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SPECTRAL PROPERTIES FOR THE EQUATION OF VIBRATING BEAM

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Abstract. In this paper we study the properties of the natural frequencies and the corresponding harmonics of transverse vibrations of a rod is exposed to tracking and axial forces. It is known that problems of this type leads to a spectral fourth-order problem with spectral parameter in the boundary conditions. We study the general characteristics of the location of the eigenvalues on the real axis and oscillation properties of eigenfunctions of these problems.

1. Introduction

We consider the following boundary value problem problem

$$(p(x)y''(x))'' - (q(x)y'(x))' + r(x)y(x) = \lambda\tau(x)y(x), \;\; 0 < x < l, \tag{1.1}$$

$$y'(0)\cos\alpha - (py'')(0)\sin\alpha = 0, \tag{1.2}$$

$$y(0)\cos\beta + Ty(0)\sin\beta = 0, \tag{1.3}$$

$$y'(l)\cos\gamma + (py'')(l)\sin\gamma = 0, \tag{1.4}$$

$$(a\lambda + b)y(l) - (c\lambda + d)Ty(l) = 0, \tag{1.5}$$

where $\lambda \in \mathbb{C}$ is spectral parameter, $Ty \equiv (py'')' - qy'$, $\alpha, \beta, \gamma, a, b, c, d$ are real constants such that $0 \le \alpha, \beta, \gamma \le \pi/2$ and

$$\sigma = bc - ad \ne 0. \tag{1.6}$$

The coefficients $p(x)$, $q(x)$, $r(x)$ and $\tau(x)$ are assumed to be real-valued continuous functions on $[0, l]$, moreover, $p(x)$ is positive and has absolutely continuous derivative, $q(x)$ is non-negative and absolutely continuous on $[0, l]$, and $\tau(x)$ is positive on $[0, l]$.

Problem (1.1)-(1.5) aries when variables are separated in the dynamical boundary value problem describing small oscillation of a vibrating beam which is subject to axial and servocontrol forces (see [13, 23, 25]).

The locations, multiplicities of the eigenvalues, the oscillation properties of eigenfunctions, the basis properties in $L_p(0, l)$, $1 < p < \infty$, of the system of root functions of the boundary value problem (1.1)-(1.5) with $r \equiv 0$, $\sigma > 0$, are considered in [18, 19], and with $r \equiv 0$, $\sigma < 0$, are considered in [2, 9].

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The subject of the present paper is the investigate characteristics of eigenvalue locations in real axis, the oscillation properties of eigenfunctions, and the basis properties in $L_p(0,l)$, $1<p<\infty$, of the system of eigenfunctions of the boundary value problem (1.1)-(1.5) in the presence of potential r (the function $r(x)$ doesn't vanishes identically on any interval constituting the part of interval $[0,l]$).

2. Preliminaries

Consider the boundary condition (see [13])

$$y(l)\cos\delta - Ty(l)\sin\delta = 0, \tag{2.1}$$

where $\delta\in[0,\pi)$.

Alongside the boundary value problem (1.1)-(1.5) we shall consider the spectral problem (1.1)-(1.4), (2.1). The problem (1.1)-(1.4), (2.1) in the case $r\equiv 0$ and $\delta\in[0,\pi/2]$ has been considered in [13] (see also [17]), where in particular proved the following assertion.

Theorem 2.1. *For fixed α,β,γ the eigenvalues of this problem are real, simple and form an infinitely increasing sequence $\{\mu_n(\delta)\}_{n=1}^{\infty}$ such that $\mu_n(\delta)>0$ for all $n\in\mathbb{N}\backslash\{1\}$; $\mu_1(\delta)>0$ for $\delta\in[0,\,\pi/2)$ and $\delta=\pi/2$, $\beta\in[0,\,\pi/2)$; $\mu_1(\pi/2)=0$ for $\beta=\pi/2$. Moreover, the eigenfunction $v_n^{(\delta)}(x)$ corresponding to the eigenvalue $\mu_n(\delta)$ has $n-1$ simple zeros in the interval $(0,l)$.*

In [17] it is shown that for each fixed $\lambda\in\mathbb{C}$ there exists a unique (up to a constant factor) nontrivial solution $v(x,\lambda)$ of problem (1.1)-(1.4), (2.1) for $r\equiv 0$. For any fixed $x\in[0,l]$ the function $v(x,\lambda)$ is an entire function of λ.

It is obvious that the eigenvalues $\mu_n(0)$ and $\mu_n(\pi/2)$, $n\in\mathbb{N}$, of boundary value problem (1.1)-(1.4), (2.1) for $r\equiv 0$ are the zeros of entire functions $v(x,\lambda)$ and $Tv(x,\lambda)$, respectively. Notice that the function

$$F_0(\lambda)=Tv(l,\lambda)/v(l,\lambda)$$

is will defined for

$$\lambda\in A\equiv\left(\bigcup_{n=1}^{\infty}A_n\right)\bigcup(\mathbb{C}\backslash\mathbb{R}),$$

where $A_n\equiv(\mu_{n-1}(0),\mu_n(0))$, $n\in\mathbb{N}$, $\mu_0(0)=-\infty$ and is meromorphic function of finite order, $\mu_n(\pi/2)$ and $\mu_n(0)$, $n\in\mathbb{N}$ are the zeros and poles of these function, respectively.

Denote:

$$\delta_0=\begin{cases}\pi/2, & if\ \beta\in[0,\,\pi/2),\\ arctg\,F_0(0), & if\ \ \beta=\pi/2,\end{cases}.$$

The problem (1.1)-(1.4), (2.1) for $r\equiv 0$ and $\delta\in[0,\pi)$ was considered in [3] (see also [17]), where in particular, it was proved the following theorem.

Theorem 2.2. *For fixed α,β,γ the eigenvalues of problem* (1.1)-(1.4), (2.1) *for $r\equiv 0$ and $\delta\in[0,\pi)$ are real, simple and form an infinitely increasing sequence $\{\mu_n(\delta)\}_{n=1}^{\infty}$ such that $\mu_n(\delta)>0$ for $n\in\mathbb{N}_0$; $\mu_1(\delta_0)=0$, $\mu_1(\delta)<0$ in the case $\delta\in(\delta_0,\pi)$. Furthermore, the eigenfunction $v_n^{(\delta)}(x)$ corresponding to the eigenvalue $\mu_n(\delta)$ for $n\in\mathbb{N}_0$ has exactly $n-1$ simple zeros in the interval $(0,l)$, the* *eigenfunction* $v_1^{(\delta)}(x)$ *corresponding to the eigenvalue* $\mu_1(\delta)$ *in the case* $\delta \in (\delta_0, \pi)$ *may have arbitrary number of zeros in the interval* $(0, l)$ *which are also simple.*

The problem (1.1)-(1.4), (2.1) in the case which the function $r(x)$ doesn't vanishes identically on any interval constituting the part of interval $[0, l]$ studied in the papers [5] (see also [16]). On setting

$$r_0 = \min_{x \in [0,l]} r(x), \quad r_1 = \max_{x \in [0,l]} r(x),$$
$$\tau_0 = \min_{x \in [0,l]} r(x), \quad \tau_1 = \max_{x \in [0,l]} r(x),$$

denote by (Ψ_1) the regular Sturmian system obtained from (1.1)-(1.4), (2.1) for $\delta \in [0, \pi/2]$ by replacing $r(x)$ by r_0 and $\tau(x)$ by τ_1. The substitution

$$\lambda' = \lambda \tau_1 - r_0$$

transform (Ψ_1) into an equivalent completely regular Sturmian system (Ψ_2) of the type to which cited above assertion from [13] may be applied. Let $\lambda'_n -$ nth eigenvalue of the system (Ψ_2) which is positive, and $\lambda_{n,0} = (\lambda'_n + r_0)/\tau_1 -$ nth eigenvalue of the system (Ψ_1), $n \in \mathbb{N}$. Then the eigenfunction $y_{n,0}(x)$, corresponding to the eigenvalue $\lambda_{n,0}$, $n \in \mathbb{N}$, has $n-1$ simple zeros in the interval $(0, l)$.

Now, using the "μ-process" (see [5, 16]) we pass from (Ψ_1) to (1.1)-(1.4), (2.1) by deformation

$$r(x, \mu) \equiv (1 - \mu'')\, r_0 + \mu' r(x),$$
$$\tau(x, \mu) \equiv (1 - \mu'')\, \tau_1 + \mu'' \tau(x), \;\; x \in [0, l], \;\; \mu', \mu'' \in [0,\, 1].$$

since the coefficient $r(x, \mu)$ increases and $\tau(x, \mu)$ decreases, then by [5] the positive eigenvalues can not decrease. The condition

$$\lambda(\mu)\, \tau(x, \mu) - r(x, \mu) > 0 \tag{2.2}$$

is a fortiori satisfied by the eigenvalues

$$\lambda_{m+1}(\mu),\; \lambda_{m+2}(\mu),\; ...$$

of Sturmian system which is obtained from (1.1)-(1.4), (2.1) by replacing $r(x)$ by $r(x, \mu)$ and $\tau(x)$ by $\tau(x, \mu)$, where m is greatest of the two numbers m_0 and 2, and m_0 is defined by:

$$\lambda'_{m_0+1} \geq (r_1 \tau_1 - r_0 \tau_0)/\tau_0 \geq \lambda'_{m_0}, \quad \lambda_{m_0+1\,,0} > 0. \tag{2.3}$$

Then by Theorem 2.1 we have the following oscillation theorem:

Theorem 2.3. *The problem* (1.1)-(1.4), (2.1) *for* $\delta \in [0, \pi/2]$ *has infinitely many eigenvalues which all real. With exception of no more than* m *of them, the eigenvalues are simple and positive. If we denote them by (for fixed* α, β, γ*)*

$$0 < \lambda_{m+1}(\delta) < \lambda_{m+2}(\delta) < ...,$$

then the eigenfunction $y_n^{(\delta)}(x)$*, corresponding to the eigenvalue* $\lambda_n(\delta)$*, has exactly* $n-1$ *simple zeros in the interval* $(0, 1)$*.*

However there are no results on the multiplicities of the first m eigenvalues and on the oscillatory properties for the corresponding eigenfunctions. In future, in the paper [4, 6-8, 20] succeeded to study the structure of root subspaces corresponding of all eigenvalues and the oscillatory properties of all eigenfunctions of regular Sturmian system (1.1)-(1.4), (2.1).

Denote:

$$\mathbb{N}_0 = \begin{cases} \mathbb{N}, & if\ \delta \in [0, \delta_0], \\ \mathbb{N}\backslash\{1\}, & if\ \delta \in (\delta_0, \pi). \end{cases}$$

Theorem 2.4. *For fixed α, β, γ the boundary value problem* (1.1)-(1.4), (2.1) *for $\delta \in [0, \pi)$ has a sequence of real and simple eigenvalues*

$$\lambda_1(\delta) < \lambda_2(\delta) < \ \dots \ < \ \lambda_n(\delta) \to +\infty\,.$$

Furthermore, the eigenfunction $y_n^{(\delta)}(x)$ corresponding to the eigenvalue $\lambda_n(\delta)$ for $n \in \mathbb{N}_0$ has exactly $n-1$ simple zeros in the interval $(0, l)$*.*

By theorem 2.4 and max-min characterization of the eigenvalues [15, p.418] we have that for any $\delta_1, \delta_2 \in (0, \pi)$ such that $\delta_1 < \delta_2$ the relations are true

$$\lambda_1(\delta_2) < \lambda_1(\delta_1) < \lambda_1(0) < \lambda_2(\delta_2) < \lambda_2(\delta_1) < \lambda_2(0) < \dots\,. \tag{2.4}$$

3. Main properties of the solution of the problem (1.1)-(1.4)

Below we will need the following results of [13, Lemma 2.1 and Lemma 2.2].

Lemma 3.1. *Let $y(x)$ be a nontrivial solution of the differential equation* (1.1) *for $r \equiv 0$ and $\lambda > 0$. If $y(x)$, $y'(x)$, $y''(x)$ and $Ty(x)$ are nonnegative at $x = x_0$ (but not all zero) they are positive for all $x > x_0$. If $y(x)$, $-y'(x)$, $y''(x)$ and $-Ty(x)$ are nonnegative at $x = x_0$ (but not all zero) they are positive for all $x < x_0$.*

Lemma 3.2. *Let $y(x)$ be a nontrivial solution of the problem* (1.1), (1.2) (1.4) *for $r \equiv 0$ and $\lambda > 0$. If x_0 is zero of the function $y(x)$ or $y''(x)$ in the interval* $(0, l)$*, then and $y'(x)\,Ty(x) < 0$ in a neighborhood of x_0. If x_0 is zero of the function $y'(x)$ or $Ty(x)$ in the interval* $(0, l)$*, then and $y(x)\,y''(x) < 0$ in a neighborhood of x_0.*

The following theorem is useful in the sequel.

Theorem 3.1. *For each fixed $\lambda \in \mathbb{C}$ there exists a nontrivial solution $y(x, \lambda)$ of problem* (1.1)-(1.4)*, with unique up to a constant coefficient.*

Proof. Let $y_k(x, \lambda)$, $k = 1, \dots, 4$ be solutions of equations (1.1) normalized for $x = 0$ by the Caushy conditions

$$y_k^{(s-1)}(0, \lambda) = \delta_{ks}, \ \ s = 1, \dots, 3, \quad Ty_k(0, \lambda) = \delta_{k4}, \tag{3.1}$$

where δ_{ks} is the Kronecker delta.

We shall seek the function $y(x, \lambda)$ in the following form:

$$y(x, \lambda) = \sum_{k=1}^{4} C_k\, y_k(x, \lambda), \tag{3.2}$$

where the C_k, $k = 1, \dots, 4$ are constants.

Suppose that in boundary conditions (1.2)-(1.4) $\alpha, \beta, \gamma \neq 0$. It follows by (3.1), (3.2) and boundary conditions (1.2), (1.3) that

$$C_3 = \frac{C_2}{p(0)} ctg\alpha, \; C_4 = -C_1 ctg\beta\,. \tag{3.3}$$

Using (3.3), from (3.2) we obtain

$$y(x,\lambda) = C_1\left\{y_1(x,\lambda) - y_4(x,\lambda) ctg\beta\right\} + C_2\left\{y_2(x,\lambda) + y_3(x,\lambda)\frac{ctg\alpha}{p(0)}\right\}. \tag{3.4}$$

Taking into account (3.4) and (1.4), to determine C_1 and C_2 we obtain the relation

$$c_1\alpha^*(\lambda) + c_2\beta^*(\lambda) = 0,$$

where

$$\alpha^*(\lambda) = \left\{y_1(l,\lambda)\, ctg\gamma + p(l)y_1''(l,\lambda)\right\} - ctg\beta\left\{y_4'(l,\lambda)\, ctg\gamma + p(l)y_4''(l,\lambda)\right\}, \tag{3.5}$$

$$\beta^*(\lambda) = \left\{y_2'(l,\lambda)\, ctg\gamma + p(l)y_2''(l,\lambda)\right\} + \frac{ctg\alpha}{p(0)}\left\{y_3'(l,\lambda)\, ctg\gamma + p(l)y_3''(l,\lambda)\right\}. \tag{3.6}$$

For the completion of the proof of Theorem 3.1 it is sufficient to demonstrate that

$$|\alpha^*(\lambda)| + |\beta^*(\lambda)| > 0\,. \tag{3.7}$$

It follows by Lemma 3.1 that $y_k'(l,\lambda) > 0$, $y_k''(l,\lambda) > 0$, $k = 1, ..., 4$ for $\lambda > \lambda^*$, where $\lambda^* = \inf\{\lambda \in \mathbb{R} : \lambda\,\tau(x) - r(x) > 0,\ x \in [0,l]\}$. Hence, by (3.6) the relation (3.7) holds for $\lambda > \lambda^*$.

Let $\lambda \in \mathbb{C}\backslash[\lambda_1(\delta_0), +\infty)$. If (3.7) fails for such λ, then the functions

$$\phi_1(x,\lambda) = y_1(x,\lambda) - ctg\beta\, y_4(x,\lambda) \text{ and } \phi_2(x,\lambda) = y_2(x,\lambda) + \frac{ctg\alpha}{p(0)}\, y_3(x,\lambda)$$

solve the problem (1.1)-(1.4). We define the function $v(x,\lambda)$:

$$\varphi(x,\lambda) = T\phi_2(l,\lambda)\phi_1(x,\lambda) - T\phi_1(l,\lambda)\phi_2(x,\lambda).$$

Since $\varphi(l,\lambda) = 0$, the function $\varphi(x,\lambda)$ is an eigenfunction of the problem (1.1)-(1.5) with $\delta = \pi/2$ corresponding to the eigenvalue $\lambda \in \mathbb{C}\backslash[\lambda_1(\delta_0), +\infty)$, with is impossible, because in this case must be $\lambda = \lambda_k(\pi/2)$ for some $k \in \mathbb{N}$ and by (2.4) we have that $\lambda \geq \lambda_1(\pi/2) \geq \lambda_1(\delta_0)$.

Now let $\lambda \in [\lambda_1(\delta_0), \lambda^*)$. It follows from (2.3) and Theorem 2.4 that $\lambda^* = \lambda_m(0)$. We define the function $\psi(x,\lambda)$:

$$\psi(x,\lambda) = \phi_2(l,\lambda)\phi_1(x,\lambda) - \phi_1(l,\lambda)\phi_2(x,\lambda),$$

Since $\psi(l,\lambda) = 0$ and $T\varphi(l,\lambda) = 0$, the functions $\psi(x,\lambda)$ and $\varphi(x,\lambda)$ are eigenfunctions of the problem (1.1)-(1.5) with $\delta = 0$ and $\delta = \pi/2$ corresponding to the same eigenvalue $\lambda \in [\lambda_1(\pi/2), \lambda_m(0))$. However, this contradicts the relation (2.4).

The remaining cases are considered similarly. Theorem 2.1 is proved.

In fact, the functions $y_k(x,\lambda)$, $k = 1, ..., 4$, and their derivatives are entire functions of λ (see [21, Ch. 1]), and therefore $y(x,\lambda)$ is also an entire function of λ for each fixed $x \in [0,l]$.

It is obvious that the eigenvalues $\lambda_n(0)$ and $\lambda_n(\pi/2)$, $n \in \mathbb{N}$, of boundary value problem (1.1)-(1.4), (2.1) are the zeros of entire functions $y(x,\lambda)$ and $Ty(x,\lambda)$, respectively. We observe that the function

$$F_r(\lambda) = Ty(l,\lambda)/y(l,\lambda)$$

is will defined for

$$\lambda \in \mathbf{A} \equiv \left(\bigcup_{n=1}^{\infty} \mathbf{A}_n \right) \bigcup (\mathbb{C} \backslash \mathbb{R}),$$

where $\mathbf{A}_n \equiv (\lambda_{n-1}(0), \lambda_n(0))\,,\ n \in \mathbb{N},\ \lambda_0(0) = -\infty$ and is meromorphic function of finite order, $\lambda_n(\pi/2)$ and $\lambda_n(0)$, $n \in \mathbb{N}$ are the zeros and poles of these function, respectively.

In equation (1.1) we set $\lambda = \rho^4$. By Theorem 1 in [21, p.59], in each subdomain of the complex $\rho-$ plane equation (1.1) has four linearly independent solutions $z_k(x,\rho),\ k = \overline{1,4}$, which are regular with respect to ρ (for sufficiently large ρ) and satisfying the relations

$$z_k^{(s)}(x,\rho) = \left(\rho\omega_k \left(r/p\right)^{\frac{1}{4}} \right)^s e^{\rho\omega_k X} \left[1 + O\left(1/\rho\right)\right], k = \overline{1,4}, s = \overline{0,3}, \tag{3.8}$$

where $\omega_k,\ k = \overline{1,4}$, are the distinct fourth roots of unity, and $X = \int\limits_0^x (r/p)^{\frac{1}{4}} dt$.

For brevity, we introduce the notation $s(\delta_1,\delta_2) = \mathrm{sgn}\delta_1 + \mathrm{sgn}\delta_2$. Let $\omega_1 = -i, \omega_2 = i, \omega_3 = -1, \omega_4 = 1$, and $h = \int\limits_0^l (r/p)^{\frac{1}{4}} dt$. We shall seek the solution $y(x,\lambda)$ in the following form:

$$y(x,\lambda) = \sum_{k=1}^{4} c_k z_k(x,\rho),$$

where $c_k,\ k = 1,\ 2,\ 3,\ 4$, are constants depending only on λ . Taking into account (3.8) and boundary conditions (1.2)-(1.4), we obtain for large $|\lambda\,|$ the asymptotic estimate

$$y\left(x,\lambda\right) = \begin{cases} \left(\sin\left(\rho X + \frac{\pi}{2}\mathrm{sgn}\beta\right) - \cos\left(\rho h + \frac{\pi}{2}s(\beta,\gamma)\right) e^{\rho(X-h)}\right)[1], \\ \hfill if\ \ s(\alpha,\beta) = 1, \\ \left(\sin\rho X - \cos\rho X + (-1)^{\mathrm{sgn}\alpha} e^{-\rho X} + (-1)^{1-\mathrm{sgn}\gamma}\sqrt{2} \right. \\ \quad \left. \times\ \sin(\rho \mathrm{h} + \frac{\pi}{4}(-1)^{\mathrm{sgn}\gamma})\, e^{\rho(X-h)}\right)[1],\ \ if\ \ s(\alpha,\beta) \neq 1, \end{cases} \tag{3.9}$$

where $[1] = 1 + O\left(\frac{1}{\rho}\right)$. Similarly, for the function $Ty(x,\lambda)$ we obtain the following asymptotic estimate

$$Ty(x,\lambda) = \begin{cases} -\rho^3(pr^3)^{\frac{1}{4}}\left(\cos\left(\rho X + \frac{\pi}{2}sgn\beta\right) + \cos\left(\rho h + \frac{\pi}{2}s(\beta,\gamma)\right)\right. \\ \hfill \left. \times\, e^{\rho(X-h)}\right)[1],\ \ if\ \ s(\alpha,\beta) = 1, \\ -\rho^3(pr^3)^{\frac{1}{4}}\left(\cos\rho X + \sin\rho X + (-1)^{\mathrm{sgn}\alpha} e^{-\rho X} - (-1)^{1-\mathrm{sgn}\gamma}\right. \\ \left. \sqrt{2}\sin\left(\rho h + \frac{\pi}{4}(-1)^{\mathrm{sgn}\gamma}\right) e^{\rho(X-h)}\right)[1],\quad if\ \ s(\alpha,\beta) = 1, \end{cases} \tag{3.10}$$

Remark 3.1. As an immediate consequence of (3.9), we obtain that the number of zeros in the interval $(0,l)$ of function $y(x,\lambda)$ tends to $+\infty$ as $\lambda \to \pm\infty$.

Taking into account relations (3.9) and (3.10), we obtain the asymptotic formulas

$$F_r(\lambda) = \begin{cases} -\rho^3 (p(1) r^3(1))^{\frac{1}{4}} \frac{(\sqrt{2})^{1-2\mathrm{sgn}\gamma} cos\left(\rho h + \frac{\pi}{2}\mathrm{sgn}\beta + \frac{\pi}{4}\mathrm{sgn}\gamma\right)}{cos\left(\rho l + \frac{\pi}{2}\mathrm{sgn}\beta + \frac{\pi}{4}(1+\mathrm{sgn}\gamma)\right)} [1], \\ \qquad\qquad\qquad\qquad if \;\; s(\alpha,\beta) = 1, \\ \\ -\rho^3 (p(1) r^3(1))^{\frac{1}{4}} \frac{(\sqrt{2})^{1-2\mathrm{sgn}\gamma} cos\left(\rho h - (1-\mathrm{sgn}\gamma)\frac{\pi}{4}\right)}{cos\left(\rho h + \frac{\pi}{4}\mathrm{sgn}\gamma\right)} [1], \\ \qquad\qquad\qquad\qquad if \;\; s(\alpha,\beta) = 1, \end{cases} \tag{3.11}$$

In turn, from (3.11) should the asymptotic formula

$$F_r(\lambda) = -\left(\sqrt{2}\right)^{1-2sgn\gamma} (p(1)r^3(1))^{\frac{1}{4}} \sqrt[4]{|\lambda|^3} \left(1 + O\left(1 \Big/ \sqrt[4]{|\lambda|}\right)\right), \text{ as } \lambda \to -\infty\,. \tag{3.12}$$

From (3.12) follows immediately

Lemma 3.3. *The following relation holds:*

$$\lim_{\lambda \to -\infty} F_r(\lambda) = -\infty\,. \tag{3.13}$$

We also have the following result.

Lemma 3.4. *The following formula holds:*

$$\frac{dF_r(\lambda)}{d\lambda} = \frac{1}{y^2(l,\lambda)} \int\limits_0^l r y^2(x,\lambda) dx, \lambda \in A \tag{3.14}$$

The proof is similar to that of [17, lemma 5; 3, Lemma 1].

Now we investigate the problem on the number of zeros of function $y(x,\lambda)$.

Lemma 3.5. *Every zero $x(\lambda)$ of the function $y(x,\lambda)$ is simple and is a differentiable function of $\lambda \in [\lambda_1(\delta_0), +\infty)$.*

Proof. If $\lambda > \lambda^*$, then by Lemma 3.2 the function $y(x,\lambda)$ does not have multiple zero in the interval $(0,l)$. Let $\lambda \in [\lambda_1(\delta_0), \lambda_m(0)]$. We define the angle δ_λ by the equality $\delta_\lambda = \cot^{-1}(Ty(l,\lambda)/y(l,\lambda)) \in [0,\pi)$. Then λ is a eigenvalue of the problem (1.1)-(1.4), (2.1) with the serial number of the $[1,\, m] \cap \mathbb{N}_0$. Hence, by Theorem 2.4 the zeros of function $y(x,\lambda)$ contained in $(0,l)$ are simple. The rest of the proof concerning the smoothness of $x(\lambda)$ follows the well-known implicit function theorem. The Lemma 3.5 is proved.

By $\tau(\lambda)$, we denote the number of zeros of $y(x,\lambda)$ in the interval $(0,l)$. Lemma 3.5 and Theorem 2.4 readily imply the following assertion.

Lemma 3.6. *Let $\lambda > \lambda_1(\delta_0)$. If $\lambda \in (\lambda_{n-1}, \lambda_n]$ for $n > 1$, then $\tau(\lambda) = n-1$.*

4. Oscillatory and basis properties of eigenfunctions of the problem (1.1)-(1.5)

The considered problem (1.1)-(1.5) can be reduced to the eigenvalue problem for the linear operator L in the Hilbert space $H = L_2(0,l) \oplus \mathbb{C}$ with inner product

$$(\hat{y}, \hat{u}) = (\{y,k\}, \{u,s\}) = (y,u)_{L_2^\tau} + |\sigma|^{-1}\, k\, \bar{s}, \tag{4.1}$$

where $(\cdot\,,\,\cdot)_{L_2^\tau}$ is an inner product in $L_2^\tau(0,l)$ and

$$L\hat{y} = L\{y(x),k\} = \left\{\frac{1}{\tau(x)}\left((Ty(x))' + r(x)y(x)\right),\, dTy(l) - by(l)\right\}$$

is an operator with the domain

$$D(L) = \left\{\{y(x),k\} \in H\ :\ y(x) \in W_2^4(0,l),\ (Ty(x))' +\ r(x)y(x) \in L_2(0,l)\ ,\right. \\ \left. y(x) \in B.C_0.\ ,\ k = ay(l) - cTy(l)\right\}.$$

dense everywhere in H (see [24]). Obviously, the operator L is well defined in H. Problem (1.1)-(1.5) acquires the form

$$L\hat{y} = \lambda\hat{y},\ \ \hat{y} \in D(L),$$

i.e., the eigenvalues λ_n, $n \in \mathbb{N}$, of the operator L and problem (1.1)-(1.5) coincide together with their multiplicities, and between the root functions, there is a one-to-one correspondence

$$\hat{y}_n = \{y_n(x), k_n\} \leftrightarrow y_n(x), \quad k_n = ay_n(l) - c\,Ty_n(l).$$

Problem (1.1)-(1.5) is strongly regular in the sense of [24]; in particular, this problem has discrete spectrum.

We define a number $\varkappa$ and an operator $J : H \to H$ as

$$\varkappa = \begin{cases} 0, & \text{if } \sigma > 0, \\ 1, & \text{if } \ \sigma < 0, \end{cases}$$

$$J\{y,k\} = \{y, k\,\mathrm{sign}\,\sigma\}.$$

The operator J is unitary and symmetric on H, and its spectrum consists of two eigenvalues, 1 with multiplicity $\varkappa$ and +1 with infinite multiplicity. Therefore, this operator generates the Pontryagin space $\Pi_\varkappa = L^2(0,l) \oplus \mathbb{C}$ with the inner product ($J-$ metric) [11, 12]

$$(\hat{y},\hat{u})_{\Pi_\varkappa} = [\{y,k\},\{u,s\}] = (y,u)_{L_2} + \sigma^{-1}\,k\,\bar{s}. \tag{4.2}$$

Theorem 4.1. *The operator L is $J-$self-adjoint in $\Pi_\varkappa$.*

The proof is similar to that of Proposition 1 in [10] (see also [14, Theorem 2.2]).

Remark 4.1. In the case $\sigma > 0$ (i.e., if $\varkappa = 0$) $J = I$ and $\Pi_\varkappa = \Pi_0 = H$, where I denotes the identity operator on H. Hence, in this case the operator L is self-adjoint on H.

Let λ be an eigenvalue of L of algebraic multiplicity ν. We set $\rho(\lambda)$ to be equal to ν if $\mathrm{Im}\lambda \neq 0$ and to the integer part $\nu/2$ if $\mathrm{Im}\lambda = 0$.

Theorem 4.2. *The eigenvalues of the operator L are arranged symmetrically around the real axis, and $\sum\limits_{\tau=1}^{n} \rho(\lambda_\tau) \le \varkappa$ for any system $\{\lambda_\tau\}_{\tau=1}^n\,(n \le +\infty)$ of eigenvalues with nonnegative imaginary parts.*

The proof of this theorem follows from [22].

Remark 4.2. In the case $\sigma > 0$ (i.e., if $\varkappa = 0$) the all eigenvalues of problem (1.1)-(1.5) are real and simple. In the case $\sigma < 0$ (i.e., if $\varkappa = 1$) this problem may have either one pair of complex conjugate nonreal eigenvalues or one real multiple eigenvalue whose multiplicity does not exceed 3.

Lemma 4.1. [11]. *Let L^* be the adjoint operator of L in H. Then $L^* = JLJ$.*

The proof of this lemma follows from [11, Section 3, Proposition 5].

For $c \neq 0$ let N be an integer such that $\mu_{N-1} < d/c \leq \mu_N$.

Theorem 4.3. *Let $\sigma > 0$. Then the eigenvalues of the boundary value problem* (1.1)-(1.5) *are form an infinitely increasing sequence $\lambda_1 < \lambda_2 < ... < \lambda_n < ...,$ where $\lambda_n \in (\lambda_{n-1}(0), \lambda_n(0))$ if $c = 0$ or $c \neq 0$ and $n < N$; $\lambda_N, \lambda_{N+1} \in (\lambda_{N-1}(0), \lambda_N(0)]$; $\lambda_n \in (\lambda_{n-2}(0), \lambda_{n-1}(0))$ if $c \neq 0$ and $n > N + 1$ Moreover, the corresponding eigenfunctions $y_1(x), y_2(x), ..., y_n(x)$, ... have the following oscillatory properties :(a) if $c = 0$ and $n > 1$, then $y_n(x)$ has exactly $n - 1$ simple zeros in the interval $(0, l)$; (b) if $c \neq 0$ and $n > 1$, then $y_n(x)$ has exactly $n - 1$ simple zeros for $n \leq N$ and exactly $n - 2$ simple zeros for $n > N$ in the interval $(0, l)$.*

The proof is similar to that [19, Theorem 2.2] (see, also [18, Theorem 2]) in view formula (3.12) and Lemmas 3.3, 3.4 and 3.6.

If $\sigma > 0$, then by theorems 4.1 - 4.3 and remarks 4.1, 4.2 L is a self-adjoint discrete lower-semibounded operator in H and hence the system of eigenvectors $\{y_n(x), k_n\}$, $n \in \mathbb{N}$, of this operator forms an orthogonal basis in H.

Throughout the following, we assume that the condition $\sigma > 0$. By (4.1) (or (4.2)) we have

$$(\hat{y}_n, \hat{y}_n)_H = \|y_n\|^2_{L_2^r} + \sigma^{-1} k_n^2. \tag{4.3}$$

We denote:

$$\delta_n = ||y_n||^2_{L_2^r} + \sigma^{-1} k_n^2, \ n = 1, 2, \dots . \tag{4.4}$$

Then, by $\sigma > 0$, from (4.4), we obtain

$$\delta_n > 0, \ n = 1, 2, \dots . \tag{4.5}$$

Note that,

$$k_n = a y_n(l) - cTy_n(l) \neq 0, \ n = 1, 2, \dots . \tag{4.6}$$

Indeed, if $k_n = 0$ for some $n \in \mathbb{N}$, then $y_n(l) = Ty_n(l) = 0$ by (1.5) and (1.6), which contradicts the relation (2.4). Then, by virtue of (4.3), (4.5) and (4.6), the elements $\hat{\vartheta}_n = \{\vartheta_n(x), s_n\}$ of the system $\{\hat{\vartheta}_n\}_{n=1}^{\infty}$ conjugated to the system $\{\hat{y}_n\}_{n=1}^{\infty}$ are defined by the relation

$$\hat{\vartheta}_n = \delta_n^{-1} \hat{y}_n, \ n \in \mathbb{N}. \tag{4.7}$$

Hence, from (4.7), by (4.5) and (4.6), we find that

$$s_n = \delta_n^{-1} k_n \neq 0, n \in \mathbb{N}. \tag{4.8}$$

Let ς be an arbitrary fixed positive integer.

Theorem 4.4. *The system of eigenfunctions $\{y_n(x)\}_{n=1, n\neq\varsigma}^{\infty}$ of problem* (1.1)-(1.5) *forms a Riesz basis in the space $L_2^r(0, l)$ and basis in the space $L_p^r(0, l)$*, $1 <$

$p < \infty$, and the conjugate system $\{u_n(x)\}_{n=1, n\neq\varsigma}^{\infty}$ of the system $\{y_n(x)\}_{n=1,\, n\neq\varsigma}^{\infty}$ is given by the formula

$$u_n(x) = y_n(x) - s_n s_\varsigma^{-1} y_\varsigma(x)$$

The basicity of the system $\{y_n(x)\}_{n=1,\, n\neq\varsigma}^{\infty}$ in the space $L_2^r(0,l)$ follows from Corollary 3.1 of [1] by (4.8). Next, the basis property of this system in the space $L_p^r(0,l)$, $p \in (1, \infty)\backslash\{2\}$ can be proved in accordance with the scheme of the proof of Theorem 5.1 in [2].

Remark 4.3. In the case $\sigma < 0$ the spectral properties of problem (1.1)-(1.5) are studied similar to that of [2].

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