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ISOSPECTRALITY PROBLEM FOR DIRAC SYSTEM

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In memory of M. G. Gasymov on his 75th birthday

Abstract. In this paper, we investigate the isospectrality problem for Dirac system by using Gelfand-Levitan integral equation and transmutation operators.

1. Introduction

Inverse problems of spectral analysis has an important place. The first result about inverse problems was obtained by Ambartsumyan [1]. Sweden mathematician Borg firstly called attention to this result [2]. Borg showed that Sturm-Liouville operator didn't define with one spectrum in general case. He also showed that Sturm-Liouville operator was defined as one-to-one when different boundary conditions were satisfied. After this study, when potential provided symmetry condition $q(x) = q(\pi - x)$, N. Levinson [13, 14] proved that one spectrum defined Sturm-Liouville operator. The basic study about inverse problems of spectral analysis was done by Gelfand and Levitan [8].

For Sturm-Liouville operators, exact solution for two spectra of inverse problem was obtained by Gasymov and Levitan [16]. In this study necessary and sufficient conditions were defined for the solution of inverse problem according to two spectra. The determinated problem of regular and singular Dirac operator for two spectra was considered by Gasymov and Dzabiev [5].

Then the next years, Hochstadt [9], Levitan [15] and Panakhov [18] investigated inverse problem for Sturm-liouville and Dirac operator for partially non-coincide spectrum with different methods.

For scalar Sturm-Liouville equations, to recover potential another method was proposed by Pöschel and Trubowitz in [19]. Then Jodeit and Levitan [10, 11, 12], for constructing isospectral problems of the classical Sturm-Liouville differential equations in scalar and vectorial cases, proposed another method that is based on the Gelfand-Levitan (GL) integral equation and transmutation (transformation) operators. Chern [3] extended this idea of Jodeit and Levitan for classical Sturm-Liouville equations in vectorial cases.

In this paper we investigate the isospectral problem for Dirac operators. Research of inverse problems for Dirac operators have been investigated by many

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mathematicians (see [4, 6, 7, 17, 20]. In this paper we give a relatively proof of the GL integral equation and a proof of the existence of transmutation operator and investigate isospectral problem of Dirac operator.

2. Preliminary Results

Let $p_0(x)$ and $q_0(x)$, $0 \leq x \leq \pi$, be real differentiable functions and h_0 and H_0 fixed real numbers. Consider Dirac system

$$By' + Q_0(x)y = \lambda y,\ 0 \leq x \leq \pi, \tag{1}$$

$$y_2(0) - h_0 y_1(0) = 0, \tag{2}$$

$$y_2(\pi) + H_0 y_1(\pi) = 0.$$

Here

$$Q_0(x) = \begin{pmatrix} p_0(x) & q_0(x) \\ q_0(x) & -p_0(x) \end{pmatrix},\ B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Let $\varphi_0(x,\lambda) = (\varphi_{01}(x,\lambda),\ \varphi_{02}(x,\lambda))^\top$ be the solution of (1) which satisfies $\varphi_{01}(0,\lambda) = 1, \varphi_{02}(0,\lambda) = h_0$ initial conditions. Thus for every λ, $\varphi_0(x,\lambda)$ satisfies the first boundary condition (2). The eigenvalues of (1)-(2) are the roots of the equation

$$\varphi_{02}(\pi) + H_0 \varphi_{01}(\pi) = 0. \tag{3}$$

Corresponding eigenfunctions are $\varphi_0(x,\lambda), n \geq 0$.

Let constitute the kernel for an integral equation system, as following. The components of matrix

$$F(x,y) = \sum_{-\infty}^{\infty} \frac{1}{\alpha_{n,0}^2} \varphi_0(x,\lambda)\varphi_0(y,\lambda)^\top, \tag{4}$$

are continuous and have two continuous derivatives. Also we accept that this series is uniform convergence with respect to x and y. Integral equation system

$$K(x,y) + F(x,y) + \int_0^x K(x,t)F(t,y)dt = 0,\ 0 \leq y \leq x \leq \pi, \tag{5}$$

plays an important role in this paper. Here $F(x,y)$ is a known matrix function of y, x is a parameter and $K(x,y)$ (as a unknown matrix function of y) plays the role of the matrix function in [17].

Theorem 2.1. *Suppose that, for all $n \geq 0$,*

$$\alpha_{n,0}^2 > 0, \tag{6}$$

where

$$\alpha_{n,0}^2 = \int_0^\pi [\varphi_{01}(x,\lambda)^2 + \varphi_{01}(x,\lambda)^2]dx.$$

Then integral equation system (5) has only one solution for every x, $0 \leq x \leq \pi$ [17].

Proof. It is enough to show that the following homogeneous equation system has only trivial solution:

$$h_x(y)+\int_0^x F(t,y)h_x(t)dt=0_{2\times 2}. \tag{7}$$

Multiplying the equation system (7) with $h_x(y)$ and integrating in y from 0 to x, we obtain

$$\int_0^x h_x^2(y)dy+\int_0^x\int_0^x F(t,y)h_x(t)h_x(y)dtdy=0_{2\times 2}.$$

From (4), we obtain

$$\int_0^x h_x^2(y)dy+\int_0^x\int_0^x [\sum_{-\infty}^{\infty}\frac{1}{\alpha_{n,0}^2}\varphi_0(t,\lambda)\varphi_0(y,\lambda)^\top]h_x(t)h_x(y)dtdy=0_{2\times 2}.$$

From the definition of Dirac-Delta function

$$\int_0^x h_x^2(y)dy+\int_0^x\int_0^x I\delta(x-t)h_x(t)h_x(y)dtdy=0_{2\times 2}.$$

Therefore we deduce

$$\int_0^x h_x^2(y)dy=0_{2\times 2}.$$

Thus we obtain that $h_x(y)\equiv 0_{2\times 2}$. The proof is completed. □

Theorem 2.2. *The solution of the integral equation system (5) satisfies differential equation system*

$$B\frac{\partial K}{\partial x}+Q(x)K(x,y)=-\frac{\partial K}{\partial y}B+Q_0(y)K(x,y). \tag{8}$$

Here

$$Q(x)=Q_0(x)+BK(x,x)-K(x,x)B. \tag{9}$$

Also, $K(x,y)$ satisfies the following conditions [17]*:*

$$BK(x,x)-K(x,x)B=Q(x)-Q_0(x), \tag{10}$$

$$K_{21}(x,0)=K_{11}(x,0)=0. \tag{11}$$

Proof. Let

$$J=K(x,y)+F(x,y)+\int_0^x K(x,t)F(t,y)dt=0_{2\times 2}\ (0\le y\le x\le\pi),$$

$$\mathfrak{J}\equiv BJ_x+J_yB+Q(x)J-Q_0(y)J=0_{2\times 2}, \tag{12}$$

where $Q(x)$ has form (9) and $Q_0(x)$ is potential of the unperturbed problem (1),

$$BJ_x=B\frac{\partial K}{\partial x}+B\frac{\partial F}{\partial x}+BK(x,x)F(x,y)+\int_0^x B\frac{\partial}{\partial x}(K(x,t))F(t,y)dt, \tag{13}$$

$$J_yB=\frac{\partial K}{\partial y}B+\frac{\partial F}{\partial y}B+\int_0^x K(x,t)\frac{\partial}{\partial y}(F(t,y))Bdt. \tag{14}$$

In integral term of (14), getting $-B\frac{\partial}{\partial t}(F(t,y))$ instead of $\frac{\partial}{\partial y}(F(t,y))B$ and integrating by parts we obtain

$$-\int_0^x K(x,t)B\frac{\partial}{\partial t}(F(t,y))dt = -(K(x,t)BF(t,y))|_0^x + \int_0^x \frac{\partial}{\partial t}(K(x,t))BF(t,y)dt$$

$$= -K(x,x)BF(x,y) + K(x,0)BF(0,y) + \int_0^x \frac{\partial}{\partial t}(K(x,t))BF(t,y)dt.$$

From definition of J, we get $K(x,0) = -F(x,0)$. Thus we can write $K_{11}(x,0) = K_{21}(x,0) = 0$ and $K(x,0)BF(0,y) = 0_{2\times 2}$. From (14)

$$J_yB = \frac{\partial K}{\partial y}B + \frac{\partial F}{\partial y}B - K(x,x)BF(x,y) + \int_0^x \frac{\partial}{\partial t}(K(x,t))BF(t,y)dt. \tag{15}$$

On the other hand,

$$Q(x)J = Q(x)K + Q(x)F + Q(x)\int_0^x K(x,t)F(t,y)dt, \tag{16}$$

$$-Q_0(y)J = -Q_0(y)K - Q_0(y)F - Q_0(y)\int_0^x K(x,t)F(t,y)dt. \tag{17}$$

Replacing (13),(15),(16) and (17) in (12), we obtain

$$\begin{aligned}\mathfrak{J} \equiv 0_{2\times 2} \equiv [B\frac{\partial K}{\partial x} + \frac{\partial K}{\partial y}B + Q(x)K - Q_0(y)K] \qquad (18)\\ + \int_0^x [B\frac{\partial K}{\partial x} + \frac{\partial K}{\partial y}B + Q(x)K - Q_0(y)K]F(t,y)dt.\end{aligned}$$

On the other hand, $K(x,y)$ satisfies conditions (10) and (11). Equation system (18) means that matrix function

$$h_x(y) = B\frac{\partial K}{\partial x} + \frac{\partial K}{\partial y}B + Q(x)K - Q_0(y)K,$$

is a solution of homogeneous integral equation system and therefore is equal to zero matrix. This proves that the differential equation system (8) holds. □

Theorem 2.3. *If $K(x,y)$ is a solution of integral equation system (5), is, thus, because of theorem 2.2, a solution of the problem (8)-(10)-(11),*
i) for every complex λ, the vector-valued function

$$\varphi(x,\lambda) = \varphi_0(x,\lambda) + \int_0^x K(x,t)\varphi_0(t,\lambda)dt, \tag{19}$$

is a solution of the differential equation system

$$By' + Q(x)y = \lambda y,\ 0 \le x \le \pi, \tag{20}$$

where

$$Q(x) = Q_0(x) + BK(x,x) - K(x,x)B;$$

ii) vector-valued function $\varphi(x,\lambda)$ satisfies the initial conditions

$$\varphi_1(0,\lambda) = 1, \varphi_2(0,\lambda) = h_0 = h. \tag{21}$$

Proof. The proof of *i)* is simple [17]. Let prove to *ii)*.

From (19) we get

$$\varphi_1(x,\lambda)=\varphi_{01}(x,\lambda)+\int_0^x[K_{11}(x,t)\varphi_{01}(t,\lambda)+K_{12}(x,t)\varphi_{02}(t,\lambda)]dt,$$

$$\varphi_2(x,\lambda)=\varphi_{02}(x,\lambda)+\int_0^x[K_{21}(x,t)\varphi_{01}(t,\lambda)+K_{22}(x,t)\varphi_{02}(t,\lambda)]dt.$$

Getting $x=0$ in above equations, we obtain $\varphi_1(0,\lambda)=\varphi_{01}(0,\lambda)$, $\varphi_2(0,\lambda)=\varphi_{02}(0,\lambda)$ since $\varphi_0(x,\lambda)$ is a solution of (1)-(2), we follow the initial conditions (21). □

Theorem 2.4. *Let λ_n, $n\geq 0$, be eigenvalues of unperturbed problem (1)-(2). Thus vector-valued function $\varphi(x,\lambda)$, λ_n instead of λ, described by (19), can be expressed by the formula*

$$\varphi(x,\lambda_n)=\varphi_0(x,\lambda_n)-\sum_{k=-\infty}^{\infty}c_k\varphi(x,\lambda_k)\int_0^x\varphi_0(t,\lambda_k)^\top\varphi_0(t,\lambda_n)dt. \tag{22}$$

Proof. From (4) and integral equation system (5) we obtain

$$K(x,t)=-F(x,t)-\int_0^x K(x,s)F(s,t)ds,$$

$$K(x,t)=-\sum_{-\infty}^{\infty}c_k[\varphi_0(x,\lambda_k)+\int_0^x K(x,s)\varphi_0(s,\lambda_k)ds]\varphi_0(t,\lambda_k)^\top.$$

If we consider (19), we obtain

$$K(x,t)=-\sum_{-\infty}^{\infty}c_k\varphi(x,\lambda_k)\varphi_0(t,\lambda_k). \tag{23}$$

Getting λ_n instead of λ in (19)

$$\varphi(x,\lambda_n)=\varphi_0(x,\lambda_n)+\int_0^x K(x,t)\varphi_0(t,\lambda_n)dt.$$

Therefore from (23) it follows that

$$\varphi(x,\lambda_n)=\varphi_0(x,\lambda_n)-\sum_{k=-\infty}^{\infty}c_k\varphi(x,\lambda_k)\int_0^x\varphi_0(t,\lambda_k)^\top\varphi_0(t,\lambda_n)dt.$$

The proof is completed. □

The formulas (19) and (22) allow us to consider the isospectrality problem. We suppose that h_0 and H_0 are finite. As shown in Theorem 2.3 vector-valued functions $\varphi(x,\lambda_n)$, $n\geq 0$ are solutions of equation system (20) and satisfy conditions (21) and thus the boundary condition

$$\varphi_2(0,\lambda_n)-h\varphi_1(0,\lambda_n)=0,$$

where

$$h = h_0. \tag{24}$$

To obtain an operator isospectrality with operator (1)-(2) it is necessary to show that for some real H, $\varphi(x, \lambda_n)$ satisfies, at the point $x = \pi$, the boundary condition

$$\varphi_2(\pi, \lambda_n) + H\varphi_1(\pi, \lambda_n) = 0.$$

At the same time we will obtain a formula for H.

Put $x = \pi$ into (22) it follows that

$$\varphi(\pi, \lambda_n) = \varphi_0(\pi, \lambda_n) - \sum_{k=-\infty}^{\infty} c_k \varphi(\pi, \lambda_k) \int_0^{\pi} \varphi_0(t, \lambda_k)^{\top} \varphi_0(t, \lambda_n) dt,$$

$$\varphi(\pi, \lambda_n) = \varphi_0(\pi, \lambda_n) - c_n \alpha_{n,0}^2 \varphi(\pi, \lambda_n), \tag{25}$$

where

$$\alpha_{n,0}^2 = \int_0^{\pi} \varphi_0(t, \lambda_n)^{\top} \varphi_0(t, \lambda_n) dt = \int_0^{\pi} [\varphi_{01}^2(t, \lambda_n) + \varphi_{02}^2(t, \lambda_n) dt.$$

From formula (25) we obtain

$$\varphi(\pi, \lambda_n) = \frac{\varphi_0(\pi, \lambda_n)}{1 + c_n \alpha_{n,0}^2} \tag{26}$$

and from here

$$\varphi_1(\pi, \lambda_n) = \frac{\varphi_{01}(\pi, \lambda_n)}{1 + c_n \alpha_{n,0}^2}, \ \ \varphi_2(\pi, \lambda_n) = \frac{\varphi_{02}(\pi, \lambda_n)}{1 + c_n \alpha_{n,0}^2}.$$

Since $\varphi_{02}(\pi, \lambda_n) + H_0 \varphi_{01}(\pi, \lambda_n) = 0$ getting $\varphi_{02}(\pi, \lambda_n) = -H_0 \varphi_{01}(\pi, \lambda_n) = 0$ we deduce

$$\varphi_2(\pi, \lambda_n) = -H_0 \frac{\varphi_{01}(\pi, \lambda_n)}{1 + c_n \alpha_{n,0}^2} = -H_0 \varphi_1(\pi, \lambda_n),$$

$$\varphi_2(\pi, \lambda_n) + H_0 \varphi_1(\pi, \lambda_n) = 0.$$

Therefore we obtain

$$H = H_0.$$

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