y_2(x,\lambda) = e^{-\lambda(1-x)};$$

$$A_1(x,\lambda) = U_2(y_2)y_1(x,\lambda) - U_2(y_1)y_2(x,\lambda);$$

$$A_2(x,\lambda) = U_1(y_1)y_2(x,\lambda) - U_1(y_2)y_1(x,\lambda)\,.$$

For problem 2, choosing the domain Ω and the line $\mathcal{L}$ by formula (0.3) by the way stated in [6], it is easy to prove the following

Theorem 1. *At restrictions* 1^0 *and* 2^0*:*

(1) *if problem (4)-(6) has a solution, then this solution is unique and is represented by the integral (analytic) formula*

$$u(x,t) = \frac{1}{2\pi\sqrt{-1}}\int_{\mathrm{L}}\left\{\delta\left(x,\lambda,\int_0^t e^{\lambda(t-\tau)}\varphi_1(\tau)d\tau\ ,\int_0^t e^{\lambda(t-\tau)}\varphi_2(\tau)d\tau\right)+\right.$$
$$+\frac{1}{\lambda\Delta(\lambda)}[A_1(x,\lambda)\varphi_1(t)+A_2(x,\lambda)\varphi_2(t)] - \int_0^1 G(x,\xi,\lambda)[e^{\lambda\, t}f_1(\xi)+$$
$$\left.+\lambda e^{\lambda\, t}f_0(\xi)+\int_0^t e^{\lambda(t-\tau)}F(\xi,\tau)d\tau]d\xi\right\}d\lambda,\ 0<x<1,\ 0<t<\infty, \tag{10}$$

(2) *if the integrals contained in (10) diverge, or the function* $u(x,t)$ *defined by formula (10) is not the solution of problem (4)-(6), this problem has no solution.*

Imposing the sufficient smoothness conditions on the functions $F(x,t),\ \varphi_i(t)\,,$ $(i=1,2)\,,\ f_k(x),\ (k=0,1)$, by the method stated in [6], it is easy to be convinced that the function $u(x,t)$ determined by formula (10) is the solution of problem (4)-(6).

Problem 3. Find the solution of the hyperbolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + F(x,t)\,,\ \ 0<x<1\,,\ \ 0<t\le T, \tag{11}$$

satisfying the boundary condition

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} - \left.u(x,t)\right|_{x=1} = \gamma(t)\,,\ \ 0<t\le T, \tag{12}$$

[2] For $G(x,\xi,\lambda)$, $\Delta(\lambda)$, $\delta(x,\lambda,\gamma_1,\gamma_2)$ see [6].

and the initial condition

$$u(x,t)|_{t=0} = f(x)\,, \quad 0 < x < 1, \tag{13}$$

where $u \equiv u(x,t)$ is the sought for classic solution, and the remaining ones are the known functions, T is some positive number.

3^0. Let the functions $F(x,t), f(x), \gamma(t)$ be continuous for $0 \le x \le 1$,$0 \le t \le T$.

4^0. Let problem (11)-(13) have the classic solution $u \equiv u(x,t)$ possessing the derivatives of the form $\dfrac{\partial^k u(x,t)}{\partial x^k}$, $\dfrac{\partial^k u(x,t)}{\partial t^k}$, $(k = 1,2)$ including $u(x,t)$, $\dfrac{\partial^k u(x,t)}{\partial x^k}, \dfrac{\partial^k u(x,t)}{\partial t^k} \in C([0,1]\times[0,T])$, $k = 1,2$.

Applying the finite integral transformation (7) to (11)-(13), we have

$$\left(\frac{\partial}{\partial x} - \lambda\right)\tilde{u}(x,t,\lambda) = e^{-\lambda t}u(x,t) - f(x) - \int_0^t e^{-\lambda\tau}F(x,\tau)d\tau, \tag{14}$$

$$\left.\frac{\partial}{\partial x}\tilde{u}(x,t,\lambda)\right|_{x=0} - \tilde{u}(x,t,\lambda)|_{x=1} = \int_0^t e^{-\lambda\tau}\gamma(\tau)d\tau, \tag{15}$$

where $\tilde{u}(x,t,\lambda) \equiv \int\limits_0^t e^{-\lambda\tau}u(x,\tau)d\tau$.

For solving problem (14)-(15), at first we solve the following parametric problem:[3]

$$y' - \lambda y = \psi(x)\,, \quad x \in (0,1), \tag{16}$$

$$y'|_{x=0} - y|_{x=1} = \beta, \tag{17}$$

where $\psi(x) \in C([0,1])$, β is some number.

The denominator of the Green function of problem (16)-(17) will be

$$\Delta(\lambda) = e^{\lambda} - \lambda. \tag{18}$$

From (18) we have

$$\Delta(\lambda) = e^{\lambda}\cdot\rho(\lambda), \tag{19}$$

$$\rho(\lambda) = 1 - \sigma(\lambda), \tag{20}$$

where $\sigma(\lambda) = \dfrac{\lambda}{e^{\lambda}}$.

In the right half-plane we try to find some appropriative domain Ω that has the inequality

$$|\sigma(\lambda)| \le \frac{1}{2}, \quad for \quad \lambda \in \Omega. \tag{21}$$

If $\lambda = r\,e^{\sqrt{-1}\varphi}$, then

$$|\sigma(\lambda)| = \frac{r}{e^{r\cos\varphi}}. \tag{22}$$

We take the sequence of numbers φ_n so that

$$0 < \varphi_1 < \varphi_2 < ... < \varphi_n < ... < \pi/2; \tag{23}$$

$$\cos\varphi_1 < 2/e; \qquad \lim_{n\to\infty}\varphi_n = \pi/2.$$

[3]The not self-adjoint boundary value problem was studied in [3] and in other papers

The function $g_n(x) = \dfrac{x}{e^{x\cos\varphi_n}}$, $x \in [0,\infty)$ for $x = \dfrac{1}{\cos\varphi_n}$ attains its greatest value

$$\max_{x\in[0,\infty)} g_n(x) = g_n\left(\frac{1}{\cos\varphi_n}\right) = \frac{1}{e\cos\varphi_n} > \frac{1}{2}, \ \ for \ \ n \geq 1, \tag{24}$$

and in the interval $\left[\frac{1}{\cos\varphi_n}, \infty\right)$ it strongly decreases and

$$\lim_{x\to\infty} g_n(x) = 0. \tag{25}$$

This means that in the interval $\left[\frac{1}{\cos\varphi_n}, \infty\right)$ there exists such a unique number for which $x = r_n$

$$g_n(r_n) = \frac{r_n}{e^{r_n\cos\varphi_n}} = \frac{1}{2}; \tag{26}$$

$$\frac{x}{e^{x\cos\varphi_n}} \leq \frac{1}{2} \ \ for \ \ x \in [r_n, \infty). \tag{27}$$

It follows from (22) and (27) that

$$|\sigma(\lambda)| \leq \frac{1}{2} \ \ for \ \ r_n \leq r = |\lambda| < \infty, \ \ |\varphi| \leq \varphi_n, \tag{28}$$

where $\varphi = \arg\lambda$.

Let

$$A_n = r_n\cos\varphi_n + \sqrt{-1}r_n\sin\varphi_n = \bar{B}_n; \ \ n = 1, 2, ...;$$
$$C_n = r_{n+1}\cos\varphi_n + \sqrt{-1}r_{n+1}\sin\varphi_n = \bar{D}_n; \ \ n = 0, 1, ...;$$
$$(C_nA_{n+1}) = \{\lambda : |\lambda| = r_{n+1};\ \varphi_n \leq \arg\lambda \leq \varphi_{n+1}\}, \ \ n = 0, 1, ..., ;$$
$$(A_nC_n) = \{\lambda : r_n \leq |\lambda| \leq r_{n+1}; \ \ \arg\lambda = \varphi_n\}, \ \ n = 1, 2, ..., ;$$

$\mathcal{L}$ is a symmetric line (see fig. 1) with respect to the real axis λ of the plane whose upper part is determined in the form

$$(\text{upper part } \mathcal{L}) = \bigcup_{n=0}^{\infty}(C_nA_{n+1}) \bigcup \bigcup_{n=1}^{\infty}(A_nC_n);$$

Ω is the closed domain with the boundary $\mathcal{L}$ remained at the right side of this line (see fig.1).

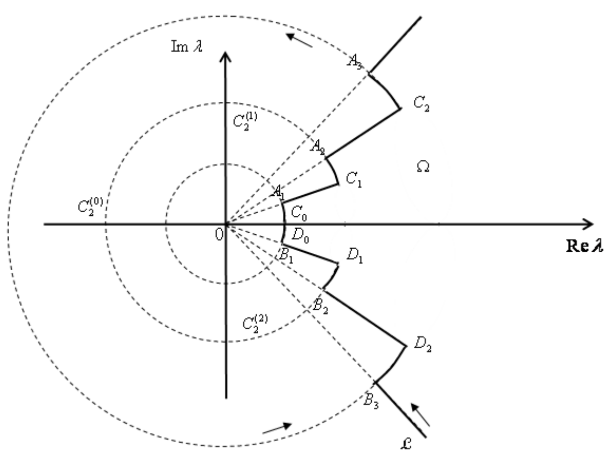

Fig. 1.

Validity of (21) follows from (28).

Taking into account (21) in (20), we have

$$|\rho(\lambda)| \geq \frac{1}{2}, \quad for \quad \lambda \in \Omega. \tag{29}$$

Taking into attention (29), from (19) we get

$$\Delta(\lambda) \neq 0, \quad for \quad \lambda \in \Omega. \tag{30}$$

Thus, we establish the following

Lemma 1. *For $\lambda \in \Omega$ and $\psi(x) \in C([0,1])$ problem (16)-(17) has a unique solution and this solution is represented by the formula*

$$y(x,\lambda) = J(x,\lambda;\psi(x)) - \frac{e^{\lambda x}}{\Delta(\lambda)}\beta, \quad 0 \leq x \leq 1\,, \quad \lambda \in \Omega, \tag{31}$$

where

$$J(x,\lambda;\psi(x)) = \frac{e^{\lambda x}}{\Delta(\lambda)}\left\{\psi(0) - \int_0^1 e^{\lambda(1-\xi)}\psi(\xi)d\xi\right\} + \int_0^x e^{\lambda(x-\xi)}\psi(\xi)d\xi. \tag{32}$$

For $\lambda \in \Omega$ according to lemma 1, from (14)-(15) we get

$$\int_0^t e^{-\lambda\,\tau}u(x,\tau)d\tau = e^{-\lambda\,t}J(x,\lambda;u(x,t)) + \Phi(x,t,\lambda), \tag{33}$$

where

$$\Phi(x,t,\lambda) = \frac{e^{\lambda x}}{\Delta(\lambda)}\left\{-f(0) - \int_0^t e^{-\lambda\,\tau}F(0,\tau)d\tau + \right.$$
$$+\int_0^1 e^{\lambda(1-\xi)}\left[f(\xi) + \int_0^t e^{-\lambda\,\tau}F(\xi,\tau)d\tau\right]d\xi -$$
$$\left. -\int_0^t e^{-\lambda\,\tau}\gamma(\tau)d\tau\right\} - \int_0^x e^{\lambda(x-\xi)}\left[f(\xi) + \int_0^t e^{-\lambda\,\tau}F(\xi,\tau)d\tau\right]d\xi. \tag{34}$$

Multiplying the both sides of (33) by $e^{\lambda\,t}$, we have

$$e^{\lambda\,t}\int_0^t e^{-\lambda\,\tau}u(x,\tau)d\tau - J(x,\lambda;u(x,t)) = e^{\lambda\,t}\Phi(x,t,\lambda),$$
$$0 \leq x \leq 1\,,\ 0 \leq t \leq T\,,\ \lambda \in \Omega. \tag{35}$$

Now show that

$$\int_{\mathcal{L}} \frac{e^{\lambda t}}{\lambda^2}d\lambda = 2\pi\sqrt{-1}\,t, \quad for \quad t > 0. \tag{36}$$

Let C_n^* be a circle in λ -plane, of radius r_n centered at the point $\lambda = 0$. In what follows, let

$$C_n^{(1)} = \{\lambda : \lambda = r_n e^{\sqrt{-1}\,\varphi},\ \varphi_n \leq \varphi \leq \pi/2\};$$
$$C_n^{(2)} = \{\lambda : \lambda = r_n e^{-\sqrt{-1}\,\varphi},\ \varphi_n \leq \varphi \leq \pi/2\};$$
$$C_n^{(0)} = \{\lambda : |\lambda| = r_n,\ Re\lambda \leq 0\}. \tag{37}$$

It is clear that

$$2\pi\sqrt{-1} = \int_{C_n^*} \frac{e^{\lambda}}{\lambda^2}d\lambda = \int_{\mathrm{L}_n} + \int_{C_n^{(1)}} + \int_{C_n^{(2)}} + \int_{C_n^{(0)}} \equiv I_{1,n} + I_{2,n} + I_{3,n} + I_{4,n}, \tag{38}$$

where L_n is the part of $\mathcal{L}$ remained interior to the circle C_n^*.

From (37) and (38) we get

$$I_{2,n}=\sqrt{-1}\int_{\varphi_n}^{\pi/2}\frac{e^{r_n(\cos\varphi+\sqrt{-1}\sin\varphi)}}{r_n(\cos\varphi+\sqrt{-1}\sin\varphi)}d\varphi.$$

Consequently [4]

$$|I_{2,n}|\le\int_{\varphi_n}^{\pi/2}\frac{e^{r_n\cos\varphi}}{r_n}d\varphi=\frac{1}{r_n}\int_{\varphi_n}^{\pi/2}e^{r_n\sin\left(\frac{\pi}{2}-\varphi\right)}d\varphi\le\frac{1}{r_n}\int_{\varphi_n}^{\pi/2}e^{r_n\left(\frac{\pi}{2}-\varphi\right)}d\varphi=$$

$$=\frac{1}{r_n^2}\left\{e^{r_n\left(\frac{\pi}{2}-\varphi_n\right)}-1\right\}\le\frac{1}{r_n^2}\left\{e^{r_n\frac{\pi}{2}\sin\left(\frac{\pi}{2}-\varphi_n\right)}-1\right\}=\frac{1}{r_n^2}\left\{e^{r_n\frac{\pi}{2}\cos\varphi_n}-1\right\}=$$

$$=\frac{1}{r_n^2}\left\{\left(e^{r_n\cos\varphi_n}\right)^{\frac{\pi}{2}}-1\right\}.$$

Hence, using (26), we get

$$|I_{2,n}|\le\frac{1}{r_n^2}\left\{(2r_n)^{\frac{\pi}{2}}-1\right\},$$

that yields the validity of

$$\lim_{n\to\infty}I_{2,n}=0. \tag{39}$$

Similarly, it is established that

$$\lim_{n\to\infty}I_{3,n}=0,\quad \lim_{n\to\infty}I_{4,n}=0. \tag{40}$$

Taking into account (39) and (40) in (38), we get

$$\int_{\mathcal{L}}\frac{e^{\lambda}}{\lambda^2}d\lambda=2\pi\sqrt{-1}, \tag{41}$$

that shows the validity of (36) for $t=1$. For $t>0$ and $t\neq1$, taking into account the identity

$$\int_C\frac{e^{\lambda t}}{\lambda^2}d\lambda\equiv\ t\int_C\frac{e^{\lambda}}{\lambda^2}d\lambda,\quad for\quad t>0,$$

where C is an arbitrary circle in the λ plane centered at the point $\lambda=0$, from equality (41) we get the validity of (36).

It holfs

Lemma 2. *If $\varphi(t)\in C^2([0,T])$, then it holds the following inversion formula*

$$\frac{1}{\pi\sqrt{-1}}\int_{\mathcal{L}}\left\{e^{\lambda t}\tilde{\varphi}(t,\lambda)-\frac{e^{\lambda t}}{\lambda}\varphi(0)\right\}d\lambda=\varphi(t)-2\varphi(0)\ ,\ \ 0<t\le T, \tag{42}$$

where $\tilde{\varphi}(t,\lambda)$ is form (7).

Proof. According to the lemma conditions, we have

$$\tilde{\varphi}(t,\lambda)-\frac{e^{\lambda t}}{\lambda}\varphi(0)=-\frac{1}{\lambda}\varphi(t)-\frac{\varphi'(t)}{\lambda^2}+\frac{e^{\lambda t}}{\lambda^2}\varphi'(0)+\frac{1}{\lambda^2}\int_0^t e^{\lambda(t-\tau)}\varphi''(\tau)d\tau.$$

Consequently, using (36) we have

$$\int_{\mathcal{L}}\left\{e^{\lambda t}\tilde{\varphi}(t,\lambda)-\frac{e^{\lambda t}}{\lambda}\varphi(0)\right\}d\lambda=-\pi\sqrt{-1}\varphi(t)+2\pi\sqrt{-1}\,t\varphi'(0)+$$

[4] Here we use the known inequality $\frac{2}{\pi}x\le\sin x\le x$, for $0\le x\le\pi/2$. (see for instance Fichtenholts G.M. Course of differential and integral calculus. M.: Nauka, 1969, vol.1).

$$+2\pi\sqrt{-1}\ \int_0^t (t-\tau)\,\varphi''(\tau)d\tau\,,$$

that yields the validity of lemma 2.

It holds

Lemma 3. *If $\psi(x) \in C^2([0,1])$, then it holds the following inversion formula*

$$\int_{\mathcal{L}} J(x,\lambda;\psi(x))d\lambda = -\pi\sqrt{-1}\psi(x)\ ,\quad 0<x<1, \tag{43}$$

where $J(x,\lambda;\psi(x))$ is from (32).

Proof. According to the lemma conditions, from (32) we get

$$J(x,\lambda;\psi(x)) = -\frac{1}{\lambda}\psi(x) + \frac{e^{-\lambda(1-x)}}{\lambda}\psi(1) + \frac{\sigma(\lambda)}{\lambda\cdot\rho(\lambda)}e^{-\lambda(1-x)}\psi(1)+$$
$$+\frac{e^{-\lambda(2-x)}}{\lambda\cdot\rho(\lambda)}\psi'(1) - \frac{e^{-\lambda(1-x)}}{\lambda\cdot\rho(\lambda)}\psi'(0) - \frac{1}{\lambda}\int_x^1 e^{-\lambda(\xi-x)}\psi'(\xi)d\xi -$$
$$-\frac{e^{-\lambda(1-x)}}{\lambda\cdot\rho(\lambda)}\int_0^1 e^{-\lambda\,\xi}\psi''(\xi)d\xi. \tag{44}$$

Using the Jordan lemma [2] and estimations (29) and (21), from (44) we easily get the validity of formula (43). The lemma is proved.

Remark. In lemma 2 and 3, the restrictions imposed on the functions $\varphi(t)$ and $\psi(x)$ may be weakened.

From (35) we have

$$\left\{e^{\lambda\,t}\int_0^t e^{-\lambda\,\tau}u(x,\tau)d\tau - \frac{e^{\lambda\,t}}{\lambda}u(x,0)\right\} - J(x,\lambda;u(x,t)) = \Phi_1(x,t,\lambda),$$
$$0\le x\le 1\,,\ \ 0\le t\,,\ \ \lambda\in\Omega, \tag{45}$$

where $\Phi_1(x,t,\lambda) = e^{\lambda\,t}\Phi(x,t,\lambda) - \frac{e^{\lambda\,t}}{\lambda}f(x)$.

Integrating (45) and using the inversion formulas (42) and (43) we get

$$u(x,t) = f(x) + \frac{1}{2\pi\sqrt{-1}}\int_{\mathcal{L}}\Phi_1(x,t,\lambda)d\lambda\ ,\quad 0<x<1\,,\ 0<t\le T. \tag{46}$$

Thus, we established the following

Theorem 2. *Under restrictions 3^0 and 4^0:*

1. *if problem (11)-(13) has a solution, this solution is unique and it may represented by formula (46);*
2. *if the integrals contained in (46) diverge or the function $u(x,t)$ defined by formula (46) is not the solution of problem (11)-(13), then this problem has no solution.*

Imposing the sufficient smoothness conditions on the functions $F(x,t)$, $f(x)$, $\gamma(t)$, by the method stated in [6], it is easy to be convinced that the function $u(x,t)$ determined by formula (46) is the solution of problem (11)-(13).

Remark 1. Denote by λ_k the roots of the equation

$$e^{\lambda} - \lambda = 0.$$

At the same time note that for all λ_k it holds the inequality $Re\lambda_k > 0$, and for large k they have the asymptotics

$$Re\lambda_k = \ln(2k\pi)\,;\ Im\lambda_k = \pm 2k\pi\,,\quad k = N, N+1, ...,$$

where N is a rather large natural number.

Let L be the Laplace straight line passing through the point a, where a is any positive fixed number and $Re\lambda_k \neq a$, for all λ_k.

Then integrating[5] (45) with respect to the Laplace straight line L, we have

$$\int_L e^{\lambda t} d\lambda \int_0^t e^{-\lambda\tau} u(x,\tau)d\tau - \int_L \frac{e^{\lambda t}}{\lambda} d\lambda\, u(x,0) -$$

$$- \int_L J(x,\lambda; u(x,t))d\lambda = \int_L \Phi_1(x,t,\lambda)d\lambda. \tag{47}$$

Note that

$$\int_L e^{\lambda t} d\lambda \int_0^t e^{-\lambda\tau} u(x,\tau)d\tau = \pi\sqrt{-1}\, u(x,t)\,, \quad t > 0\,;$$

$$\int_L \frac{e^{\lambda t}}{\lambda} d\lambda = 2\pi\sqrt{-1};$$

$$\int_L J(x,\lambda; u(x,t))d\lambda = \pi\sqrt{-1}u(x,t) + 2\pi\sqrt{-1}\times$$

$$\times \sum_{Re\lambda_k < a} \frac{e^{\lambda_k x}}{\lambda_k - 1} \left\{ u(0,t) - \int_0^1 e^{\lambda_k(1-\xi)} u(\xi,t)d\xi \right\}\,,\ 0 < x < 1. \tag{48}$$

Substituting (48) in (47), we get

$$u(x,0) + \sum_{Re\lambda_k < a} \frac{e^{\lambda_k x}}{\lambda_k - 1} \left\{ u(0,t) - \lambda_k \int_0^1 e^{-\lambda_k \xi} u(\xi,t)d\xi \right\} =$$

$$= -\frac{1}{2\pi\sqrt{-1}} \int_L \Phi_1(x,t,\lambda)d\lambda\,, \quad 0 < x < 1\,,\ 0 < t \le T. \tag{49}$$

(49) is the Fredholm integral equation of second kind. It is clear that it is more difficult to solve the Fredholm equation of second kind than problem 3 Consequently, for solving problem 3 it is not appropriate to solve problem 3 by using the Laplace straight line.

Remark 2. The method used here for finding the "appropriate" domain Ω and line $\mathcal{L}$ may be successfully used while solving a wide range of mixed problems for hyperbolic equations with irregular boundary conditions.

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