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SOME SPECTRAL PROPERTIES OF THE BOUNDARY VALUE PROBLEM WITH SPECTRAL PARAMETER IN THE BOUNDARY CONDITIONS

ZIYATKHAN S. ALIYEV AND FATIMA-KHANIM I. ALLAHVERDI-ZADA

Abstract. In this paper we consider the Sturm-Liouville problem with spectral parameter in the boundary conditions. We study the structure of root subspaces and location of eigenvalues on the complex plane of this problem.

1. Introduction

In this paper we consider the following boundary value problem spectral problem

$$-y''(x) = \lambda y(x),\ \ x \in (0,1), \tag{1.1}$$

$$y'(0) = -(a_0\lambda + b_0)y(0), \tag{1.2}$$

$$y'(1) = (a_1\lambda + b_1)y(1), \tag{1.3}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, a_0, a_1, b_0, b_1 are real constants, and $a_0 \neq 0$, $a_1 \neq 0$. This problem arising in a mathematical model of torsional vibrations of a rod with pulleys at the ends [16].

The structure of root subspaces and location of eigenvalues on the real axis of problem (1.1)-(1.3) were studied by Kapustin [10] (see also [11]) for the case where $a_0 > 0$, $a_1 > 0$, $b_1 = 0$ and by Aliev [1, 3] for the cases where $a_0 > 0$, $a_1 < 0$, $b_0 = 0$, $b_1 = 0$ and $a_0 < 0$, $a_1 < 0$, $b_0 = 0$, $b_1 = 0$ and by Dunyamaliyeva [9] for the case $a_0 < 0$, $a_1 < 0$, $b_0 = 0$, $b_1 \in \mathbb{R}$. In these papers, studied also basis properties in the space $L_p(0,1)$, $1 < p < \infty$, of the system of root functions, where obtained necessary and sufficient conditions for the basicity of subsystems of root functions of problem (1.1)-(1.3) in the space $L_p(0,1)$, $1 < p < \infty$. In [5, 6] studied the eigenvalue problem for a second order differential equation with spectral parameter in the boundary conditions in the more general case, where investigate oscillation properties of eigenfunctions and obtained the sufficient condition for basicity of subsystem of eigenfunctions in the space $L_p(0,1)$, $1 < p < \infty$. To study the basis properties of systems of root functions of problem (1.1)-(1.3) in the space $L_p(0,1)$, $1 < p < \infty$, one needs a general characteristic of the arrangement of eigenvalues on the real axis (the complex plane) and the structure

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of root subspaces. In the present paper we study these problems in the case $a_0 < 0$, $a_1 < 0$, $b_0 < 1$, $b_1 \in \mathbb{R}$.

2. Operator interpretation of problem (1.1)-(1.3)

The considered problem (1.1)-(1.3) can be reduced to the eigenvalue problem for the linear operator L in the Hilbert space $H = L_2(0, l) \oplus \mathbb{C}^2$ with inner product

$$(\hat{u}, \hat{v})_H = (\{u(x), m, n\}, \{v(x), s, t\})_H = (u, v)_{L_2} + |a_0|^{-1} m\bar{s} + |a_1|^{-1} n\bar{t} \quad (2.1)$$

where $(\cdot, \cdot)_{L_2}$ is an inner product in $L_2(0, 1)$ and

$$L\hat{y} = L\{ y(x), m, n\} = \{-y''(x),\, y'(0) + b_0 y(0),\, y'(1) - b_1 y(1)\}$$

is an operator with the domain

$$D(L) = \{ \hat{y} \in H \mid y(x),\, y'(x) \in AC[0, 1],\, m = -a_0 y(0),\, n = a_1 y(1)\}$$

dense everywhere in H [13, 15]. With this framework it is easily seen that the eigenvalue problem (1.1)-(1.3) is equivalent to eigenvalue problem

$$L\hat{y} = \lambda\hat{y}, \;\; \hat{y} \in D(L),$$

i.e. the eigenvalues λ_n, $n \in \mathbb{N}$ of the operator L and problem (1.1)-(1.3) coincide together with their multiplicities, and between the root functions, there is a one-to-one correspondence

$$y_n(x) \leftrightarrow \hat{y}_n = \{y_n(x),\, m_n, k_n\}, \;\; m_n = -a_0 y_n(0), \;\; k_n = a_1 y_n(1).$$

Problem (1.1)-(1.3) is strongly regular in the sense of [15]; in particular, this problem has discrete spectrum.

We now introduce the the operator $J : H \to H$ by

$$J\{ y, m, n\} = \{ y, -m, -n\}.$$

The operator J is unitary and symmetric in H with spectrum consisting of two eigenvalues, -1 with multiplicity 2 and 1 with infinite multiplicity [9]. Consequently, this operator generates the Pontryagin space $\Pi_2 = L_2(0, l) \oplus \mathbb{C}^2$ with inner product (J-metric) [8]

$$(\hat{u}, \hat{v})_{\Pi_2} = (\{u(x), m, n\}, \{v(x), s, t\})_{\Pi_2} = (u, v)_{L_2} + a_0^{-1} m\bar{s} + a_1^{-1} n\bar{t}. \quad (2.2)$$

Theorem 2.1 . *The operator JL is self-adjoint, bounded below and has compact resolvent in H.*

The proof is similar to that of Theorem 2.2 in [9].

From Theorem 2.2 implies that

Corollary 2.1. *L is a self-adjoint operator on Π_2.*

Let λ be an eigenvalue of L of algebraic multiplicity ν. We set $\rho(\lambda)$ to be equal to ν if $\mathrm{Im}\lambda \neq 0$ and to the integer part $\nu/2$ if $\mathrm{Im}\lambda = 0$.

Theorem 2.3 . *The eigenvalues of the operator L are arranged symmetrically around the real axis, and $\sum\limits_{k=1}^{n} \rho(\lambda_k) \leq 2$ for any system $\{\lambda_k\}_{k=1}^{n} (n \leq +\infty)$ of eigenvalues with nonnegative imaginary parts.*

The proof of this theorem follows from [12] (see also [13]).

It follows from Theorem 2.3 that problem (1.1)-(1.3) may have either at most two pair of complex conjugate non-real eigenvalues, or have at most two real multiple eigenvalues whose sum of the algebraic multiplicities not exceeding 5.

3. Some auxiliary facts and contentions

Along with problem (1.1)-(1.3), consider the following eigenvalue problems

$$\begin{gathered} -y''(x) = \lambda y(x),\ x \in (0,1), \\ y'(0) = -(a_0\lambda + b_0)y(0),\ \ y(1) = 0, \end{gathered} \tag{3.1}$$

and

$$\begin{gathered} -y''(x) = \lambda y(x),\ x \in (0,1), \\ y'(0) = -(a_0\lambda + b_0)y(0),\ \ y'(1) = 0. \end{gathered} \tag{3.2}$$

The substitution $x = x(t) = 1 - t$ transforms $[0,1]$ into the interval $[0,1]$ and problems (3.1) and (3.2) into

$$\begin{gathered} -u''(t) = \lambda u(t),\ t \in (0,1), \\ u(0) = 0,\ \ u'(1) = (a_0\lambda + b_0)u(1), \end{gathered} \tag{3.3}$$

and

$$\begin{gathered} -u''(t) = \lambda u(t),\ t \in (0,1), \\ u'(0) = 0,\ \ u'(1) = (a_0\lambda + b_0)u(1), \end{gathered} \tag{3.4}$$

respectively, where $u(t) = y(x(t)) = y(1-t)$.

The solution of equation $-u''(t,\lambda) = \lambda u(t,\lambda)$, $t \in (0,1)$, satisfying the initial conditions $u(0,\lambda) = 0$ and $u'(0,\lambda) = 1$ is

$$u(t,\lambda) = \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}},$$

and the solution of this equation satisfying the initial conditions $u(0,\lambda) = 1$ and $u'(0,\lambda) = 0$ is

$$\tilde{u}(t,\lambda) = \cos\sqrt{\lambda}t.$$

The eigenvalues $\mu'_k = k^2\pi^2$ and $\nu'_k = \left(k - \frac{1}{2}\right)^2\pi^2$, $k = 1, 2, \ldots$, of the boundary value problems

$$\begin{gathered} -u''(t) = \lambda u(t),\ \ 0 < x < 1, \\ u(0) = 0,\ u(1) = 0, \end{gathered}$$

and

$$\begin{gathered} -u''(t) = \lambda u(t),\ \ 0 < x < 1, \\ u(0) = 0,\ u'(1) = 0, \end{gathered}$$

are the poles and zeros of a meromorphic function $F(\lambda) = \frac{\sqrt{\lambda}\cos\sqrt{\lambda}}{\sin\sqrt{\lambda}}$, respectively, defined on

$$D' \equiv (\mathbb{C}\backslash\mathbb{R}) \cup \left(\bigcup_{k=1}^{\infty} (\mu'_{k-1}, \mu'_k)\right);$$

the eigenvalues $\mu''_k = \left(k - \frac{1}{2}\right)^2\pi^2$ and $\nu''_k = (k-1)^2\pi^2$, $k = 1, 2, \ldots$, of the boundary value problems

$$\begin{gathered} -u''(t) = \lambda u(t),\ \ 0 < x < 1, \\ u'(0) = 0,\ u(1) = 0, \end{gathered}$$

and

$$-u''(t) = \lambda u(t),\ 0 < x < 1,$$
$$u'(0) = 0,\ u'(1) = 0,$$

are the poles and zeros of a meromorphic function $\tilde{F}(\lambda) = \frac{\sqrt{\lambda}\sin\sqrt{\lambda}}{\cos\sqrt{\lambda}}$, respectively, defined on

$$D'' \equiv (\mathbb{C}\backslash\mathbb{R}) \cup \left(\bigcup_{k=1}^{\infty}(\mu''_{k-1},\mu''_k)\right).$$

Lemma 3.1. *The following relations hold*

$$\frac{dF(\lambda)}{d\lambda} = -\frac{\int\limits_0^1 u^2(x,\lambda)dx}{u^2(1,\lambda)},\quad \lambda\in D', \tag{3.5}$$

$$\frac{d\tilde{F}(\lambda)}{d\lambda} = -\frac{\int\limits_0^1 \tilde{u}^2(x,\lambda)dx}{\tilde{u}^2(1,\lambda)},\quad \lambda\in D''. \tag{3.6}$$

The proof follows from (2.3) of [4].

Lemma 3.2. *The following relations hold*

$$\lim_{\lambda\to-\infty} F(\lambda) = +\infty, \tag{3.7}$$

$$\lim_{\lambda\to-\infty} \tilde{F}(\lambda) = +\infty. \tag{3.8}$$

Proof. For $\lambda < 0$ we have

$$F(\lambda) = \frac{\sqrt{\lambda}\cos\sqrt{\lambda}}{\sin\sqrt{\lambda}} = \frac{i\sqrt{|\lambda|}\cos i\sqrt{|\lambda|}}{\sin i\sqrt{|\lambda|}} = \frac{\sqrt{|\lambda|}\,ch\sqrt{|\lambda|}}{sh\sqrt{|\lambda|}}$$

with implies that

$$F(\lambda) = \sqrt{|\lambda|}\left(1 + O\left(\frac{1}{\sqrt{|\lambda|}}\right)\right),\ \lambda\to-\infty.$$

Relation (3.8) is proved similarly.

Lemma 3.3. *The function $F(\lambda)$ $(\tilde{F}(\lambda)))$ is concave on the interval $(-\infty,\mu'_1)$ $((-\infty,\mu''_1))$.*

The proof of this lemma is similar to that of [7, Sentense 4].

We also have the following relations

$$F(0) = 1,\ \ \tilde{F}(0) = 0. \tag{3.9}$$

Set $B'_k = (\mu'_{k-1},\mu'_k)$ and $B''_k = (\mu''_{k-1},\mu''_k)$, $k = 1, 2, \ldots$.

Notice that, the eigenvalues (counted with multiplicities) of problems (3.1) and (3.2) are roots of the equations

$$F(\lambda) = a_0\lambda + b_0, \tag{3.10}$$

and

$$\tilde{F}(\lambda) = a_0\lambda + b_0. \tag{3.11}$$

respectively.

Lemma 3.4. *If* $b_0 < 0$, *then the equation* (3.10) ((3.11)) *can has a unique solution in each interval* B'_k (B''_k), $k = 2, 3, 4, \ldots$.
Proof. By (3.1) we have

$$-u'u|_0^1 + \int_0^1 u'^2(t,\lambda)\,dt = \lambda \int_0^1 u^2(t,\lambda)\,dt \tag{3.12}$$

with implies that

$$\int_0^1 u'^2(t,\lambda)\,dt \; - b_0 u^2(1,\lambda) = \lambda \left(\int_0^1 u^2(t,\lambda)\,dt + a_0 u^2(1,\lambda) \right). \tag{3.13}$$

Let $\lambda \in B'_k$, $k \in \mathbb{N}\backslash\{1\}$ is an eigenvalue of problem (3.10). Then by $b_0 < 0$ from (3.13) we obtain

$$\int_0^1 u^2(t,\lambda)\,dt + a_0 u^2(1,\lambda) > 0.$$

By virtue (3.5) from this relation follows that

$$\frac{d}{d\lambda}\left(F(\lambda) - (a_0\lambda + b_0)\right) > 0.$$

Then the function $F(\lambda) - (a_0\lambda + b_0)$ is takes a value zero only strictly increasing in the interval B'_k, $k \in \mathbb{N}\backslash\{1\}$. Hence, the equation (3.1) can has a unique solution in each interval B'_k, $k \in \mathbb{N}\backslash\{1\}$.

The assertion of this lemma for the problem (3.2) can be proved similarly. The proof of Lemma 3.4 is completed.

Lemma 3.5. *If* $b_0 \in (0,1)$, *then the equation* (3.11) *can has a unique solution in each interval* B''_k, $k = 2, 3, 4, \ldots$.
Proof. From the proof of Lemma 3.4 it follows that if equation (3.11) has more than one solution in some interval $B''_{\tilde{k}}$, $\tilde{k} \geq 2$, then is for one of these solutions, which is denoted $\tilde{\lambda}$, we have the inequality

$$\tilde{F}'(\tilde{\lambda}) - a_0 \geq 0,$$

from whence, by (3.6) and (3.13), we obtain

$$\int_0^1 \tilde{u}'^2(t,\tilde{\lambda})\,dt \; < b_0 \tilde{u}^2(1,\tilde{\lambda}). \tag{3.14}$$

By $\tilde{u}(x,\lambda) = \cos\sqrt{\lambda}x$ we have

$$\int_0^1 \tilde{u}'^2(t,\tilde{\lambda})\,dt \; = \frac{\tilde{\lambda}}{2}\left(1 - \frac{\sin 2\sqrt{\tilde{\lambda}}}{2\sqrt{\tilde{\lambda}}}\right) \text{ and } \tilde{u}^2(1,\tilde{\lambda}) = \cos^2\tilde{\lambda}. \tag{3.15}$$

Direct calculations show that

$$\frac{\lambda}{2}\left(1 - \frac{\sin 2\sqrt{\lambda}}{2\sqrt{\lambda}}\right) > 1 \text{ for } \lambda > \frac{\pi^2}{4} = \mu'_1.$$

Consequently, by (3.15) we have the following relations

$$\int_0^1 \tilde{u}'^2(t, \tilde{\lambda})\, dt > 1 \text{ and } b_0 \tilde{u}^2(1, \tilde{\lambda}) < 1,$$

which contradicts inequality (3.14). The proof of Lemma 3.5 is complete.

Theorem 3.1. *If $b_0 < 0$, then the eigenvalues of the boundary value problems* (3.1) *and* (3.2) *are simple and forms are infinitely increasing sequences of* $\{\mu_k\}_{k=1}^{\infty}$ *and* $\{\nu_k\}_{k=1}^{\infty}$ *respectively, and the following relation holds*

$$\mu_1 < \nu_1 < 0 < \nu_2 < \mu_2 < \nu_3 < \mu_3 < \nu_4 < \mu_4 < \ \dots . \tag{3.16}$$

Proof. By (3.5)-(3.8) and the relations $u\,(1, \mu_k') = 0,\ \tilde{u}\,(1, \mu_k'') = 0,\ k \in \mathbb{N}$, we have

$$\lim_{\lambda \to \mu_{n-1}' + 0} F(\lambda) = +\infty, \quad \lim_{\lambda \to \mu_n' - 0} F(\lambda) = -\infty;$$
$$\lim_{\lambda \to \mu_{n-1}'' + 0} \tilde{F}(\lambda) = +\infty, \quad \lim_{\lambda \to \mu_n'' - 0} \tilde{F}(\lambda) = -\infty.$$

Hence, the functions $F(\lambda)$ and $\tilde{F}(\lambda)$ takes each value in $(-\infty, +\infty)$ at are unique points (μ_{k-1}', μ_k') and (μ_{k-1}', μ_k') respectively. Since $a_0 < 0$, it follows that the function $a_0\lambda + b_0$ is strictly decreasing on the interval $(-\infty, +\infty)$.

By Lemmas 3.3, 3.4 and relations (3.9) it follows from the preceding considerations that in the intervals $(-\infty, \mu_1')$ and $(-\infty, \mu_1'')$, equations (3.10) and (3.11) have two simple roots $\mu_1 < 0 < \mu_2$ and $\nu_1 < 0 < \nu_2$ respectively, and $\mu_1 < \nu_1 < 0 < \nu_2 < \mu_2$; in the intervals (μ_{k-1}', μ_k') and (μ_{k-1}', μ_k'), $k \geq 2$, equations (3.10) and (3.11) have one roots μ_{k+1} and ν_{k+1} respectively, and $\nu_{k+1} < \mu_{k+1}$. The proof of Theorem 3.1 is complete.

Theorem 3.2. *Let $b_0 \in (0, 1)$. Then the eigenvalues of the boundary value problem* (3.1) *are real and simple and form an infinitely increasing sequence of* $\{\mu_k\}_{k=1}^{\infty}$, *and* $\mu_1 < 0 < \mu_2 < \mu_3 < \ \dots$ *; for the equation* (3.2) *one of the following assertions holds: (i) all eigenvalues of problem* (3.2) *are real; in this case,* $(-\infty, \mu_1'')$ *contains algebraically two eigenvalues (either two simple eigenvalues or one double eigenvalue), and* (μ_{k-1}'', μ_k''), $k = 2, 3, \dots$, *contains one simple eigenvalue, and either* $\nu_1 \leq \nu_2 < 0 < \nu_3 < \nu_4 < \dots$ *or* $0 < \nu_1 \leq \nu_2 < \nu_3 < \nu_4 < \dots$*; (ii) problem* (3.2) *has one pair of nonreal complex conjugate eigenvalues; in this case,* $(-\infty, \mu_1'')$ *contains no eigenvalues, and* (μ_{k-1}'', μ_k''), $k = 2, 3, \dots$, *contains one simple eigenvalue, i.e.* $\nu_1,\ \nu_2 \in \mathbb{C}\backslash\mathbb{R}$, $0 < \nu_3 < \nu_4 < \dots$. *Moreover, the following relations hold:*

$$\begin{aligned} &\mu_1 < \nu_1 \leq \nu_2 < \mu_2 < \nu_3 < \mu_3 < \ \nu_4 < \mu_4 < \ \dots, \text{ if } \nu_1, \nu_2 \in \mathbb{R}, \\ &\mu_1 < \mu_2 < \nu_3 < \mu_3 < \ \nu_4 < \mu_4 < \ \dots, \text{ if } \nu_1, \nu_2 \in \mathbb{C}\backslash\mathbb{R}. \end{aligned} \tag{3.17}$$

The proof of the first part of this theorem is similar to that of Theorem 3.1, and the second part is similar to that of Theorem 4.1 from [2] with use Lemmas 3.1-3.5 and relation (3.9).

4. The structure of root subspaces and location of eigenvalues on the real axis of problem (1.1)-(1.3)

The solution of equation (1.1) satisfying the initial conditions

$$y(0,\lambda)=-1 \ \text{ and } \ y'(0,\lambda)=a_0\lambda+b_0 \tag{4.1}$$

is

$$y(x,\lambda)=\left(a_0\sqrt{\lambda}+\frac{b_0}{\sqrt{\lambda}}\right)\sin\sqrt{\lambda}\,x-\cos\sqrt{\lambda}\,x. \tag{4.2}$$

The function

$$G(\lambda)=\frac{y'(1,\lambda)}{y(1,\lambda)}$$

is defined in the set

$$D\equiv(\mathbb{C}\backslash\mathbb{R})\cup\left(\bigcup_{k=1}^{\infty}(\mu_{k-1},\mu_k)\right),$$

where is assumed $\mu_0=-\infty$. This is a meromorphic function of finite order, and μ_k and ν_k, $k\in\mathbb{N}$, are the zeros and poles of this function, respectively. Notice that, the eigenvalues (counted with multiplicities) of problem (1.1)-(1.3) are roots of the equation

$$G(\lambda)=a_1\lambda+b_1. \tag{4.3}$$

Lemma 4.1. *The following relations hold*

$$\frac{dG(\lambda)}{d\lambda}=-\frac{\int\limits_0^1 y^2(x,\lambda)dx+a_0}{y^2(1,\lambda)},\ \ \lambda\in D, \tag{4.4}$$

$$G(\lambda)=\sqrt{|\lambda|}\left(1+O\left(\frac{1}{\sqrt{|\lambda|}}\right)\right)\ \ \textit{for}\ \ \lambda\to-\infty. \tag{4.5}$$

The proof of this lemma is similar to that of [9, Lemma 2.2].

Lemma 4.2. *If $b_0\in(-\infty,0)\cup(0,1)$, then the following relations hold:*

$$\begin{array}{l}
G(\mu_1-0)=+\infty,\ G(\mu_1+0)=-\infty;\\
G(\lambda)>0\ \ for\ \ \lambda\in(\nu_1,\nu_2)\ \ in\ the\ case\ \ b_0<0\ \ or\ \ b_0\in(0,1)\ \ and\ \ \nu_1,\nu_2\,(\nu_1\neq\nu_2)\in\mathbb{R},\\
G(\lambda)<0\ \ for\ \ \lambda\in(\mu_1,\mu_2)\backslash\{\nu_1\}\ \ in\ the\ case\ \ b_0\in(0,1)\ \ and\ \ \nu_1=\nu_2,\\
G(\lambda)<0\ \ for\ \ \lambda\in(\mu_1,\mu_2)\ \ in\ the\ case\ \ b_0\in(0,1)\ \ and\ \ \nu_1,\nu_2\in\mathbb{C};\\
G(\mu_k-0)=-\infty, G(\mu_k+0)=+\infty,\ k=2,3,\,...\,.
\end{array} \tag{4.6}$$

Proof. By (3.16) and (3.17) it follows from (4.5) that

$$G(\lambda)>0\ \text{ for }\ \lambda\in(-\infty,\mu_1),$$

with implies that

$$G(\mu_1-0)=+\infty.$$

Since μ_1 is simple pole of the function $G(\lambda)$, then we have

$$G(\mu_1+0)=-\infty.$$

In the case $b_0 < 0$ or $b_0 \in (0,1)$ and $\nu_1, \nu_2 \in \mathbb{R}$, $\nu_1 \neq \nu_2$, by the relations (3.16) and (3.17) it follows from last equality that

$$\begin{gathered} G(\nu_1 - 0) < 0,\; G(\nu_1) = 0, \\ G(\lambda) > 0 \text{ for } \lambda \in (\nu_1, \nu_2), \\ G(\nu_2) = 0,\; G(\nu_2 + 0) < 0. \end{gathered}$$

In the cases $b_0 \in (0,1)$, $\nu_1 = \nu_2 \in \mathbb{R}$ and $b_0 \in (0,1)$, $\nu_1, \nu_2 \in \mathbb{C}$, by the relations (3.17) it follows from last equality that

$$G(\nu_1 - 0) < 0,\; G(\nu_1) = 0,\; G(\nu_1 + 0) < 0,\; G(\lambda) < 0 \text{ for } \lambda \in (\mu_1, \mu_2)\backslash\{\nu_1\},$$

$$G(\lambda) < 0 \text{ for } \lambda \in (\mu_1, \mu_2),$$

respectively. Consequently, we obtain

$$G(\mu_2 - 0) = -\infty.$$

From simplicity of the pole μ_2 it follows also that

$$G(\mu_2 + 0) = +\infty.$$

Further, since $\nu_k \in (\mu_{k-1}, \mu_k)$, $k \geq 3$, is a simple zeros of the function $G(\lambda)$, then by (4.4) we have

$$G(\nu_k - 0) > 0 \text{ and } G(\nu_k - 0) < 0,$$

with implies that

$$G(\mu_k - 0) = -\infty \text{ and } G(\mu_k + 0) = +\infty.$$

The proof of Lemma 4.2 is complete.

Theorem 4.1. *The following representation holds:*

$$G(\lambda) = \sum_{k=1}^{\infty} \frac{\lambda\, c_k}{\mu_k(\lambda - \mu_k)}, \tag{4.7}$$

where

$$c_k = \underset{\lambda=\mu_k}{res}\, G(\lambda) = \frac{y'_x(1,\lambda)}{y'_\lambda(1,\lambda)},\; k \in N, \tag{4.8}$$

$c_1 < 0,\; c_k > 0,\, k \in \mathbb{N}\backslash\{1\}$.

Proof of this theorem is similar to that of [9, Theorem 2.4] with the use of Lemma 4.1.

From (4.7) we obtain

$$\frac{d^2G(\lambda)}{d\lambda^2} = 2\sum_{k=1}^{\infty} \frac{c_k}{(\lambda - \mu_k)^3},$$

with implies that

$$\frac{d^2G(\lambda)}{d\lambda^2} > 0, \text{ if } \lambda \in (\mu_1, \mu_2). \tag{4.9}$$

Hence, the function $G(\lambda)$ is convex upward in the interval (μ_1, μ_2).

Lemma 4.3. *If $b_0 < 0$ and $b_1 < 0$, then the problem* (1.1)-(1.3) *does not have nonreal eigenvalues.*

Proof. Let $\mu \in \mathbb{C}\backslash\mathbb{R}$ be an eigenvalue of problem (1.1)-(1.3). Then $\bar{\mu}$ is also an eigenvalue of this problem, since the coefficients a_0, a_1, b_0 and b_1 are real; moreover $y(x, \bar{\mu}) = \overline{y(x, \mu)}$. Multiplying the both parts of equation (1.1) by the function $\overline{y(x,\mu)}$ and integrating the obtained equality by parts in the range from 0 to 1, and also taking into account (1.2)-(1.3) we get

$$\begin{gathered}\int_0^1 |y'(x,\mu)|^2\,dx - b_0|y(0,\mu)|^2 - b_1|y(1,\mu)|^2 = \\ \mu\left\{\int_0^1 |y(x,\mu|^2 + a_0|y(0,\mu)|^2 + a_1|y(1,\mu)|^2\right\}.\end{gathered} \tag{4.10}$$

On the other hand by virtue of (1.1), we have

$$-y''(x,\mu)\overline{y(x,\mu)} + \overline{y''(x,\mu)}\,y(x,\mu) = (\mu-\bar{\mu})|y(x,\mu)|^2.$$

Integrating this relation from 0 to 1, using the formula for the integration by parts, and taking into account conditions (1.2)-(1.3), we obtain

$$-(\mu-\bar{\mu})\{a_1|y(1,\mu)|^2 + a_0|y(0,\mu)|^2 = (\mu-\bar{\mu})\int_0^1 |y(x,\mu)|^2\,dx,$$

from where implies that

$$\int_0^1 |y(x,\mu)|^2\,dx + a_0|y(0,\mu)|^2 + a_1|y(1,\mu)|^2 = 0. \tag{4.11}$$

By (4.10) and (4.11) we have

$$\int_0^1 |y'(x,\mu)|^2\,dx - b_0|y(0,\mu)|^2 - b_1|y(1,\mu)|^2 = 0,$$

which contradicts conditions $b_0 < 0$ and $b_1 < 0$. The proof of Lemma 4.3 is complete.

Lemma 4.4. *If $b_0 < 0$ and $b_1 < 0$, then the eigenvalues of the boundary value problem* (1.1)-(1.3) *are simple.*

Proof. If λ^* is multiple root of the equation (4.3), then by (4.4) and (4.1) we obtain

$$\int_0^1 y^2(x,\lambda^*)\,dx + a_0y^2(0,\lambda^*) + a_1y^2(1,\lambda^*) = 0. \tag{4.12}$$

Multiplying the both parts of equation (1.1) by the function $y(x,\lambda^*)$ and integrating the obtained equality by parts in the range from 0 to 1, and also taking into account the boundary conditions (1.2)-(1.3) we have

$$\begin{gathered}\int_0^1 y'^2(x,\lambda^*)\,dx - b_0y^2(0,\lambda^*) - b_1y^2(1,\lambda^*) = \\ \lambda^*\left\{\int_0^1 y^2(x,\lambda^*)\,dx + a_0y^2(0,\lambda^*) + a_1y^2(1,\lambda^*)\right\}.\end{gathered} \tag{4.13}$$

By virtue (4.12) from (4.13) we obtain

$$\int_0^1 y'^2(x,\lambda^*)\,dx - b_0y^2(0,\lambda^*) - b_1y^2(1,\lambda^*) = 0,$$

which contradicts conditions $b_0 < 0$ and $b_1 < 0$. The proof of Lemma 4.5 is complete.

Let $D_k = (\mu_{k-1}, \mu_k)$, $k = 1, 2, \dots,$ where $\mu_0 = -\infty$.

Lemma 4.5. *If* $b_0 < 0$ *and* $b_1 < 0$*, then the problem* (1.1)-(1.3) *can has a unique eigenvalue in each interval* D_k*,* $k = 1, 3, 4, \dots$ *.*

Proof. If $\lambda^* \in D_k,\ k \in \mathbb{N}\backslash\{2\}$ is an eigenvalue of the problem (1.1)- (1.3), then by virtue of (4.13) we have

$$\int_0^1 y^2(x,\lambda^*)\,dx\, + a_0 y^2(0,\lambda^*) + a_1 y^2(1,\lambda^*) < 0, \text{ if } \tilde{\lambda} \in B_1,$$

and

$$\int_0^1 y^2(x,\lambda^*)\,dx\, + a_0 y^2(0,\lambda^*) + a_1 y^2(1,\lambda^*) > 0, \text{ if } \lambda^* \in B_k,\ k \in \mathbb{N}\backslash\{1,2\}.$$

By (4.4) it follows from these relations that $\frac{d}{d\lambda}\left(G(\lambda) - (a_1\lambda + b_1)\right)|_{\lambda=\lambda^*}$ is positive, if $\lambda^* \in D_1$ and is negative, if $\lambda^* \in D_k,\ k \in \mathbb{N}\backslash\{1,2\}$. Thus, the function $G(\lambda) - (a_1\lambda + b_1)$ is assumes zero values only strictly increasing (decreasing) in the interval D_1 ($D_k,\ k \in \mathbb{N}\backslash\{1,2\}$). Consequently, equation (4.3) can has a unique solution in each interval D_k, $k = 1, 3, 4, \dots$. The proof of Lemma 4.5 is complete.

Let λ and μ, $\lambda \neq \mu$, be eigenvalues of the operator L. By Theorem 2.1 and Corollary 2.1 operator L is J-self-adjoint in Π_2. Then the eigenvectors

$\hat{y}(\lambda) = \{y(x,\lambda), -a_0 y(0,\lambda),\, a_1 y(1,\lambda)\}$ and $\hat{y}(\mu) = \{y(x,\mu), -a_0 y(0,\mu),\, a_1 y(1,\mu)\}$

corresponding to eigenvalues λ and μ are J-orthogonal in Π_2; consequently, by (2.2) and (4.1), we have

$$\int_0^1 y(x,\lambda)\overline{y(x,\mu)}\,dx + a_0 y(0,\lambda)\overline{y(0,\mu)} + a_1 y(1,\lambda)\overline{y(1,\mu)} = 0\,. \tag{4.14}$$

Lemma 4.6. *Let* $b_0 < 0$ *or* $b_1 < 0$ *and let* λ *be an eigenvalue of problem* (1.1)-(1.3) *such that* $(\operatorname{sgn}\lambda)(G'(\lambda) - a_1) \geq 0$*. Then this problem does not have nonreal eigenvalues.*

Proof. Let $\mu \in \mathbb{C}\backslash\mathbb{R}$ be an eigenvalue of problem $\overline{(1.1)\text{- }(1.3)}$. Then multiplying the both parts of equation (1.1) by the function $\overline{y(x,\mu)}$ and integrating the obtained equality by parts in the range from 0 to 1 and taking (1.2)-(1.3) and (4.14) into account, we obtain

$$\left.\begin{array}{l} \int_0^1 y'(x,\lambda)\overline{y'(x,\mu)}\,dx - b_0 y(0,\lambda)\overline{y(0,\mu)} - b_1 y(1,\lambda)\overline{y(1,\mu)} = \\ \lambda\left\{\int_0^1 y(x,\lambda)\overline{y(x,\mu)}\,dx + a_0 y(0,\lambda)\overline{y(0,\mu)} + a_1 y(1,\lambda)\overline{y(1,\mu)}\right\} \end{array}\right. \tag{4.15}$$

Let $b_0 < 0$. By (4.14) and (4.15), we have

$$\int_0^1 y'(x,\lambda)\overline{y'(x,\mu)}\,dx - b_0 y(0,\lambda)\overline{y(0,\mu)} = b_1 y(1,\lambda)\overline{y(1,\mu)}. \tag{4.16}$$

From (4.16) it follows that

$$\begin{aligned}&\int\limits_0^1 \tfrac{y'(x,\lambda)}{y(1,\lambda)}\overline{\left(\tfrac{y'(x,\mu)}{y(1,\mu)}\right)}\,dx - b_0\tfrac{y(0,\lambda)}{y(1,\lambda)}\overline{\left(\tfrac{y(0,\mu)}{y(1,\mu)}\right)} = b_1,\\ &\int\limits_0^1 \tfrac{y'(x,\lambda)}{y(1,\lambda)}\,\tfrac{y'(x,\mu)}{y(1,\mu)}dx - b_0\tfrac{y(0,\lambda)}{y(1,\lambda)}\tfrac{y(0,\mu)}{y(1,\mu)} = b_1\end{aligned} \tag{4.17}$$

By adding the first relation in (4.17) to the second one, we obtain

$$2\int\limits_0^1 \frac{y'(x,\lambda)}{y(1,\lambda)}\,\mathrm{Re}\frac{y'(x,\mu)}{y(1,\mu)}\,dx - 2b_0\frac{y(0,\lambda)}{y(1,\lambda)}\mathrm{Re}\frac{y(0,\mu)}{y(1,\mu)} = 2b_1. \tag{4.18}$$

By (4.10), (4.11) and (4.13), we have

$$\int\limits_0^1 \left|\frac{y'(x,\mu)}{y(1,\mu)}\right|^2 dx - b_0\left|\frac{y(0,\mu)}{y(1,\mu)}\right|^2 = b_1 \tag{4.19}$$

and

$$\begin{aligned}&\int\limits_0^1 y'^2(x,\lambda)\,dx\ - b_0y^2(0,\lambda) - b_1y^2(1,\lambda) =\\ &\lambda\left\{\int\limits_0^1 y^2(x,\lambda)\,dx\ + a_0y^2(0,\lambda) + a_1y^2(1,\lambda)\right\},\end{aligned} \tag{4.20}$$

respectively.

If $G'(\lambda) \leq a_1$ at $\lambda < 0$ or $G'(\lambda) \geq a_1$ at $\lambda > 0$, then by (4.4) and (4.20) we get

$$\int\limits_0^1 \left(\frac{y'(x,\lambda)}{y(1,\lambda)}\right)^2 dx - b_0\left(\frac{y(0,\lambda)}{y(1,\lambda)}\right)^2 \leq b_1 \tag{4.21}$$

By virtue of relations (4.18), (4.19) and (4.21), we have

$$\begin{aligned}&\int\limits_0^1 \left\{\left(\tfrac{y'(x,\lambda)}{y(1,\lambda)} - \mathrm{Re}\tfrac{y'(x,\mu)}{y(1,\mu)}\right)^2 + \mathrm{Im}^2\tfrac{y'(x,\mu)}{y(1,\mu)}\right\} -\\ &b_0\left\{\left(\tfrac{y(0,\lambda)}{y(1,\lambda)} - \mathrm{Re}\tfrac{y(0,\mu)}{y(1,\mu)}\right)^2 + \mathrm{Im}^2\tfrac{y(0,\mu)}{y(1,\mu)}\right\} < 0,\ \text{ if } (\mathrm{sgn}\,\lambda)(G'(\lambda) - a_1) < 0,\end{aligned}$$

$$\begin{aligned}&\int\limits_0^1 \left\{\left(\tfrac{y'(x,\lambda)}{y(1,\lambda)} - \mathrm{Re}\tfrac{y'(x,\mu)}{y(1,\mu)}\right)^2 + \mathrm{Im}^2\tfrac{y'(x,\mu)}{y(1,\mu)}\right\} -\\ &b_0\left\{\left(\tfrac{y(0,\lambda)}{y(1,\lambda)} - \mathrm{Re}\tfrac{y(0,\mu)}{y(1,\mu)}\right)^2 + \mathrm{Im}^2\tfrac{y(0,\mu)}{y(1,\mu)}\right\} = 0,\ \text{ if } (\mathrm{sgn}\,\lambda)(G'(\lambda) - a_1) = 0.\end{aligned}$$

Since $b_0 < 0$, then it follows from the second relation that $\mathrm{Im}\frac{y'(x,\mu)}{y(1,\mu)} = 0$, together with (1.1), contradicts the condition $\mu \in \mathbb{C}\backslash\mathbb{R}$.

The case $b_1 < 0$ an be considered in a similar way. The proof of Lemma 4.6 is completed.

Lemma 4.7. *Let $b_0 < 0$ or $b_1 < 0$ and let $\lambda, \mu \in \mathbb{R}$, $\lambda \neq \mu$, be eigenvalues of problem* (1.1)- (1.3) *and* $(\mathrm{sgn}\,\lambda)(G'(\lambda) - a_1) \geq 0$. *Then* $(\mathrm{sgn}\,\mu)(G'(\mu) - a_1) < 0$.

Proof. Let $b_0 < 0$ and $(\operatorname{sgn}\mu)(G'(\mu) - a_1) \geq 0$. Then, by following the corresponding argument in the proof of Lemma 4.6, we obtain

$$\int_0^1 \left(\frac{y'(x,\lambda)}{y(1,\lambda)} - \frac{y'(x,\mu)}{y(1,\mu)}\right)^2 - b_0 \left(\frac{y(0,\lambda)}{y(1,\lambda)} - \frac{y(0,\mu)}{y(1,\mu)}\right)^2 < 0,$$
$$\text{if either } \operatorname{sgn}(\lambda)\,(G'(\lambda) - a_1) > 0 \text{ or } \operatorname{sgn}(\mu)\,(G'(\mu) - a_1) > 0,$$

$$\int_0^1 \left(\frac{y'(x,\lambda)}{y(1,\lambda)} - \frac{y'(x,\mu)}{y(1,\mu)}\right)^2 - b_0 \left(\frac{y(0,\lambda)}{y(1,\lambda)} - \frac{y(0,\mu)}{y(1,\mu)}\right)^2 = 0,$$
$$\text{if } \ (G'(\lambda) - a_1) = \ (G'(\mu) - a_1) = 0.$$

It follows from the second relation that $\frac{y'(x,\lambda)}{y(1,\lambda)} = \frac{y'(x,\mu)}{y(1,\mu)}$, $x \in (0,1)$. Consequently, $y(1,\mu)y'(x,\lambda) \equiv y(1,\lambda)y'(x,\mu)$. Since $\lambda \neq \mu$, it follows from (1.1) that $y(x,\lambda) \equiv 0$.

The case $b_1 < 0$ an be considered in a similar way. The resulting contradictions completes the proof of Lemma 4.7.

Theorem 4.2. *Let $b_0 < 1$ and $b_1 \in \mathbb{R}$. Then one of the following assertions holds: (i) all eigenvalues of problem* (1.1)-(1.3) *are real; in this case, D_2 contains algebraically two eigenvalues (either two simple eigenvalues or one double eigenvalue), and D_k, $k = 1, 3, 4, \ldots$, contains one simple eigenvalue; (ii) all eigenvalues of problem* (1.1)-(1.3) *are real; in this case, D_2 contains no eigenvalues, while there exists a positive integer M ($M > 2$) such that D_M contains algebraically three eigenvalues (either three simple eigenvalues, or one double eigenvalue and one simple eigenvalue, or one triple eigenvalue), and D_k, $k = 1, 3, 4, \ldots, k \neq m$, contains one simple eigenvalue; (iii) problem* (1.1)- (1.3) *has one pair of non-real complex conjugate eigenvalues; in this case, D_2 contains no eigenvalues, and D_k, $k = 1, 3, 4, \ldots$, contains one simple eigenvalue.*

Proof. Let $b_0 < 0$ and $b_1 < 0$. Then, by Lemmas 4.2-4.5 and Theorem 4.1, the equation (4.3) has unique root in the interval D_k, $k = 1, 3, 4, \ldots$, and two simple roots in the interval D_2.

Let $b_0 \in (0,1)$. If either $0 < \nu_1 \leq \nu_2$ and $b_1 < 0$ or $\nu_1 \leq \nu_2 < 0$ and $b_1 < \frac{b_0}{b_0-1}$ or $\nu_1, \nu_2 \in \mathbb{C}\backslash\mathbb{R}$ and $b_1 < \frac{b_0}{b_0-1}$, then by Lemmas 4.2, 4.6, 4.7 and Theorem 4.1, the equation (4.3) has unique root in the interval D_k, $k = 1, 3, 4, \ldots$, and two simple roots in the interval D_2.

Now let $b_0 < 0$ and $b_1 > 0$. In this case it follows from (3.16) that $\mu_1 < \nu_1 < 0 < \nu_2 < \mu_2$. Moreover, by Lemma 4.2 we have $G(\lambda) > 0$ for $\lambda \in (\nu_1, \nu_2)$. Let

$$b_1^* = \begin{cases} \max\limits_{\lambda \in (\nu_1,\nu_2)} G(\lambda), & if \ \max\limits_{\lambda \in (\nu_1,\nu_2)} G(\lambda) = G(\lambda_0), \ \ \lambda_0 > 0, \\ \frac{b_0}{b_0-1}, & if \ \max\limits_{\lambda \in (\nu_1,\nu_2)} G(\lambda) = G(\lambda_0), \ \ \lambda_0 < 0. \end{cases}$$

By (4.9) for each given number $b > b_1^*$, there exists a number a_1^* such that the line $a_1^*\lambda + b_1$, $\lambda \in \mathbb{R}$, is tangent to the graph of the function $G(\lambda)$ at some point λ^* on the interval (ν_1, ν_2). Consequently, in the interval D_2 equation (4.3) has two simple roots if $a_1 < a_1^*$, one double root if $a_1 = a_1^*$ and no root if $a_1 > a_1^*$.

By virtue of Lemma 4.2, equation (4.3) has at least one solution in each interval D_k, $k = 1, 3, \ldots$.

There exists sufficiently large number $k_0 \in \mathbb{N}$ such that for $k \geq k_0$ the relation

$$\sum_{\lambda_k \in \mathrm{B}_{R_k}} \varkappa(\lambda_k) = k + 1. \qquad (4.22)$$

is true, which can be proved similar to that of [9, formula (3.10)], where $R_k = \nu_k + \delta$, δ is a small number, $B_{R_k} = \{ z \in \mathbb{C} : |z| < R_k \}$ and $\varkappa(\lambda_k)$ is the multiplicity of the zero λ_k of the function $G(\lambda) - (a_1\lambda + b_1)$.

Let $0 < b_1 \leq b_1^*$. If $0 < b_1 < b_1^*$, then equation (4.3) has two simple roots in the interval D_2, and if $b_1 = b_1^*$, then this equation has either one double root or two simple roots in the interval D_2. Furthermore, the equation (4.3) has at least one root in each interval D_k, $k = 1, 3, 4, \ldots$. Then, by formula (3.10) this equation has exactly one simple root in each interval D_k, $k = 1, 3, 4, \ldots$. Note that, this facts is true in the case $b_1 > b_1^*$ and $a_1 \leq a_1^*$.

Now let $b_1 > b_1^*$ and $a_1 > a_1^*$. In this case the equation (4.3) has no root in the interval D_2, while has at least one root in each interval D_k, $k = 1, 3, 4, \ldots$. We show that in the interval D_1 equation (4.3) has a unique solution. Indeed, if this equation has two solutions in the interval D_1, then there exists $\tilde{a}_1 < 0$ and $\tilde{b}_1 < 0$ such that equation $G(\lambda) = \tilde{a}_1\lambda + \tilde{b}_1$ has at least three solutions in the interval D_1, which contradicts lemma 4.5. Then by formula (3.10) this equation has either one pair of nonreal complex conjugate eigenvalues, or there exists a positive integer M ($M > 2$) such that D_M contains algebraically three eigenvalues (either three simple eigenvalues, or one double eigenvalue and one simple eigenvalue, or one triple eigenvalue).

The case $b_0 \in (0, 1)$ and $b_1 > 0$ is consider similarly. The prof of Theorem 4.2 is complete.

References

[1] Z.S.Aliev, Basis properties of the root functions of an eigenvalue problem with a spectral parameter in the boundary conditions, *Doklady Mathematics,* **82** (1) (2010), 583-586.

[2] Z.S.Aliev, Basis properties in L_p of systems of root functions of a spectral problem with spectral parameter in a boundary condition *Differential Equations,* **47** (6) (2011), 766-777.

[3] Z.S.Aliev, On basis properties of root functions of a boundary value problem containing a spectral parameter in the boundary conditions, *Doklady Mathematics,* **87** (2) (2013), 137-139.

[4] Z.S.Aliyev, E.A.Aghayev, The basis properties of the system of root functions of Sturm-Liouville problem with spectral parameter in the boundary condition, *Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan,* **7** (4) (2009), 63-72.

[5] Z.S. Aliyev, A.A. Dunyamaliyeva, Basis properties of root functions of the Sturm-Liouville problem with a spectral parameter in the boundary conditions, *Doklady Mathematics,* **88** (1) (2013), 441-445.

[6] Z.S. Aliyev, A.A. Dunyamaliyeva, Defect basicity of the root functions of the Sturm - Liouville problem with spectral parameter in the boundary conditions, *Differential Equations,* **51** (10) (2015), 1263-1280.

[7] J. Ben Amara, A. A. Shkalikov, A Sturm-liouville problem with physical and spectral parameters in boundary conditions, *Mathematical Notes,* **66** (2) (1999), 127-134.

[8] T.Ya. Azizov, I.S. Iokhvidov, *Foundations of the theory of linear operators in spaces with indefinite metric,* Nauka, Moscow, 1986.

[9] A.A. Dunyamaliyeva, Some spectral properties of the boundary value problem with spectral parameter in the boundary conditions, *Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan,* **40** (2) (2014), 52-64.

[10] N. Yu. Kapustin, On a spectral problem arising in a mathematical model of torsional vibrations of a rod with pulleys at the ends, *Differential Equations,* **41** (10) (2005), 1490-1492.

[11] N.B. Kerimov, R.G. Poladov, Basis properties of the system of eigenfunctions in the Sturm-Liouville problem with a spectral parameter in the boundary conditions, *Doklady Mathematics,* **85** (1) (2012), 8-13.

[12] L.S. Pontryagin, Hermitian operators in a space with indefinite metric, *Izv. Akad. Nauk SSSR Ser. Mat.,* **8** (1944), 243-280.

[13] E.M. Russakovskii, Operator treatment of boundary problems with spectral parameters entering via polynomials in the boundary conditions, *Functional Analysis and Its Applications,* **9** (4) (1975), 358-359.

[14] B.V. Shabat, *Introduction to complex analysis,* Nauka, Moscow, 1969.

[15] A.A. Shkalikov, Boundary problems for ordinary differential equations with parameter in the boundary conditions, *Journal of Soviet Mathematics* **33** (6) (1986), 1311-1342.

[16] A.N. Tikhonov and A.A. Samarskii, *Equations of mathematical physics,* Nauka, Moscow, 1972 (in Russian).

Ziyatkhan S. Aliyev
Baku State University, Baku AZ 1148, Azerbaijan
Institute of Mathematics and Mechanics, NAS of Azerbaijan, Baku AZ 1141, Azerbaijan
E-mail address: `z_aliyev@mail.ru`

Fatima-khanim I. Allahverdi-zada
Baku State University, Baku AZ 1148, Azerbaijan
E-mail address: `f.allahverdizade@mail.ru`

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