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# ON THE PROPERTIES OF $Q$- AND $Q'$-INTEGRALS OF THE FUNCTION MEASURABLE ON THE REAL AXIS

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**Abstract**. In the same paper Titchmarsh established that, when studying the properties of trigonometric series conjugate to Fourier series of Lebesgue integrable functions, $Q$-integration leads to a series of natural results. A very uncomfortable fact impeding the application of $Q$-integrals and $Q'$-integrals when studying diverse problems of function theory is the absence of the additivity property. If one adds the some condition to the definition of $Q$-integrability ($Q'$-integrability) of a function $f$, then the $Q$-integral and $Q'$-integral become additive. In this paper, we give the definition of $Q$- and $Q'$-integrals for the function, measurable on the real axis $R$, and study its additivity properties.

## 1. Introduction

For a measurable complex function $f$ on an interval $[a, b] \subset R$ we set
$[f(x)]_n = [f(x)]^n = f(x)$ for $|f(x)| \leq n$,
$[f(x)]_n = n \cdot sgnf(x)$, $[f(x)]^n = 0$ for $|f(x)| > n$, $n \in N$,
where $sgnz = \frac{z}{|z|}$ for $z \neq 0$ and $sgn0 = 0$.

In 1929, E.Titchmarsh [10] introduced the notions of $Q$- and $Q'$-integrals.

**Definition 1.1.** If a finite limit $\lim\limits_{n\to\infty} \int_a^b [f(x)]_n\, dx$ ($\lim\limits_{n\to\infty} \int_a^b [f(x)]^n\, dx$, respectively) exists, then $f$ is said to be *$Q$-integrable* (*$Q'$-integrable*, respectively) on $[a, b]$, that is $f \in Q[a, b]$ ($f \in Q'[a, b]$), and the value of this limit is referred to as the *$Q$-integral* (*$Q'$-integral*) of this function and is denoted by

$$(Q)\int_a^b f(x)\, dx \left( (Q')\int_a^b f(x)\, dx \right).$$

In the same paper, Titchmarsh established that, when studying the properties of trigonometric series conjugate to Fourier series of Lebesgue integrable functions, $Q$-integration leads to a series of natural results. A very uncomfortable fact impeding the application of $Q$-integrals and $Q'$-integrals when studying diverse problems of function theory is the absence of the additivity property, that

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is, the $Q$-integrability ($Q'$-integrability) of two functions does not imply the $Q$-integrability ($Q'$-integrability) of their sum. If one adds the condition

$$\lambda\, m\,\{\, x \in [a,b]:\ |f(x)| > \lambda \,\} = o(1),\ \lambda \to +\infty, \tag{1.1}$$

where $m$ stands for the Lebesgue measure, to the definition of $Q$-integrability ($Q'$-integrability) of a function $f$ on the interval $[a,b]$, then the $Q$-integral and $Q'$-integral coincide ($Q\,[a,b] = Q'\,[a,\,b]$), and these integrals become additive.

**Definition 1.2.** If $f \in Q'\,[a,b]$ (or $f \in Q\,[a,b]$) and condition (1.1) holds, then $f$ is said to be $A$-*integrable* on $[a,b]$, $f \in A\,[a,b]$, and the limit $\lim\limits_{n\to\infty} \int_a^b [f(x)]^n\,dx$ (or the limit $\lim\limits_{n\to\infty} \int_a^b [f(x)]_n\,dx$) is denoted in this case by

$$(A)\int_a^b f(x)\,dx.$$

As we noted above, the $Q$-integral and the $Q'$-integral do not have the additivity property. Titchmarsh in [10] for real functions and the author in [8] for complex functions established that, if $f \in Q\,[a,b]$ and $g \in L\,[a,b]$ (that is, $g$ is Lebesgue integrable on the interval $[a,b]$), then $f+g \in Q\,[a,b]$ and

$$(Q)\int_a^b [f(x)+g(x)]\,dx = (Q)\int_a^b f(x)\,dx + (L)\int_a^b g(x)\,dx.$$

The properties of $Q$- and $Q'$-integrals were investigated in [8]-[10], and in [1]-[7], [11]-[15] given the applications of $A$-, $Q$- and $Q'$-integrals in theory of functions of real and complex variable.

In this paper, similar to the definitions 1.1 and 1.2, we give the definition of $Q$- , $Q'$- and $A$-integrals for the function, measurable on the real axis $R$, and we study its properties.

## 2. Main results

For a complex function $f$ measurable on the real axis $R$ we assume
$[f(x)]_{\delta,\lambda} = [f(x)]^{\delta,\lambda} = f(x)$ for $\delta \le |f(x)| \le \lambda$,
$[f(x)]_{\delta,\lambda} = [f(x)]^{\delta,\lambda} = 0$ for $|f(x)| < \delta$,
$[f(x)]_{\delta,\lambda} = \lambda\, sgn f(x)$, $[f(x)]^{\delta,\lambda} = 0$ for $|f(x)| > \lambda$, $0 < \delta < \lambda$.

**Definition 2.1.** If a finite limit $\lim\limits_{\substack{\delta\to 0+\\ \lambda\to+\infty}} \int_R [f(x)]_{\delta,\lambda}\,dx$ ( $\lim\limits_{\substack{\delta\to 0+\\ \lambda\to+\infty}} \int_R [f(x)]^{\delta,\lambda}\,dx$ respectively) exists, then $f$ is said to be $Q$-*integrable* ($Q'$-*integrable*) on $R$, that is $f \in Q(R)$ ($f \in Q'(R)$), and the value of this limit is referred to as the $Q$-*integral* ($Q'$-*integral*) of this function and is denoted by

$$(Q)\int_R f(x)\,dx \left( (Q')\int_R f(x)\,dx \right).$$

*Remark* 2.1. Let $h > 0$ be any positive number. From equalities

$$\lim_{\substack{\delta\to 0+\\ \lambda\to+\infty}} \int_R [f(x)]_{\delta,\lambda}\,dx = \lim_{\delta\to 0+} \int_{\{x\in R:\ \delta\le|f(x)|\le h\}} f(x)\,dx +$$

$$+\lim_{\lambda\to+\infty}\int_{\{x\in R:\ |f(x)|>h\}}[f(x)]_{\lambda}\,dx, \tag{2.1}$$

$$\lim_{\substack{\delta\to0+\\ \lambda\to+\infty}}\int_R [f(x)]^{\delta,\lambda}\,dx=\lim_{\delta\to0+}\int_{\{x\in R:\ \delta\le|f(x)|\le h\}} f(x)\,dx+$$

$$+\lim_{\lambda\to+\infty}\int_{\{x\in R:\ |f(x)|>h\}}[f(x)]^{\lambda}\,dx \tag{2.2}$$

follows that if for some $h>0$ there exists the integral $\int_{\{x\in R:\ |f(x)|\le h\}} f(x)\,dx$, then $Q$- and $Q'$-integrals of the function $f$ may be determined as follows:

$$(Q)\int_R f(x)\,dx=\lim_{\lambda\to+\infty}\int_R [f(x)]_{\lambda}\,dx,\ (Q')\int_R f(x)\,dx=\lim_{\lambda\to+\infty}\int_R [f(x)]^{\lambda}\,dx,$$

where $[f(x)]_{\lambda}$ and $[f(x)]^{\lambda}$ are determined as in definition 1, and if there exists the integral $\int_{\{x\in R:\ |f(x)|>h\}} f(x)\,dx$, then $Q$- and $Q'$-integrals of the function $f$ may be determined as follows:

$$(Q)\int_R f(x)\,dx=(Q')\int_R f(x)\,dx=\lim_{\delta\to0+}\int_{\{x\in R:\,|f(x)|\ge\delta\}} f(x)\,dx.$$

Note that as in case of an interval $Q$- and $Q'$-integrals of the functions measurable on the real axis also doesn't satisfy additivity property, that is from $Q$-integrability ($Q'$-integrability) of two functions $Q$-integrability ($Q'$-integrability) of their sums doesn't follow yet. If one adds the conditions

$$\delta\, m\{\,x\in R:\ |f(x)|>\delta\,\}=o(1)\,,\delta\to0+, \tag{2.3}$$

$$\lambda\, m\{\,x\in R:\ |f(x)|>\lambda\,\}=o(1)\,,\lambda\to+\infty, \tag{2.4}$$

to the definition of $Q$-integrability ($Q'$-integrability) of a function $f$ on $R$, then $Q$-integral and $Q'$-integral coincide ($Q(R)=Q'(R)$) and these integrals become additive (see [4]).

**Definition 2.2.** If $f\in Q'(R)$ (or $f\in Q(R)$) and the conditions (2.3) and (2.4) are holds, then $f$ is said to be *$A$-integrable* on $R$, $f\in A(R)$ and the limit $\lim\limits_{\substack{\delta\to0+\\ \lambda\to+\infty}}\int_R [f(x)]^{\delta,\lambda}\,dx$ (or the limit $\lim\limits_{\substack{\delta\to0+\\ \lambda\to+\infty}}\int_R [f(x)]_{\delta,\lambda}\,dx$) is denoted in this case by

$$(A)\int_R f(x)\,dx.$$

For the real function $f$ measurable on $R$ we assume

$$(f>\lambda)=\{\,t\in R:\ f(t)>\lambda\,\},$$

$$(f<\lambda)=\{\,t\in R:\ f(t)<\lambda\,\},\ \ (f\ge\lambda)=\{\,t\in R:\ f(t)\ge\lambda\,\},$$

$$(f\le\lambda)=\{\,t\in R:\ f(t)\le\lambda\,\},\ (\delta\le f\le\lambda)=\{\,t\in R:\ \delta\le f(t)\le\lambda\,\}.$$

**Definition 2.3.** We denote by $M(R;C)$ the class of measurable complex-valued functions $f$ on $R$ which are finite limits $\lim\limits_{\lambda\to+\infty}\lambda\, m(|f|>\lambda)$ and $\lim\limits_{\lambda\to+\infty}\int_{(|f|>h)}[f(x)]_{\lambda}\,dx$ exists.

**Lemma 2.1.** *If a function $f$ belongs to $Q$ and the function $g$ satisfies the conditions* (2.3) *and* (2.4) *on $R$, then their sum $f+g$ belongs to $M(R;C)$; here the following equation holds:*

$$\lim_{\lambda\to+\infty}\lambda\, m\left(|f+g|>\lambda\right)=\lim_{\lambda\to+\infty}\lambda\, m\left(|f|>\lambda\right),$$

$$\lim_{\delta\to 0+}\delta\, m\left(|f+g|>\delta\right)=\lim_{\delta\to 0+}\delta\, m\left(|f|>\delta\right).$$

The proof is similar to the proof of the [8, lemma 1].

**Theorem 2.1.** *Let $f\in Q'(R)$. Then $f\in Q(R)$ and the following equation holds:*

$$\left(Q'\right)\int_R\left[f(x)+g(x)\right]dx=\left(Q'\right)\int_R f(x)\,dx+(A)\int_R g(x)\,dx. \qquad (2.5)$$

**Proof.** Let $h>0$ be any positive number. It follows from $f\in Q'(R)$ and from (2.3) that there exists are finite limits $\lim\limits_{\delta\to 0+}\int_{(\delta\le|f|\le h)}f(x)\,dx$ and $\lim\limits_{\lambda\to+\infty}\int_{(|f|>h)}\left[f(x)\right]^{\lambda}dx$. The similar to the proof of the [8, theorem 1] it is proved that from the existence of the limit $\lim\limits_{\lambda\to+\infty}\int_{(|f|>h)}\left[f(x)\right]^{\lambda}dx$ follows the existence of the limit $\lim\limits_{\lambda\to+\infty}\int_{(|f|>h)}\left[f(x)\right]_{\lambda}dx$ and their equality. Hence, from (2.1) it follows that the function $f$ is $Q$-integrable and equation (2.5) holds. This completes the proof of the theorem.

**Theorem 2.2.** *The $Q$-integral and the $Q'$-integral coincide on the function class $M(R;C)$, that is, if $f\in M(R;C)$, then for the existence of the integral $(Q)\int_R f(x)\,dx$ it is necessary and sufficient that the integral $(Q')\int_R f(x)\,dx$ exist, and in that case equation* (2.5) *holds.*

**Proof.** By Theorem 2.1, it follows from the condition $f\in Q'(R)$ that $f\in Q(R)$ and the equation (2.5) holds. It remains to prove that, in the function class $M(R;C)$, it follows from $f\in Q(R)$ that $f\in Q'(R)$. Let $h>0$ be any positive number. It follows from (2.1) that if $f\in Q(R)$ and $f\in M(R;C)$, then there exists are finite limits $\lim\limits_{\delta\to 0+}\int_{(\delta\le|f|\le h)}f(x)\,dx$, $\lim\limits_{\lambda\to+\infty}\int_{(|f|>h)}\left[f(x)\right]_{\lambda}dx$ and $\lim\limits_{\lambda\to+\infty}\lambda\, m\left(|f|>\lambda\right)$. The similar to the proof of the [8, theorem 2] it is proved that from the existence of the limit $\lim\limits_{\lambda\to+\infty}\int_{(|f|>h)}\left[f(x)\right]_{\lambda}dx$ follows the existence of the limit $\lim\limits_{\lambda\to+\infty}\int_{(|f|>h)}\left[f(x)\right]^{\lambda}dx$. Hence, from (2.3) it follows that the function $f$ is $Q'$-integrable and equation (2.5) holds. This completes the proof of the theorem.

**Theorem 2.3.** *If a function $f\in M(R;C)$ is $Q'$-integrable on $R$ and a function $g$ is $A$-integrable on $R$, then their sum $f+g\in M(R;C)$ is $Q'$-integrable on $R$, and the following equation holds:*

$$\left(Q'\right)\int_R\left[f(x)+g(x)\right]dx=\left(Q'\right)\int_R f(x)\,dx+(A)\int_R g(x)\,dx. \qquad (2.6)$$

**Proof.** Since $g \in A\,(R)$, it follows from Lemma 2.1 that $f + g \in M\,(R;\, C)$. We claim that the following relations hold:

$$\lambda\, m\,((|f| > \lambda) \cap (|f+g| \le \lambda)) = o\,(1)\,, \lambda \to +\infty, \tag{2.7}$$

$$\lambda\, m\,((|f| \le \lambda) \cap (|f+g| > \lambda)) = o\,(1)\,, \lambda \to +\infty, \tag{2.8}$$

$$\delta\, m\,((|f| > \delta) \cap (|f+g| \le \delta)) = o\,(1)\,, \delta \to 0+, \tag{2.9}$$

$$\delta\, m\,((|f| \le \delta) \cap (|f+g| > \delta)) = o\,(1)\,, \delta \to 0+, \tag{2.10}$$

Let $\lim\limits_{\lambda\to\infty} \lambda m\,(|f| > \lambda) = \alpha$, $\lim\limits_{\delta\to 0+} \delta m\,(|f| > \delta) = \beta$. For every $\varepsilon > 0$the inclusions

$$(|f| > \lambda) \cap (|f+g| \le \lambda) \subset (\lambda < |f| \le (1+\varepsilon)\lambda) \cup$$

$$\cup\,((|f| > (1+\varepsilon)\,\lambda) \cap (|f+g| \le \lambda)) \subset ((|f| > \lambda) \setminus (|f| > (1+\varepsilon)\,\lambda)) \cup (|g| > \varepsilon\lambda)$$

lead to the inequality

$$m\,((|f| > \lambda) \cap (|f+g| \le \lambda)) \le m\,(|f| > \lambda) - m\,(|f| > (1+\varepsilon)\,\lambda) + m\,(|g| > \varepsilon\lambda)\,,$$

which implies that

$$\overline{\lim_{\lambda\to+\infty}}\, \lambda \cdot m\,(|f| > \lambda) \cap (|f+g| \le \lambda) \le \alpha - \frac{\alpha}{1+\varepsilon} = \frac{\varepsilon\alpha}{1+\varepsilon}.$$

Hence, since $\varepsilon > 0$ is arbitrary, we obtain (2.7). Equations (2.8)-(2.10) can be proved in a similar way in view of Lemma 2.1.

We claim now that $f + g \in Q'\,(R)$. Let us take an arbitrary numbers $\varepsilon > 0$ and $\lambda > \delta > 0$. It follows from (2.7)-(2.10) and from the $A$-integrability of $g$ that

$$\begin{aligned}
\int_{(\delta\le|f+g|\le\lambda)} (f\,(x) + g\,(x))\, dx &= \int_{(\delta\le|f+g|\le\lambda)\cap(|f|\le\lambda)} (f\,(x) + g\,(x))\, dx + o\,(1) = \\
&= \int_{(\delta\le|f+g|\le\lambda)\cap(|f|\le\lambda)\cap(|g|\le\varepsilon\lambda)} (f\,(x) + g\,(x))\, dx + o\,(1) = \\
&= \int_{(\delta\le|f+g|\le\lambda)\cap(\delta\le|f|\le\lambda)\cap(|g|\le\varepsilon\lambda)} (f\,(x) + g\,(x))\, dx + \\
&+ \int_{(\delta\le|f+g|\le\lambda)\cap(|f|<\delta)\cap(|g|\le\varepsilon\lambda)} g\,(x)\, dx + o\,(1) = \\
&= \int_{(\delta\le|f+g|\le\lambda)\cap(\delta\le|f|\le\lambda)\cap(|g|\le\varepsilon\lambda)} f\,(x)\, dx + \\
&+ \int_{(\delta\le|f+g|\le\lambda)\cap(\delta\le|f|\le\lambda)\cap(\varepsilon\delta\le|g|\le\varepsilon\lambda)} g\,(x)\, dx + \\
&+ \int_{(\delta\le|f+g|\le\lambda)\cap(\delta\le|f|\le\lambda)\cap(|g|<\varepsilon\delta)} g\,(x)\, dx + \\
&+ \int_{(\delta\le|f+g|\le\lambda)\cap(|f|<\delta)\cap(\varepsilon\delta\le|g|\le\varepsilon\lambda)} g\,(x)\, dx + o\,(1)\,, \lambda \to +\infty, \delta \to 0+,
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
\int_{(\delta\le|f|\le\lambda)} f\,(x)\, dx &= \int_{(\delta\le|f|\le\lambda)\cap(|f+g|\le\lambda)} f\,(x)\, dx + o\,(1) = \\
&= \int_{(\delta\le|f|\le\lambda)\cap(|f+g|\le\lambda)\cap(|g|\le\varepsilon\lambda)} f\,(x)\, dx + o\,(1) = \\
&+ \int_{(\delta\le|f|\le\lambda)\cap(\delta\le|f+g|\le\lambda)\cap(|g|\le\varepsilon\lambda)} f\,(x)\, dx -
\end{aligned}$$

$$-\int_{(\delta\le|f|\le\lambda)\cap(|f+g|<\delta)\cap(\varepsilon\delta\le|g|\le\varepsilon\lambda)} g(x)\,dx+o(1),\lambda\to+\infty,\delta\to 0+. \tag{2.12}$$

Since

$$\left|\int_{(\delta\le|f+g|\le\lambda)\cap(\delta\le|f|\le\lambda)\cap(|g|<\varepsilon\delta)} g(x)\,dx\right|\le\varepsilon\delta\left[m\left(|f|\ge\delta\right)+m\left(|f+g|\ge\delta\right)\right],$$

$$\left|\int_{(\varepsilon\delta\le|g|\le\varepsilon\lambda)\cap[(|f|>\lambda)\bigcup(|f+g|>\lambda)]} g(x)\,dx\right|\le\varepsilon\lambda\left[m\left(|f|>\lambda\right)+m\left(|f+g|>\lambda\right)\right],$$

$$\left|\int_{(\varepsilon\delta\le|g|\le\varepsilon\lambda)\cap(|f|<\delta)\cap(|f+g|<\delta)} g(x)\,dx\right|\le 2\delta m\left(|g|\ge\varepsilon\delta\right),$$

it follows from (2.11), (2.12) that the following inequality holds for sufficiently large values of $\lambda$ and sufficiently small values of $\delta$:

$$\left|\int_{(\delta\le|f+g|\le\lambda)}(f(x)+g(x))\,dx-\int_{(\delta\le|f|\le\lambda)} f(x)\,dx-\int_{(\varepsilon\delta\le|g|\le\varepsilon\lambda)} g(x)\,dx\right|\le$$

$$\le\varepsilon+2\varepsilon(\beta+\varepsilon)+2\varepsilon(\alpha+\varepsilon)+2\varepsilon.$$

Hence, since $\varepsilon>0$ is arbitrary, it follows that the function $f+g$ is $Q'$-integrable and that (2.6) holds. This completes the proof of the theorem.

Theorems 2.2 and 2.3 imply the following corollary.

**Corollary 2.1.** *If a function $f\in M(R;C)$ is $Q$-integrable on $R$ and the function $g$ is $A$-integrable on $R$, then their sum $f+g$ belongs to the class $M(R;C)$ and is $Q$-integrable on $R$ and the following equation holds:*

$$(Q)\int_R[f(x)+g(x)]\,dx=(Q)\int_R f(x)\,dx+(A)\int_R g(x)\,dx.$$

**Theorem 2.4.** *Let $f\in M(R;C)$ and let $g$ be a measurable function on $R$ such that the difference $f-g$ satisfies the condition* (1.1)*. If $f$ and $g$ are $Q'$-integrable on $R$, then $f-g$ is $A$-integrable on $R$, and*

$$(A)\int_R[f(x)-g(x)]\,dx=\left(Q'\right)\int_R f(x)\,dx-\left(Q'\right)\int_R g(x)\,dx.$$

The proof of Theorem 2.4 is carried out in a similar fashion to that of Theorem 2.2 (one should consider the function $g-f$ instead of $g$).

Theorems 2.2 and 2.4 imply the following corollary.

**Corollary 2.2.** *Let $f\in M(R;C)$ and a function $g$ measurable on $R$ be such that the difference $f-g$ satisfies the condition* (1.1)*. If $f$ and $g$ are $Q$-integrable on $R$, then $f-g$ is $A$-integrable on $R$, and*

$$(A)\int_R[f(x)-g(x)]\,dx=(Q)\int_R f(x)\,dx-(Q)\int_R g(x)\,dx.$$

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