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USING OF B.M. LEVITAN AND M.G. GASYMOV'S SOLVABILITY THEOREM TO THE INVERSE PROBLEM WITH NONSEPARATED BOUNDARY CONDITIONS

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In memory of M. G. Gasymov on his 75th birthday

Abstract. To uniquely reconstruct the inverse self-adjoint Sturm-Liouville problem with the real numbers in the nonseparated boundary conditions, in addition to the spectrum of the problem itself, the spectra of additional boundary value problems, and a certain sequence of signs were used before. If the problems considered in the work of Levitan and Gasymov are used instead of the additional problem, then the inverse self-adjoint Sturm-Liouville problem with nonseparated boundary conditions can be uniquely reconstructed by a few number of spectral data, that is by two spectra and two eigenvalues. Uniqueness theorems and a solvability theorem are proved. The corresponding examples and counterexample are considered.

1. Introduction

Let L denote the Sturm-Liouville problem

$$ly = -y'' + q(x)\,y = \lambda\,y = s^2\,y, \tag{1.1}$$

$$U_i(y) = a_{i1}\,y(0) + a_{i2}\,y'(0) + a_{i3}\,y(\pi) + a_{i4}\,y'(\pi) = 0, \qquad i = 1, 2, \tag{1.2}$$

where $q(x)$is a real continous function on $[0, \pi]$; and a_{ij} with $i = 1,\ 2,\ j = 1,\ 2,\ 3,\ 4$ are complex constants.

The inverse Sturm-Liouville problem for L in the case of separated boundary conditions ($a_{13} = a_{14} = a_{21} = a_{22} = 0$) was first considered in [2, 3, 7, 22] and has been well studied since then (see [8, 9, 10, 11, 15, 20, 24]). The inverse problem with asymmetric and symmetric potentials and nonseparated boundary conditions was studied by M. G. Gasymov, I. M. Guseinov, V. A. Marchenko, I. M. Nabiev, O. A. Plaksina, V. A. Sadovnichii, I. V. Stankevich, V. A. Yurko, and other authors (see [1, 4, 5, 6, 12, 14, 16, 17, 18, 19, 21, 23, 25]).

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Note that general self-adjoint nonseparated boundary conditions (1.2) can be reduced to one of the two following types:

(i) boundary conditions

$$V_1(y) = a_{11}\, y(0) + y'(0) + a_{13}\, y(\pi) = 0, \tag{1.3}$$

$$V_2(y) = a_{21}\, y(0) + a_{23}\, y(\pi) + y'(\pi) = 0, \tag{1.4}$$

where a_{11} and a_{23} are any real numbers, $a_{13} \neq 0$ is any complex number, and $a_{21} = -\overline{a_{13}}$;

(ii) boundary conditions

$$P_1(y) = y(0) + \omega\, y(\pi) = 0, \tag{1.5}$$

$$P_2(y) = \overline{\omega}\, y'(0) + y'(\pi) + \alpha\, y(\pi) = 0, \tag{1.6}$$

where $\omega \neq 0$ is any complex number and α is any real number.

To uniquely reconstruct the boundary value problems with self-adjoint nonseparated boundary conditions (1.3), (1.4), in addition to the spectrum of the problem itself, the spectra of two more boundary value problems, a certain sequence of signs, and a certain real number were used (see, e.g., [14, 25]).

Let Y_1 and Y_2 denote the following spectral Sturm-Liouville problems.

Problem Y_1 :

$$-y''+q(x)\,y = \lambda\, y,\;\; a_{11}\, y(0)+y'(0)+a_{13}\, y(\pi) = 0,\;\; -a_{13}\, y(0)+a_{23}\, y(\pi)+y'(\pi) = 0.$$

Problem Y_2 :

$$-y'' + q(x)\, y = \lambda\, y, \quad a_{11}\, y(0) + y'(0) = 0, \quad y(\pi) = 0.$$

Here a_{11}, a_{13}, and a_{23} are real numbers.

The boundary conditions of Problem Y_1 are a special case of boundary conditions (1.3), (1.4).

In [25], to uniquely reconstruct Problem Y_1, in addition to the spectrum of the problem itself, the spectra $\{z_n\}$ of Problem Y_2, the sequence of signs $\omega_n = \operatorname{sign}(|\theta'(\pi, z_n)| - |a_{13}|)$, where $\theta(x, \lambda)$ is the solution of equation (1.2) with the boundary conditions $\theta(0, \lambda) = 1$, $\theta'(0, \lambda) = -a_{11}$, were used.

As showed below, if the problems considered in the work of Levitan and Gasymov are used instead of the additional problem Y_2 with separated boundary conditions, then the problem Y_1 can be uniquely reconstructed by a few number of spectral data, that is by two spectra and two eigenvalues. Two eigenvalues can be used instead of infinite consequence of sigs mentioned in [25].

2. Uniqueness of reconstruction Problem Y_1 from three spectra

In what follows, we denote a problem of type L but with different coefficients in the equation and different parameters in the boundary forms by $\widetilde{\mathrm{L}}$. Throughout the paper, we assume that if some symbol denotes an object from Problem L then the same symbol with the tilde $\widetilde{\ }$ denotes the corresponding object from Problem $\widetilde{\mathrm{L}}$.

Along with Problem Y_1, we consider the following two problems with decomposable boundary conditions.

Problem $\mathbf{G}_1$:

$$\begin{aligned} ly &= -y'' + q(x)\,y = \lambda\,y, \\ U_{1,1}(y) &= a_{11}\,y(0) + y'(0) = 0, \\ U_{2,1}(y) &= a_{23}\,y(\pi) + y'(\pi) = 0. \end{aligned}$$

Problem $\mathbf{G}_2$:

$$\begin{aligned} ly &= -y'' + q(x)\,y = \lambda\,y, \\ U_{1,1}(y) &= a\,y(0) + y'(0) = 0, \qquad a \neq a_{11}, \\ U_{2,1}(y) &= a_{23}\,y(\pi) + y'(\pi) = 0. \end{aligned}$$

Theorem 2.1. *If the eigenvalues of Problems Y_1 and $\widetilde{Y}_1$, G_1 and $\widetilde{G}_1$, G_2 and $\widetilde{G}_2$ coincide, and their respective algebraic multiplicities coincide, then these boundary value problems coincide as well, i.e., $q(x) = \widetilde{q}(x)$, $a_{11} = \widetilde{a}_{11}$, $a_{13} = \widetilde{a}_{13}$, $a_{23} = \widetilde{a}_{23}$.*

Proof of Theorem 2.1. When applying Borg's uniqueness theorem [3], [10, c. 9] to Problems G_1 and G_2, we see that

$$q(x) = \widetilde{q}(x), \qquad a_{11} = \widetilde{a}_{11}, \qquad a_{23} = \widetilde{a}_{23}, \qquad a = \widetilde{a}. \tag{2.1}$$

Let us demonstrate that $a_{13} = \widetilde{a}_{13}$.

Let $y_1(x,\,\lambda)$ and $y_2(x,\,\lambda)$ be linearly independent solutions of equation (1.1) satisfying the conditions

$$y_1(0,\,\lambda) = 1, \quad y_1'(0,\,\lambda) = 0, \quad y_2(0,\,\lambda) = 0, \quad y_2'(0,\,\lambda) = 1. \tag{2.2}$$

Then we have the asymptotic formulas

$$\begin{aligned} y_1(x,\,\lambda) &= \cos sx + \tfrac{1}{s}\,u(x)\,\sin sx + O\left(\tfrac{1}{s^2}\right), \\ y_2(x,\,\lambda) &= \tfrac{1}{s}\,\sin sx - \tfrac{1}{s^2}\,u(x)\,\cos sx + O\left(\tfrac{1}{s^3}\right), \\ y_1'(x,\,\lambda) &= -s\,\sin sx + u(x)\,\cos sx + O\left(\tfrac{1}{s}\right), \\ y_2'(x,\,\lambda) &= \cos sx + \tfrac{1}{s}\,u(x)\,\sin sx + O\left(\tfrac{1}{s^2}\right), \end{aligned} \tag{2.3}$$

where $u(x) = \frac{1}{2}\int_0^x q(t)\,dt$, for sufficiently large $\lambda = s^2 \in \mathbb{R}$ ([13, p. 62–65]).

The eigenvalues λ_k of problem Y_1 are the roots of the entire function ([14])

$$\Delta(\lambda) = 2\,a_{13} - a_{23}\,y_1(\pi,\,\lambda) - y_1'(\pi,\,\lambda) + (a_{11}\,a_{23} + a_{13}^2)\,y_2(\pi,\,\lambda) + a_{11}\,y_2'(\pi,\,\lambda), \tag{2.4}$$

and the following are true:

$$\begin{gathered} \lambda_k = k^2 + \pi^{-1}\left(2\,b + (-1)^{k+1} 4\,a_{13}\right) + \sigma_k, \\ \{\sigma_k\} \in l_2, \quad \lambda_k \leq \lambda_{k+1}, \quad \lambda_k < \lambda_{k+2}, \quad b = -a_{11} + a_{23} + \tfrac{1}{2}\int_0^\pi q(t)\,dt. \end{gathered} \tag{2.5}$$

Substituting the asymptotic formulas for $y_1(x,\lambda)$ and $y_2(x,\lambda))$ in (2.4) yields

$$\Delta(\lambda) = 2\,a_{13} - a_{23}\,\cos\sqrt{\lambda}\pi + \sqrt{\lambda}\,\sin\sqrt{\lambda}\pi - u(\pi)\,\cos\sqrt{\lambda}\pi + a_{11}\,\cos\sqrt{\lambda}\pi + O\left(\frac{1}{\sqrt{\lambda}}\right).$$

Similarly, we have

$$\widetilde{\Delta}(\lambda) = 2\,\widetilde{a}_{13} - \widetilde{a}_{23}\,\cos\sqrt{\lambda}\pi + \sqrt{\lambda}\,\sin\sqrt{\lambda}\pi - \widetilde{u}(\pi)\,\cos\sqrt{\lambda}\pi + \widetilde{a}_{11}\,\cos\sqrt{\lambda}\pi + O\left(\frac{1}{\sqrt{\lambda}}\right).$$

It can be observed that $\Delta(\lambda)$ and $\widetilde{\Delta}(\lambda)$ are an entire function of order 1/2. Besides, according to the assumptions of the theorem, the eigenvalues of Y_1 and $\widetilde{\mathrm{Y}}_1$ coincide and their corresponding algebraic multiplicities are equal. Therefore, the Hadamard factorization theorem implies that $\Delta(\lambda) \equiv C\,\widetilde{\Delta}(\lambda)$, where C is a nonzero constant. It follows that

$$\begin{aligned}\Delta(\lambda) - C\widetilde{\Delta}(\lambda) \ \equiv\ & 2\,(a_{13} - C\,\widetilde{a}_{13}) - (a_{23} - C\,\widetilde{a}_{23})\,\cos\sqrt{\lambda}\pi + (1-C)\,\sqrt{\lambda}\,\sin\sqrt{\lambda}\pi \\ & -(u(\pi) - C\,\widetilde{u}(\pi))\,\cos\sqrt{\lambda}\pi + (a_{11} - C\,\widetilde{a}_{11})\,\cos\sqrt{\lambda}\pi + (1-C)\,O\left(\tfrac{1}{\sqrt{\lambda}}\right) \ \equiv\ 0.\end{aligned} \tag{2.6}$$

Here, 1, $\sin\sqrt{\lambda}\pi$, $\cos\sqrt{\lambda}\pi$, $\sqrt{\lambda}\cdot\sin\sqrt{\lambda}\pi$, $O\left(\frac{1}{\sqrt{\lambda}}\right)$ are linearly independent functions of λ. (This can easily be verified using the definition of linearly independent functions.) Therefore, $C = 1$ and

$$2\,(a_{13} - \widetilde{a}_{13}) + O\left(\frac{1}{\lambda}\right) \equiv 0. \tag{2.7}$$

Then we have $a_{13} = \widetilde{a}_{13}$. □

Remark 2.1. Borg's Theorem [10, p. 9]) is a special case of Theorem 2.1. Indeed, in the case of separated conditions ($a_{12} = a_{21} = 0$), problem Y_1 coincides with G_1. Therefore, problems Y_1=G_1 and G_2 can only be uniquely reconstructed using two spectra (namely, those of Y_1=G_1 and G_2).

Theorem 2.1 will be used to prove theorems of unique reconstruction of problem Y_1 from two spectra and one or two eigenvalues.

3. The uniqueness of reconstructing Problem Y_1 from two spectra and one or two eigenvalues

There are theorems stronger than Theorem 2.1 which also are true. They are based on the fact that equation $\Delta(\lambda) = 0$ with respect to unknown coefficient a_{13} is quadratic.

Let λ_1 be the eigenvalue of problem Y_1, and $\widetilde{\lambda}_1$ be the eigenvalue of problem $\widetilde{\mathrm{Y}}_1$.

Theorem 3.1. *Let* $\lambda_1 = \widetilde{\lambda}_1$. *If the eigenvalues of problems* G_1 *and* $\widetilde{\mathrm{G}}_1$, G_2 *and* $\widetilde{\mathrm{G}}_2$ *coincide and their respective algebraic multiplicities coincide as well, and besides at least one of the conditions are satisfied:*

$$y_2(\pi, \lambda_1) = 0; \tag{3.1}$$

$$y_2(\pi,\, \lambda_1)\,\Big(a_{11}\,a_{23}\,y_2(\pi,\, \lambda_1) + a_{11}\,y_2'(\pi,\, \lambda_1) - a_{23}\,y_1(\pi,\, \lambda_1) - y_1'(\pi,\, \lambda_1)\Big) = 1; \tag{3.2}$$

$$\Delta(\lambda_1) = 0, \quad \frac{\Delta(\lambda)}{d\lambda}\Big|_{\lambda=\lambda_1} = 0, \quad \frac{y_2(\pi,\lambda)}{d\lambda}\Big|_{\lambda=\lambda_1} \neq 0. \tag{3.3}$$

Then $q(x) = \widetilde{q}(x)$, $a_{11} = \widetilde{a}_{11}$, $a_{13} = \widetilde{a}_{13}$, $a_{23} = \widetilde{a}_{23}$.

Proof of Theorem 3.1. When applying Borg's uniqueness theorem ([3], [10, c. 9]) to problems G_1 and G_2 similarly to Theorem 3.1, we see that:

$$q(x) = \widetilde{q}(x), \qquad a_{11} = \widetilde{a}_{11}, \qquad a_{23} = \widetilde{a}_{23}, \qquad a = \widetilde{a}.$$

Let us demonstrate that $a_{13} = \widetilde{a}_{13}$.

1) Let us assume that condition (3.1) is satisfied. Condition (3.1) means that the coefficient at a_{13}^2 in (2.4) equals zero (the square equation degenerates into a linear equation). So, from (2.4), (3.1) and equations $\lambda_1 = \widetilde{\lambda}_1$, $\Delta(\lambda_1) = \widetilde{\Delta}(\widetilde{\lambda}_1) = 0$, we have

$$2\, a_{13} = 2\, \widetilde{a}_{13} = a_{23}\, y_1(\pi,\, \lambda_1) + y_1'(\pi,\, \lambda) + a_{11}\, y_2'(\pi,\, \lambda_1). \tag{3.4}$$

Hence from (3.4), we obtain $a_{13} = \widetilde{a}_{13}$. Thus, the theorem is proved for condition (3.1).

2) Let us assume that condition (3.2) is satisfied. From (2.4) we have

$$2\, a_{13} - a_{23}\, y_1(\pi,\, \lambda_1) - y_1'(\pi,\, \lambda_1) + (a_{11}\, a_{23} + a_{13}^2)\, y_2(\pi,\, \lambda_1) + a_{11}\, y_2'(\pi,\, \lambda_1) = 0. \tag{3.5}$$

Equation (3.5) means that the coefficient at a_{13}^2 does not equal zero (the quadratic equation does not degenerate into a linear equation) and the discriminant of equation (3.5) equals zero. Consequently, we obtain uniqueness solution

$$a_{13} = -\frac{1}{y_2(\pi,\, \lambda_1)}. \tag{3.6}$$

Thus, the theorem is proved for condition (3.2).

3) Let us assume that condition (3.3) is satisfied. From condition (3.3) and equations (2.4), (3.3), we conclude that

$$\begin{gathered} 2\,(a_{13} - \widetilde{a}_{13}) + (a_{13}^2 - \widetilde{a}_{13}^2)\, y_2(\pi,\, \lambda_1) = 0, \\ (a_{13}^2 - \widetilde{a}_{13}^2)\, \frac{y_2(\pi,\lambda)}{d\lambda}\bigg|_{\lambda=\lambda_1} = 0. \end{gathered} \tag{3.7}$$

From condition $\frac{y_2(\pi,\lambda)}{d\lambda}\Big|_{\lambda=\lambda_1} \neq 0$ and equations (3.7), we obtain $a_{13} = \widetilde{a}_{13}$.

Thus, the theorem is proved for condition (3.3). □

Let λ_1 and λ_2 be arbitrary eigenvalues of Problem Y_1, and $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ be the corresponding eigenvalues of Problem $\widetilde{Y}_1$.

Theorem 3.2. *Let $\lambda_1 = \widetilde{\lambda}_1$ and $\lambda_2 = \widetilde{\lambda}_2$. If the eigenvalues of problems G_1 and $\widetilde{G}_1$, G_2 and $\widetilde{G}_2$ coincide and their respective algebraic multiplicities coincide as well, and besides the condition*

$$y_2(\pi, \lambda_2) - y_2(\pi, \lambda_1) \neq 0, \tag{3.8}$$

is satisfied, then $q(x) = \widetilde{q}(x)$, $a_{11} = \widetilde{a}_{11}$, $a_{13} = \widetilde{a}_{13}$, $a_{23} = \widetilde{a}_{23}$.

Proof of Theorem 3.2. From equation (2.4), we have

$$\begin{gathered} 2\,(a_{13} - \widetilde{a}_{13}) + (a_{13}^2 - \widetilde{a}_{13}^2)\, y_2(\pi,\, \lambda_1) = 0, \\ 2\,(a_{13} - \widetilde{a}_{13}) + (a_{13}^2 - \widetilde{a}_{13}^2)\, y_2(\pi,\, \lambda_2) = 0. \end{gathered} \tag{3.9}$$

From (3.8) we obtain the solution of this system of the linear algebraic equations with two unknown $(a_{13} - \widetilde{a}_{13})$ and $(a_{13}^2 - \widetilde{a}_{13}^2)$, and the solution is unique. So $a_{13} = \widetilde{a}_{13}$. □

Theorem 3.3. *Let us assume that eigenvalues of Problems* G_1 *and* $\widetilde{\mathrm{G}}_1$, G_2 *and* $\widetilde{\mathrm{G}}_2$ *coincide and their respective algebraic multiplicities coincide as well. Then among the eigenvalues of Problem* Y_1 *there will be one eigenvalue* $\lambda_1 = \widetilde{\lambda}_1$*, which satisfies one of conditions (3.1), (3.2), or (3.3), or two eigenvalue* $\lambda_1 = \widetilde{\lambda}_1$, $\lambda_2 = \widetilde{\lambda}_2$*, which satisfy condition (3.8).*

Proof of Theorem 3.3. Assume the converse. Then none of conditions (3.1), (3.2), (3.3) (3.8) is satisfied. Problem Y_1 has infinite set of eigenvalues and they all are either simple or twofold eigenvalues, which is seen in (2.5). Let conditions (3.1), (3.2), (3.3) and (3.8) be not satisfied for all simple eigenvalues of Problem Y_1, then for all simple eigenvalues of Problem Y_1 the following conditions are satisfied:

$$y_2(\pi, \lambda_1) \neq 0, \quad y_2(\pi, \lambda_2) \neq 0, \quad y_2(\pi, \lambda_2) - y_2(\pi, \lambda_1) = 0, \tag{3.10}$$

and for all twofold eigenvalues λ_i the following conditions are satisfied:

$$y_2(\pi, \lambda_i) \neq 0, \quad \frac{y_2(\pi, \lambda)}{d\lambda}\Big|_{\lambda=\lambda_i} = 0. \tag{3.11}$$

When applying Borg's uniqueness theorem [3], [10, p. 9] to Problems G_1 and G_2 similarly in Theorem 2.1, we obtain (2.1). From the aforementioned and (3.10), (3.11) we get that for all eigenvalues λ_i of Problem Y_1 equations $\Delta(\lambda_i) - \widetilde{\Delta}(\lambda_i) = 0$ equal the following equation

$$2\,(a_{13} - \widetilde{a}_{13}) + (a_{13}^2 - \widetilde{a}_{13}^2)\, y_2(\pi,\, \lambda_1) = 0 \qquad (y_2(\pi,\, \lambda_1) \neq 0), \tag{3.12}$$

and for all any twofold eigenvalues λ_i, the equations $\frac{\Delta(\lambda)}{d\lambda}\Big|_{\lambda=\lambda_i} - \frac{\Delta(\lambda)}{d\lambda}\Big|_{\lambda=\lambda_i} = 0$ equal the following equation

$$(a_{13}^2 - \widetilde{a}_{13}^2)\, \frac{y_2(\pi, \lambda)}{d\lambda}\Big|_{\lambda=\lambda_i} = 0 \qquad \left(y_2(\pi, \lambda_i) \neq 0, \quad \frac{y_2(\pi, \lambda)}{d\lambda}\Big|_{\lambda=\lambda_i} = 0 \right). \tag{3.13}$$

Equations (3.12) and (3.13) are satisfied if and only if $a_{13} = \widetilde{a}_{13}$ and $a_{13} = -\widetilde{a}_{13} - \frac{2}{y_2(\pi,\, \lambda_i)}$. If $a_{13} \neq -\frac{1}{y_2(\pi,\, \lambda_1)}$, then these values do not coincide and are two different solutions for a_{13}.

Let us demonstrate that $a_{13} \neq -\frac{1}{y_2(\pi,\, \lambda_1)}$. Assume the converse. Then $a_{13} = -\frac{1}{y_2(\pi,\, \lambda_1)}$. Since (3.1), (3.2) are not true, it follows that

$$\begin{gathered} y_2(\pi, \lambda_1) \neq 0, \\ y_2(\pi,\, \lambda_1) \Big(a_{11}\, a_{23}\, y_2(\pi,\, \lambda_1) + a_{11}\, y_2'(\pi,\, \lambda_1) - a_{23}\, y_1(\pi,\, \lambda_1) - y_1'(\pi,\, \lambda_1) \Big) \neq 1. \end{gathered} \tag{3.14}$$

This inequalities demonstrate that the equation discriminant square relative to a_{13} differs from zero.) As λ_1 is the eigenvalue of Problem Y_1, then it is the root of characteristic determinant $\Delta(\lambda)$ and satisfies equation (3.5). Substituting $-\frac{1}{y_2(\pi,\, \lambda_1)}$ for a_{13} in (3.5), we get

$$a_{11}\, a_{23}\, y_2(\pi,\, \lambda_1) + a_{11}\, y_2'(\pi,\, \lambda_1) - a_{23}\, y_1(\pi,\, \lambda_1) - y_1'(\pi,\, \lambda_1) = \frac{1}{y_2(\pi,\, \lambda_1)}.$$

The last equation contradicts the inequalities (3.14).

Thus, for all eigenvalues λ_i the equations $\Delta(\lambda_i) = 0$ are satisfied with two different values a_{13}. This contradicts Theorem 2.1 about uniqueness of reconstruction of Problem Y_1 by all eigenvalues. This contradiction proves the theorem. Thus, among the eigenvalues of Problem Y_1 there will be one eigenvalue $\lambda_1 = \widetilde{\lambda}_1$, which satisfies one of conditions (3.1), (3.2), or (3.3), or two eigenvalue $\lambda_1 = \widetilde{\lambda}_1$, $\lambda_2 = \widetilde{\lambda}_2$, which satisfy condition (3.8). □

4. Solvability of the Inverse Problem from two spectra and two eigenvalues

The main question in the paragraph is as follows.

Solvability question for the inverse problem. Given two real numbers λ_1 and λ_2 and two sequences of real numbers μ_k and ν_k, do there exist an absolutely continuous function $q(x)$ and numbers a, a_{11}, a_{13}, and a_{23} such that $\{\mu_k\}$ is the spectrum of Problem G_1, $\{\nu_k\}$ is the spectrum of Problem G_2, and the numbers λ_1 and λ_2 are the eigenvalues of problem Y_1?

Suppose that sequences of real numbers μ_k and ν_k satisfy the following two conditions.

Condition 1. The numbers μ_k and ν_k alternate, i.e., $\mu_0 < \nu_0 < \mu_1 < \nu_1 < \mu_2 < \nu_2 < \dots$ (or $\nu_0 < \mu_0 < \nu_1 < \mu_1 < \nu_2 < \mu_2 < \dots$).

Condition 2. The following asymptotic formulas hold:

$$\mu_k = k^2 + b_0 + o\,(1)\,, \quad \nu_k = k^2 + b_0' + o\,(1)\,,$$

moreover, $b_0' \neq b_0$.

Condition 3. The function

$$\Phi(x) = \frac{2}{\pi} \sum_{k=1}^{\infty} \left(\frac{\nu_k - \mu_k}{b_0' - b_0} \cos \sqrt{\mu_k} - \cos kx \right)$$

has integrable derivative.

When applying Theorem 3.4.2 from [9, p. 58] to Problems G_1 and G_2, we obtain the following solvability theorem for the inverse problem.

Lemma 4.1. *Two sequences of real numbers μ_k and ν_k are the eigenvalues of Problems G_1 and G_2, respectively, if and only if Conditions 1, 2, and 3 are satisfied.*

To prove the solvability of the inverse problem stated above, it remains to show that the coefficient a_{13} can be found. Let us demonstrate this.

It is already shown that $q(x)$ can be found; hence, we can consider solutions of equation (1.1).

Let $y_1(x,\, \lambda)$ and $y_2(x,\, \lambda)$ be linearly independent solutions of equation (1.1) satisfying conditions (2.2).

If the values λ_1 and λ_2 are the roots of the equation

$$2\, a_{13} - a_{23}\, y_1(\pi,\, \lambda_i) - y_1'(\pi,\, \lambda_i) + (a_{11}\, a_{23} + a_{13}^2)\, y_2(\pi,\, \lambda_i) + a_{11}\, y_2'(\pi,\, \lambda_i) = 0, \tag{4.1}$$

then the values λ_1 and λ_2 are the roots of the characteristic determinant $\Delta(\lambda)$ and the eigenvalues of Problem G_1. So, to prove the unique solvability of the Inverse

Problem we need to prove the unique solvability of equations (4.1) relative to unknown value a_{13}.

Suppose values a_{11}, a_{23} and function $q(x)$ are reconstructed. Then linearly independent solutions $y_1(x,\,\lambda)$ and $y_2(x,\,\lambda)$ of equation (1.1) under conditions (2.2) are known. So we can set the following conditions.

Condition 4. Numbers λ_1 and λ_2 satisfy equation (4.1). Besides the number λ_1 or λ_2 satisfies at least one of conditions (3.1), (3.2), or (3.3).

Condition 5. Numbers λ_1 and λ_2 satisfy equation (4.1), inequality (3.8), and the condition $D_1^2 = D\cdot D_2$, where

$$D = \begin{vmatrix} 2 & y_2(\pi,\,\lambda_1) \\ 2 & y_2(\pi,\,\lambda_2) \end{vmatrix}, \tag{4.2}$$

$$D_1 = \begin{vmatrix} a_{23}\,y_1(\pi,\,\lambda_1) + y_1'(\pi,\,\lambda_1) - a_{11}\,a_{23}\,y_2(\pi,\,\lambda_1) - a_{11}\,y_2'(\pi,\,\lambda_1) & y_2(\pi,\,\lambda_1) \\ a_{23}\,y_1(\pi,\,\lambda_2) + y_1'(\pi,\,\lambda_2) - a_{11}\,a_{23}\,y_2(\pi,\,\lambda_2) - a_{11}\,y_2'(\pi,\,\lambda_2) & y_2(\pi,\,\lambda_2) \end{vmatrix}, \tag{4.3}$$

$$D_2 = \begin{vmatrix} 2 & a_{23}\,y_1(\pi,\,\lambda_1) + y_1'(\pi,\,\lambda_1) - a_{11}\,a_{23}\,y_2(\pi,\,\lambda_1) - a_{11}\,y_2'(\pi,\,\lambda_1) \\ 2 & a_{23}\,y_1(\pi,\,\lambda_2) + y_1'(\pi,\,\lambda_2) - a_{11}\,a_{23}\,y_2(\pi,\,\lambda_2) - a_{11}\,y_2'(\pi,\,\lambda_2) \end{vmatrix}. \tag{4.4}$$

The application of Lemma 4.1 and Theorem 3.3 yields

Theorem 4.1. *If two sequences of real numbers μ_k, ν_k satisfy Conditions 1, 2 and 3, and two real numbers λ_1, λ_2 satisfy Conditions 4 or 5, then there exists a unique Problem Y_1 (with an absolutely continuous function $q(x)$ and numbers a, a_{11}, a_{13}, a_{23}) such that $\{\mu_k\}$ is the spectrum of Problem G_1, $\{\nu_k\}$ is the spectrum of Problem G_2, and numbers λ_1, λ_2 are the eigenvalues of Problem Y_1.*

Proof of Theorem 4.1. If two sequences of real numbers μ_k, ν_k satisfy Conditions 1, 2 and 3, then by Lemma 4.1 there exists a unique Problem G_1 and a unique Problem G_2 with an absolutely continuous function $q(x)$ and numbers a, a_{11}, and a_{13}. To prove the solvability of the inverse problem stated above, it remains to show that the coefficient a_{13} can be found. Let us show this. It is already shown that $q(x)$ can be found; hence, we can consider solutions of equation (1.1). Let $y_1(x,\,\lambda)$ and $y_2(x,\,\lambda)$ be linearly independent solutions of equation (1.1), satisfying conditions (2.2). With the help of $y_1(x,\,\lambda)$ and $y_2(x,\,\lambda)$ we write system (4.1) of two equations with one unknown a_{13}. Suppose the numbers λ_1 and λ_2 satisfy Condition 4 or 5. Then equations (4.1) mean that numbers λ_1 and λ_2 are eigenvalues of Problem Y_1 with one unknown coefficient a_{13}. If the numbers λ_1 and λ_2 satisfy at least one of conditions (3.1), (3.2), (3.3), or (3.8), then coefficient a_{13} is uniquely determined by the numbers λ_1 and λ_2. The formula for a_{13} depends on what kind of conditions (3.1), (3.2), (3.3) or (3.8) is satisfied.

1) If λ_1 satisfies (3.1), then using (4.1), we get

$$a_{13} = \frac{1}{2}\Big(a_{23}\,y_1(\pi,\,\lambda_1) + y_1'(\pi,\,\lambda_1) - a_{11}\,y_2'(\pi,\,\lambda_1)\Big). \tag{4.5}$$

2) If λ_1 satisfies (3.2), then from (4.1) it follows that quadratic with respect to unknown a_{13} have nonzero discriminant. So the coefficient a_{13} is uniquely determined by formula (3.6).

3) If λ_1 satisfies (3.3), then we get (4.1) and the equation $\frac{\Delta(\lambda)}{d\lambda}\Big|_{\lambda=\lambda_1} = 0$. The equation is equivalent to the equation

$$a_{13}^2 = V,$$

where

$$V = \tfrac{d}{d\lambda}\Big(a_{23}\, y_1(\pi,\, \lambda) + y_1'(\pi,\, \lambda) - a_{11}\, y_2'(\pi,\, \lambda)\Big) \cdot \big(\tfrac{d}{d\lambda}\, y_2(\pi,\, \lambda)\big)^{-1} \Bigg|_{\lambda=\lambda_1} - \\ -a_{11}\, a_{23}. \tag{4.6}$$

Substituting (4.6) for a_{13}^2 in (4.1), we get

$$a_{13} = \frac{1}{2}\Big(a_{23}\, y_1(\pi,\, \lambda_1) + y_1'(\pi,\, \lambda_1) - (a_{11}\, a_{23} + V)\, y_2(\pi,\, \lambda_1) - a_{11}\, y_2'(\pi,\, \lambda_1)\Big). \tag{4.7}$$

4) If λ_1 and λ_2 satisfy condition (3.8), then by Cramer's rule, it follows that the solution of (4.1) is

$$a_{13} = \frac{D_1}{D}, \qquad \left(a_{13}^2 = \frac{D_2}{D}\right), \tag{4.8}$$

where D, D_1, D_2 are determined by formulas (4.2), (4.3), (4.4). □

Remark 4.1. From Lemma 4.1 it follows that two sequences of real numbers μ_k, ν_k satisfying Conditions 1, 2 and 3 exist. From Theorem 3.3 it follows that two real numbers λ_1, λ_2, satisfying Conditions 4 or 5 exist.

Remark 4.2. Theorem 4.1 generalizes Levitan and Gasymov's solvability theorem [2, Theorem 3.4.2, p. 58] to the case of nonseparated boundary conditions. Indeed, in the special case where $a_{13} = 0$ (Problem Y_1=Problem G_1), the numbers λ_1 and λ_2 coincide with two terms in the sequence of μ_k. So λ_1 and λ_2 satisfy Condition 4 or 5. From Theorem 4.1 it follows that two terms in the sequence of μ_k with Condition 4 or 5 exist. Thus, in the case $a_{13} = 0$, Theorem 4.1 coincides with Levitan and Gasymov's solvability theorem.

5. Scheme for identification of Problems Y_1, G_1 G_2

By Theorem 4.1 we can give the Scheme for identification of Problems Y_1, G_1 G_2:

Step 1. The absolutely continuous function $q(x)$ and the numbers a, a_{11}, a_{23} are uniquely determined from two sequences of real numbers μ_k, ν_k under Conditions 1, 2 and 3. They, and therefore Problems G_1 G_2 are determined by well known methods of identification of inverse Sturm-Liouville problems (see [9]).

Step 2. By the function $q(x)$ we find the linearly independent solutions $y_1(x,\, \lambda)$ and $y_2(x,\, \lambda)$ of equation (1.1), satisfying conditions (2.2).

Step 3. By the numbers a_{11}, a_{23}, the functions $y_1(x,\, \lambda)$, $y_2(x,\, \lambda)$, we write characteristic determinant (2.4) of Problem Y_1 with unknown coefficient a_{13}.

Step 4. By the numbers λ_1 and λ_2 satisfying Condition 4 or 5 we uniquely determine a_{13}. The formula for a_{13} depends on what kind of conditions (3.1), (3.2), (3.3) or (3.8) is satisfied. If λ_1 satisfies condition (3.1), then we use formula

(4.5); if λ_1 satisfies condition (3.2), then we use formula (3.6); if λ_1 satisfies condition (3.3), then we use formulas (4.6) and (4.7); if λ_1 and λ_1 satisfy condition (3.8), then we use formula (4.8).

6. Examples

Example 1. Let μ_k be the roots of the equation $\sqrt{\mu}\,\sin\sqrt{\mu} = 0$, ν_k be the roots of the equation $\operatorname{ctg}\sqrt{\nu} = \sqrt{\nu}$, and $\lambda_1 = 1$. By two sequences of real numbers μ_k and ν_k and the well known methods for identification of inverse Sturm-Liouville problem with separated boundary conditions (see [9]) we obtain $q(x) = 0$, $a = -1$, $a_{11} = 0$, $a_{23} = 0$. The result is Problems G_1, G_2 and Y_1 (with unknown coefficient a_{13}):

Problem G_1: $-y'' = \lambda\, y$, $y'(0) = 0$, $y'(\pi) = 0$.

Problem G_2: $-y'' = \lambda\, y$, $y'(0) - y(0) = 0$, $y'(\pi) = 0$.

Problem Y_1: $-y'' = \lambda\, y$, $y'(0) + a_{13}\, y(\pi) = 0$, $y'(\pi) - a_{13}\, y(0) = 0$.

Consequently, linearly independent solutions $y_1(x, \lambda)$ and $y_2(x, \lambda)$ of the equation $-y'' = \lambda\, y$ satisfying conditions (2.2) are

$$y_1(x, \lambda) = \cos\sqrt{\lambda}, \quad y_2(x, \lambda) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}. \tag{6.1}$$

From (2.4) it follows that the characteristic determinant of Problem Y_1 with unknown coefficient a_{13} is

$$\Delta(\lambda) = 2\, a_{13} + \sqrt{\lambda}\,\sin\sqrt{\lambda}\,\pi + a_{13}^2\,\frac{\sin\sqrt{\lambda}\,\pi}{\sqrt{\lambda}}. \tag{6.2}$$

Since $\lambda_1 = 1$ it follows that $y_2(x, \lambda_1) = \frac{\sin\sqrt{\lambda_1}\pi}{\sqrt{\lambda_1}} = \sin\pi = 0$. Therefore, condition (3.1) is satisfied. Using (4.5), we get

$$a_{13} = -\frac{1}{2}\sqrt{\lambda_1}\,\sin\sqrt{\lambda_1}\,\pi = -\frac{1}{2}\,\sin\pi = 0.$$

Finally, we obtain

Problem Y_1: $-y'' = \lambda\, y$, $y'(0) = 0$, $y'(\pi) = 0$.

We see that $Y_1 = G_1$.

Example 2. Let μ_k be the roots of the equation $\sqrt{\mu}\,\sin\sqrt{\mu} = 0$, ν_k be the roots of the equation $\operatorname{ctg}\sqrt{\nu} = \sqrt{\nu}$, and $\lambda_1 = 1/4$. By two sequences of real numbers μ_k and ν_k similarly to Example 1 we obtain $q(x) = 0$, $a = -1$, $a_{11} = 0$, $a_{23} = 0$. Hence, the linearly independent solutions $y_1(x, \lambda)$ and $y_2(x, \lambda)$ of the equation $-y'' = \lambda\, y$ satisfying conditions (2.2) are (6.1), and the unknown coefficient a_{13} satisfies (6.2).

Since $\lambda_1 = 1/4$ it follows that $y_2(x, \lambda_1) = \frac{\sin\sqrt{\lambda_1}\pi}{\sqrt{\lambda_1}} = 2\,\sin(\pi/2) = 2 \neq 0$ and

$$y_2(\pi, \lambda_1)\Big(a_{11}\, a_{23}\, y_2(\pi, \lambda_1) + a_{11}\, y_2'(\pi, \lambda_1) - a_{23}\, y_1(\pi, \lambda_1) - y_1'(\pi, \lambda_1)\Big) = 2\cdot\frac{1}{2} = 1.$$

Therefore, condition (3.2) is satisfied. Using (3.6), we get

$$a_{13} = -\frac{1}{y_2(\pi, \lambda_1)} = -\frac{\sqrt{\lambda_1}}{\sin\sqrt{\lambda_1}\,\pi} = -\frac{1}{2}.$$

Finally, we obtain:

Problem Y_1: $-y'' = \lambda y$, $y'(0) - \frac{1}{2}y(\pi) = 0$, $y'(\pi) + \frac{1}{2}y(0) = 0$.

Example 3. Let μ_k be the roots of the equation $\sqrt{\mu}\sin\sqrt{\mu} = 0$, ν_k be the roots of the equation $\operatorname{ctg}\sqrt{\nu} = \sqrt{\nu}$, and $\lambda_1 = 1/4$. By two sequences of real numbers μ_k and ν_k similarly to Example 1 we obtain $q(x) = 0$, $a = -1$, $a_{11} = 0$, $a_{23} = 0$. Hence, the linearly independent solutions $y_1(x, \lambda)$ and $y_2(x, \lambda)$ of the equation $-y'' = \lambda y$ satisfying conditions (2.2) are (6.1), and the unknown coefficient a_{13} satisfies (6.2).

Since $\lambda_1 = 1/4$ it follows that $\frac{y_2(\pi,\lambda)}{d\lambda}\Big|_{\lambda=\lambda_1} = -4 \neq 0$. Therefore, condition (3.3) is satisfied. Using (4.6) and (4.7) , we get

$$V = \frac{d}{d\lambda}\Big(y_1'(\pi, \lambda)\Big)\cdot\left(\frac{d}{d\lambda}\,y_2(\pi, \lambda)\right)^{-1}\Bigg|_{\lambda=\lambda_1} = \frac{1}{4},$$

$$a_{13} = \frac{1}{2}\Big(y_1'(\pi, \lambda_1) - V\,y_2(\pi, \lambda_1)\Big) = -\frac{1}{2}.$$

Finally, we obtain:
Problem Y_1: $-y'' = \lambda y$, $y'(0) - \frac{1}{2}y(\pi) = 0$, $y'(\pi) + \frac{1}{2}y(0) = 0$.

Example 4. Let μ_k be the roots of the equation $\sqrt{\mu}\sin\sqrt{\mu} = 0$, ν_k be the roots of the equation $\operatorname{ctg}\sqrt{\nu} = \sqrt{\nu}$, and $\lambda_1 = 0$, $\lambda_2 = 1$. By two sequences of real numbers μ_k and ν_k similarly to Example 1 we obtain $q(x) = 0$, $a = -1$, $a_{11} = 0$, $a_{23} = 0$. Hence, the linearly independent solutions $y_1(x, \lambda)$ and $y_2(x, \lambda)$ of the equation $-y'' = \lambda y$ satisfying conditions (2.2) are (6.1), and the unknown coefficient a_{13} satisfies (6.2).

Since $\lambda_1 = 0$, $\lambda_2 = 1$ it follows that

$$y_2(x, \lambda_2) - y_2(x, \lambda_1) = \frac{\sin\sqrt{\lambda_2}\pi}{\sqrt{\lambda_2}} - \frac{\sin\sqrt{\lambda_1}\pi}{\sqrt{\lambda_1}} = 0 - \pi \neq 0, \quad D_1^2 = D\cdot D_2,$$

where

$$D = \begin{vmatrix} 2 & \pi \\ 2 & 0 \end{vmatrix}, \quad D_1 = \begin{vmatrix} -\sqrt{\lambda_1}\sin\sqrt{\lambda_1}\pi & \pi \\ -\sqrt{\lambda_2}\sin\sqrt{\lambda_2}\pi & 0 \end{vmatrix}, \quad D_2 = \begin{vmatrix} 2 & -\sqrt{\lambda_1}\sin\sqrt{\lambda_1}\pi \\ 2 & -\sqrt{\lambda_2}\sin\sqrt{\lambda_2}\pi \end{vmatrix}.$$

Therefore, the condition (3.8) is satisfied. Using (4.8), we get

$$a_{13} = \frac{D_1}{D} = 0.$$

Finally, we obtain:
Problem Y_1: $-y'' = \lambda y$, $y'(0) = 0$, $y'(\pi) = 0$.

Remark 6.1. The same Problem Y_1 can be obtained from different formulas. For example, the Problem $-y'' = \lambda y$, $y'(0) = 0$, $y'(\pi) = 0$ is obtained from (4.5) in Example 1, and is obtained from (4.8) in Example 4. The Problem $-y'' = \lambda y$, $y'(0) - \frac{1}{2}y(\pi) = 0$, $y'(\pi) + \frac{1}{2}y(0) = 0$ is obtained from (3.6) in Example 2, and is obtained from (4.6) and (4.7) in Example 3.

7. Counterexample

The equation $D_1^2 = D \cdot D_2$ in (3.8) means that arbitrary numbers λ_1 λ_2 can not be eigenvalues of Problem Y_1. If λ_1 λ_2 are eigenvalues of preassigned Problem Y_1, then the equation $D_1^2 = D \cdot D_2$ is automatically satisfied. See Example 4, where the eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$ of the problem $-y'' = \lambda\, y$, $y'(0) = 0$, $y'(\pi) = 0$ satisfy the equation $D_1^2 = D \cdot D_2 = 0$. If it is not known that eigenvalues λ_1 λ_2 are eigenvalues of Problem Y_1, then an agreement between the numbers λ_1 λ_2 is necessary. This agreement is given by the condition $D_1^2 = D \cdot D_2$. If this condition is not satisfied, then formulas (4.8) can not be used. An disagreement arises between the formulas $a_{13} = \frac{D_1}{D}$ and $a_{13}^2 = \frac{D_2}{D}$. Let us show this in an example.

Let μ_k be the roots of the equation $\sqrt{\mu}\,\sin\sqrt{\mu} = 0$, ν_k be the roots of the equation $\operatorname{ctg}\sqrt{\nu} = \sqrt{\nu}$, and $\lambda_1 = 0$, $\lambda_2 = 1/16$.

By two sequences of real numbers μ_k and ν_k similarly to Example 1 we obtain $q(x) = 0$, $a = -1$, $a_{11} = 0$, $a_{23} = 0$. Hence, the linearly independent solutions $y_1(x,\, \lambda)$ and $y_2(x,\, \lambda)$ of the equation $-y'' = \lambda\, y$ satisfying conditions (2.2) are (6.1), and the unknown coefficient a_{13} satisfies (6.2).

Since $\lambda_1 = 0$, $\lambda_2 = \frac{1}{16}$ it follows that

$$y_2(x,\, \lambda_2) - y_2(x,\, \lambda_1) = \frac{\sin\sqrt{\lambda_2}\pi}{\sqrt{\lambda_2}} - \frac{\sin\sqrt{\lambda_1}\pi}{\sqrt{\lambda_1}} = 2\sqrt{2} - \pi \neq 0,$$

$$D = \begin{vmatrix} 2 & \pi \\ 2 & 2\sqrt{2} \end{vmatrix} = 4\sqrt{2} - 2\pi,$$

$$D_1 = \begin{vmatrix} 0 & \pi \\ -\frac{1}{4}\cdot\frac{\sqrt{2}}{2} & 2\sqrt{2} \end{vmatrix} = \frac{\sqrt{2}\,\pi}{8}, \qquad D_2 = \begin{vmatrix} 2 & 0 \\ 2 & -\frac{1}{4}\cdot\frac{\sqrt{2}}{2} \end{vmatrix} = -\frac{\sqrt{2}}{4}.$$

Therefore, if we use formulas (4.8), then we get

$$a_{13} = \frac{D_1}{D} = \frac{\sqrt{2}\,\pi}{32\sqrt{2} - 16\,\pi}, \qquad a_{13}^2 = \frac{D_2}{D} = \frac{\sqrt{2}}{8\pi - 16\sqrt{2}}.$$

The result is the contrary: $a_{13}^2 \neq a_{13}^2$. The contrary arise because $D_1^2 \neq D \cdot D_2$. This happens due to the fact that numbers $\lambda_1 = 0$ $\lambda_2 = 1/16$ are eigenvalues of different Problems of Y_1-type.

Indeed, the equation $\Delta(\lambda) = 0$ with unknown a_{13} is square and has the form:

$$\Delta(\lambda) = 2\,a_{13} + \sqrt{\lambda}\,\sin\sqrt{\lambda}\,\pi + a_{13}^2\,\frac{\sin\sqrt{\lambda}\,\pi}{\sqrt{\lambda}} = 0.$$

In the case $\lambda = \lambda_1 = 0$ the quadratic equation is

$$2\,a_{13} + a_{13}^2\,\pi = 0$$

and has the roots $a_{13} = 0$ $a_{13} = -2/\pi$.

In the case $\lambda = \lambda_2 = \frac{1}{16}$ the quadratic equation is

$$2\,a_{13} + \frac{\sqrt{2}}{8} + a_{13}^2\,2\sqrt{2} = 0$$

and has the roots $a_{13} = \frac{1}{8}\left(-2\sqrt{2} + 2\right)$ and $a_{13} = -\frac{1}{8}\left(2\sqrt{2} + 2\right)$.

Thus, the numbers $\lambda_1 = 0$ and $\lambda_2 = 1/16$ are eigenvalues of different Problems of Y_1-type.

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