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CONSTRUCTING EXTREMAL ELEMENTS IN APPROXIMATION BY SUMS OF UNIVARIATE FUNCTIONS

ARZU M.-B. BABAYEV AND IBRAHIM K. MAHAROV

Abstract. The approximation problem considered in the paper is to approximate a continuous bivariate function by sums of univariate functions in the uniform norm. For certain class of bivariate functions, we obtain an explicit formula for a best approximation.

1. Introduction

It is well known that approximate representations of functions of several variables by simple combinations, thus by sums, of functions of fewer variables is of both theoretical and practical significance. Application areas range from statistics to nuclear physics (for references see [3, 7]). In applications, as a rule, it is required to evaluate or estimate the error of this approximate representation. If the representation is $f(x,y) \approx \varphi(x) + \psi(y)$, where $f(x,y), \varphi(x)$ and $\psi(y)$ are continuous functions on a compact set $Q \subset R^2$, on projections of Q onto coordinate axes x and y respectively, then the error is defined as

$$E(f) = E(f,Q) \stackrel{def}{=} \inf_{\varphi+\psi} \max_{(x,y)\in Q} |f(x,y) - \varphi(x) - \psi(y)|.$$

It should be remarked that this type of approximation has arisen in connection with the classical functional equations [5], the numerical solution of certain elliptic p.d.e. boundary value problems [4] and dimension theory [16].

In [15], Rivlin and Sibner proved that for a function $f(x,y)$ with the continuous nonnegative derivative $\frac{\partial f}{\partial x \partial y}$ on a rectangle $R = [a_1,b_1] \times [a_2,b_2]$, the above–mentioned error can be computed by the formula

$$E(f,R) = \frac{1}{4}\left[f(a_1,a_2) + f(b_1,b_2) - f(a_1,b_2) - f(b_1,a_2)\right]. \qquad (1.1)$$

Babaev [2, 3] generalized this result and proved that the formula is valid for a continuous function $f(x,y)$ with the nonnegative difference $\Delta_{h_1 h_2} f$. More precisely he considered the class $M(R)$ of continuous functions $f(x,y)$ with the property

$$\Delta_{h_1,h_1} f = f(x,y) + f(x+h_1, y+h_2) - f(x, y+h_2) - f(x+h_1, y) \geq 0$$

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for each rectangle $[x, x+h_1] \times [y, y+h_2] \subset R$, and proved that if $f(x,y)$ belongs to $M(R)$, where $R = [a_1, b_1] \times [a_2, b_2]$, then the formula (1.1) is valid. Formulas of type (1.1) were also obtained for functions with the nonnegative differences $\Delta_{h_1 h_2} f$, but defined on sets different from a rectangle (see [9, 10]).

In [8], Ismailov constructed the following classes of bivariate functions. Let $c \in (a_1, b_1]$, $R_1 = [a_1, c] \times [a_2, b_2]$ and $R_2 = [c, b_1] \times [a_2, b_2]$. It is clear that $R = R_1 \cup R_2$ and if $c = b_1$ then $R = R_1$. With each rectangle $S = [x_1, x_2] \times [y_1, y_2]$ lying in R associate the following functional:

$$L(f, S) = \frac{1}{4}\left[f(x_1, y_1) + f(x_2, y_2) - f(x_1, y_2) - f(x_2, y_1)\right].$$

Definition 1.1. (see [8]) We say that a continuous function $f(x,y)$ belongs to the class $V_c(R)$ if
1) $L(f, S) \geq 0$, for each $S \subset R_1$;
2) $L(f, S) \leq 0$, for each $S \subset R_2$;
3) $L(f, S) \geq 0$, for each $S = [a_1, b_1] \times [y_1, y_2]$, $S \subset R$.

The class $V_c(R)$ has the following obvious properties:
1) For given functions $f_1, f_2 \in V_c(R)$ and numbers $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 f_1 + \alpha_2 f_2 \in V_c(R)$. $V_c(R)$ is a closed subset of the space of continuous functions.
2) $V_{b_1}(R) = M(R)$.
3) If f is a common element of $V_{c_1}(R)$ and $V_{c_2}(R)$, $a_1 < c_1 < c_2 \leq b_1$ then $f(x,y) = \varphi(x) + \psi(y)$ on the rectangle $[c_1, c_2] \times [a_2, b_2]$.

The main result of [8] was as follows.

Theorem 1.1. *The best approximation of a function $f(x,y)$ from the class $V_c(R)$ can be calculated by the formula*

$$E(f, R) = L(f, R_1) = \frac{1}{4}\left[f(a_1, a_2) + f(c, b_2) - f(a_1, b_2) - f(c, a_2)\right].$$

Let y_0 be any solution from $[a_2, b_2]$ of the equation

$$L(f, Y) = \frac{1}{2} L(f, R_1), \quad Y = [a_1, c] \times [a_2, y]. \tag{1.2}$$

Then a function $\varphi_0(x) + \psi_0(y)$, where

$$\varphi_0(x) = f(x, y_0),$$

$$\psi_0(y) = \frac{1}{2}\left[f(a_1, y) + f(c, y) - f(a_1, y_0) - f(c, y_0)\right].$$

is a best approximating sum among all sums $\varphi(x) + \psi(y)$.

Note that in special case $c = b_1$, Theorem 1.1 turns into Babaev's result from [3]. It should be remarked that for constructing extremal elements by Babaev's or Ismailov's method one should solve the equations of type (1.2). This is a quite difficult task for complicated bivariate functions $f(x,y)$. In this paper, we give an explicit formula for a best approximating sum of univariate functions to a given function from the class $M(R)$.

2. Main result

In the theory of approximation by sums of univariate functions, the concept of "a bolt of lightning" is principal. A bolt of lightning (see [1, 6, 7, 13, 14]) is a finite ordered set $p = \{p_1, p_2, \cdots p_n\}$ on the plane such that $p_i \neq p_{i+1}$, each segment line $p_i p_{i+1}$ (unit of the bolt) is parallel to the coordinate axis x or y, and two adjacent units $p_i p_{i+1}$ and $p_{i+1} p_{i+2}$ are perpendicular. A bolt of lightning p is said to be closed if $p_n p_1 \perp p_1 p_2$ (in this case, n is an even number).

The following theorem is valid.

Theorem 2.1. *Let a function $f(x, y)$ belong to the class $M(R)$. Then the function $g(x, y) = g_1(x) + g_2(y)$, where*

$$g_1(x) = \frac{1}{2} f(x, a_2) + \frac{1}{2} f(x, b_2) - \frac{1}{4} f(a_1, a_2) - \frac{1}{4} f(b_1, b_2),$$

$$g_2(y) = \frac{1}{2} f(a_1, y) + \frac{1}{2} f(b_1, y) - \frac{1}{4} f(a_1, b_2) - \frac{1}{4} f(b_1, a_2)$$

is a best approximation for the function f.

Proof. With each rectangle $S = [u_1, v_1] \times [u_2, v_2] \subset R$ we associate the functional

$$L(h, S) = \frac{1}{4} \left(h(u_1, u_2) + h(v_1, v_2) - h(u_1, v_2) - h(v_1, u_2) \right), \quad h \in C(R).$$

This functional has the following obvious properties:

(A) $L(z, S) = 0$ for any function $z = z_1(x) + z_2(y)$ and $S \subset R$..

(B) For any point $(x, y) \in R$, $L(f, R) = \sum_{i=1}^{4} L(f, S_i)$, where $S_1 = [a_1, x] \times [a_2, y]$, $S_2 = [x, b_1] \times [y, b_2]$, $S_3 = [a_1, x] \times [y, b_2]$, $S_4 = [x, b_1] \times [a_2, y]$.

Since f belongs to the class $M(R)$, for any rectangle $S = [u_1, v_1] \times [u_2, v_2] \subset R$ we can write

$$L(f, S) \geq 0. \tag{2.1}$$

Set the function

$$p(x, y) = L(f, S_1) + L(f, S_2) - L(f, S_3) - L(f, S_4). \tag{2.2}$$

It is not difficult to verify that the function $f - p$ has the form $z_1(x) + z_2(y)$. Hence

$$E(f, R) = E(p, R). \tag{2.3}$$

Calculate the norm $\|p\|$. From the property (B), it follows that

$$p(x, y) = L(f, R) - 2(L(f, S_3) + L(f, S_4))$$

and

$$p(x, y) = 2 \left(L(f, S_1) + L(f, S_2) \right) - L(f, R).$$

From the last equalities and (2.1), we obtain that

$$|p(x, y)| \leq L(f, R), \text{ for any } (x, y) \in R.$$

On the other hand, one can check that

$$p(a_1, a_2) = p(b_1, b_2) = L(f, R) \tag{2.4}$$

and

$$p(a_1, b_2) = p(b_1, a_2) = -L(f, R). \tag{2.5}$$

Therefore,

$$\|p\| = L\left(f, R\right). \tag{2.6}$$

Note that the points $(a_1, a_2), (a_1, b_2), (b_1, b_2), (b_1, a_2)$ in the given order form a bolt of lightning. We conclude from (2.4)-(2.6) that at the points of this bolt, the function p alternatively takes its maximum and minimum. By Havinson's characterization theorem (see theorem 1 in [7]), the zero function is a best approximation to p. Hence

$$E(p, R) = L\left(f, R\right). \tag{2.7}$$

Now from (2.3) and (2.7) we obtain that

$$E\left(f, R\right) = L\left(f, R\right). \tag{2.8}$$

It is not difficult to verify that the function $p(x, y)$ has the form

$$p(x, y) = f(x, y) - g_1(x) - g_2(y).$$

On the other hand, from (2.6) and (2.8) it follows that

$$E(f, R) = \|p\|.$$

Therefore, the function $g_1(x) + g_2(y)$ is a best approximation for f. □

Until this moment we have been approximating a function $f(x, y)$ from $M(R)$ on the rectangle R by functions $\varphi(x) + \psi(y)$. As it is seen from the following theorem in some cases the formula in theorem 2.1 is true also for more general sets different from a rectangle.

Theorem 2.2. *Let $f(x, y)$ be a function from $M(R)$ and $Q \subset R$ is a compact set which contains all vertices of R (points (a_1, a_2), (b_1, b_2), (a_1, b_2), (b_1, a_2)). Besides, we assume that the projections of Q onto the coordinate axes coincide with the corresponding projections of R. Then*

$$E(f, Q) = L(f, R) = \frac{1}{4}\left[f\left(a_1, a_2\right) + f\left(b_1, b_2\right) - f\left(a_1, b_2\right) - f\left(b_1, a_2\right)\right]$$

and the function $g(x, y) = g_1(x) + g_2(y)$, where

$$g_1(x) = \frac{1}{2}f(x, a_2) + \frac{1}{2}f(x, b_2) - \frac{1}{4}f(a_1, a_2) - \frac{1}{4}f(b_1, b_2),$$

$$g_2(y) = \frac{1}{2}f(a_1, y) + \frac{1}{2}f(b_1, y) - \frac{1}{4}f(a_1, b_2) - \frac{1}{4}f(b_1, a_2)$$

is a best approximation for the function f on the set Q.

Proof. As $Q \subset R$, $E(f, Q) \leq E(f, R)$. On the other hand by theorem 1.1, $E(f, R) = L(f, R)$. Hence $E(f, Q) \leq L(f, R)$. It can be easily shown that $L(f, R) \leq E(f, Q)$ (see [8]). But then automatically $E(f, Q) = L(f, R)$. Besides, we conclude that $E(f, Q) = E(f, R)$. It follows from the last equality and theorem 2.1 that the function $g_1(x) + g_2(y)$ is a best approximation on Q. □

Remark. Some interesting properties of the best approximation $E(f, Q)$ depending on the approximation domain Q were investigated in the papers [11, 12].

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Arzu M.-B. Babayev

Institute of Mathematics and Mechanics of NAS of Azerbaijan, 9 B. Vahabzadeh str., AZ1141, Baku, Azerbaijan.

Ibrahim K. Maharov

Institute of Mathematics and Mechanics of NAS of Azerbaijan, 9 B. Vahabzadeh str., AZ1141, Baku, Azerbaijan.

E-mail address: ibrahimmaharov@gmail.com

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