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GLOBAL BIFURCATION OF SOLUTIONS OF NONLINEAR ONE-DIMENSIONAL DIRAC SYSTEM

HUMAY SH. RZAYEVA

Abstract. In this paper we investigate the structure of the solution set for a linearizable and nonlinearizable boundary value problem for one-dimensional Dirac system. We show the existence of two families of continua of solutions, corresponding to the usual nodal properties and bifurcating from the line of trivial solutions.

1. Introduction

We consider the following nonlinear Dirac equation

$$Bw'(x) = \lambda w(x) + h(x, w(x), \lambda), \ 0 < x < \pi, \tag{1.1}$$

with the boundary conditions $U(w) = \begin{pmatrix} U_1(w) \\ U_2(w) \end{pmatrix} = 0$ given by

$$U_1(w) := (\sin\alpha, \, \cos\alpha)\, w(0) = v(0)\cos\alpha + u(0)\sin\alpha = 0, \tag{1.2}$$

$$U_2(w) := (\sin\beta, \, \cos\beta)\, w(\pi) = v(\pi)\cos\beta + u(\pi)\sin\beta = 0, \tag{1.3}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ w(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix},$$

$\lambda \in \mathbb{R}$ is a spectral parameter, α and β are real constants: moreover $0 \leq \alpha, \beta < \pi$. We assume that the nonlinear term h has the form $h = f + g$, where $f = \begin{pmatrix} \bar{f} \\ \bar{\bar{f}} \end{pmatrix}$ and $g = \begin{pmatrix} \bar{g} \\ \bar{\bar{g}} \end{pmatrix}$ are continuous functions on $C\left([0,\pi] \times \mathbb{R}^2 \times \mathbb{R}\,;\, \mathbb{R}^2\right)$ and satisfies the conditions:

$$|\bar{f}(x, w, \lambda)| \leq K|u|, \ |\bar{\bar{f}}(x, w, \lambda)| \leq M|v|\,, \, x \in [0,\pi], \, 0 < |w| \leq 1, \, \lambda \in \mathbb{R}, \tag{1.4}$$

where K and M are the positive constants;

$$g(x, w, \lambda) = o(|w|) \ \text{ as } \ |w| \to 0, \tag{1.5}$$

uniformly with respect to $x \in [0, \pi]$ and $\lambda \in \Lambda$, for every compact interval $\Lambda \subset \mathbb{R}$ (here $|\cdot|$ denotes a norm in $\mathbb{R}^2$).

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The equation (1.1) is equivalent to the system of two consistent first-order ordinary differential equations

$$\begin{aligned} v' &= \lambda u + \bar{f}(x,u,v,\lambda) + \bar{g}(x,u,v,\lambda), \\ u' &= -\lambda v + \bar{\bar{f}}(x,u,v,\lambda) + \bar{\bar{g}}(x,u,v,\lambda). \end{aligned} \tag{1.6}$$

The global results for nonlinear Sturm-Liouville problems were obtained by Rabinowitz [18], Berestycki [5], Schmitt and Smith [20], Chiapinelli [7], Przybycin [16, 17], Aiyev [1], Rynne [19], Binding, Browne, Watson [6], G. Dai [8], Aliyev and Mamedova [3], Mamedova [15]. These papers prove the existence of global continua of nontrivial solutions in $\mathbb{R} \times C^1$ corresponding to the usual nodal properties and emanating from bifurcation intervals (in $\mathbb{R} \times 0$, which we identify with $\mathbb{R}$) surrounding the eigenvalues of the linear problem. Similar results for nonlinearizable Sturm-Liouville problems of fourth order were obtained Makhmudov and Aliev [13, 14], Aiev [2].

Only Schmitt and Smith [18] are studied the nonlinear problem (1.1)-(1.3) under certain restrictions on constants K and M. They show that, in certain cases, for every large $|k|$ there exists a family of unbounded subcontinua of solutions bifurcating from intervals of the line of trivial solutions corresponding to the k-th eigenvalue of linear problem.

In this paper we study the behavior of continua of solutions of problem (1.1)-(1.3) bifurcating from the points and intervals of the line of trivial solutions.

2. Preliminary

If $F \equiv 0$, then (1.1)-(1.3) is a linear canonical one-dimensional Dirac system [11, Ch. 1, § 10]

$$\begin{aligned} &Bw(x) = \lambda w(x),\ 0 < x < \pi, \\ &w \in \text{B.C.}, \end{aligned} \tag{2.1}$$

where by B.C. denoted the set of boundary conditions (1.2)-(1.3).

It is known (see [11, Ch. 1, § 11]; [4, Ch. 8]) that eigenvalues of the boundary value problem (2.1) are real, algebraically simple and the values range from $-\infty$ to $+\infty$ and can be numerated in increasing order.

The oscillation properties of eigenvector-functions of problem (2.1) is investigated in [3], where, in particular, is proved the following result.

Theorem 2.1. *The eigenvalues $\lambda_k, k \in \mathbb{Z}$, of the problem* (2.1) *can be numbered in ascending order on the real axis*

$$\ldots < \lambda_{-k} < \ldots < \lambda_{-1} < \lambda_0 < \lambda_1 < \ldots < \lambda_k < \ldots .$$

The eigenvector-functions $w_k(x) = w(x, \lambda_k) = \begin{pmatrix} u(x,\lambda_k) \\ v(x,\lambda_k) \end{pmatrix} = \begin{pmatrix} u_k(x) \\ v_k(x) \end{pmatrix}$ *have, with a suitable interpretation, the following oscillation properties: if $k > 0$, and $k = 0$, $\alpha \geq \beta$ (except the cases $\alpha = \beta = 0$ and $\alpha = \beta = \pi/2$), then*

$$\begin{pmatrix} s(u_k) \\ s(v_k) \end{pmatrix} = \begin{pmatrix} k - 1 + \chi(\alpha - \pi/2) + \chi(\pi/2 - \beta) \\ k - 1 + \text{sgn}\alpha \end{pmatrix}; \tag{2.2}$$

if $k<0$, and $k=0$, $\alpha<\beta$, then

$$\begin{pmatrix} s(u_k) \\ s(v_k) \end{pmatrix} = \begin{pmatrix} |k|-1+\chi(\pi/2-\alpha)+\chi(\beta-\pi/2) \\ |k|-1+\mathrm{sgn}\beta \end{pmatrix}, \tag{2.3}$$

where $s(g)$ the number of zeros of the function $g\in C([0,\pi];\mathbb{R})$ in the interval $(0,\pi)$ and

$$\chi(x)=\begin{cases} 0, & if\ x\le 0, \\ 1, & if\ x>0. \end{cases}$$

We define E to be the Banach space $C\left([0,\pi];\mathbb{R}^2\right)\cap$B.C. with the usual norm $||w||=\max\limits_{x\in[0,\pi]}|u(x)|+\max\limits_{x\in[0,\pi]}|v(x)|$.

Let S_k^+ be set of $w=\begin{pmatrix} u \\ v \end{pmatrix}\in E$ which satisfy the conditions:

(i) $|u(x)|+|v(x)|>0,$ for all $x\in[0,\pi]$;

(ii) the zeros of functions $u(x)$ and $v(x)$ are nodal and interspersed, and if $k>0$, and $k=0$, $\alpha\ge\beta$ (except the cases $\alpha=\beta=0$ and $\alpha=\beta=\pi/2$), then

$$\begin{pmatrix} s(u) \\ s(v) \end{pmatrix} = \begin{pmatrix} k-1+\chi(\alpha-\pi/2)+\chi(\pi/2-\beta) \\ k-1+\mathrm{sgn}\alpha \end{pmatrix},$$

if $k<0$, and $k=0$, $\alpha<\beta$, then

$$\begin{pmatrix} s(u) \\ s(v) \end{pmatrix} = \begin{pmatrix} |k|-1+\chi(\pi/2-\alpha)+\chi(\beta-\pi/2) \\ |k|-1+\mathrm{sgn}\beta \end{pmatrix};$$

(iii) the function $u(x)$ is positive in a deleted neighborhood of $x=0$.

Let $S_k^-=-S_k^+$ and $S_k=S_k^-\cup S_k^+$. It follows by Theorem 2.1] that $w_k\in S_k,\ k\in\mathbb{Z}$, i.e. the sets $S_k^-,\ S_k^+$ and S_k are nonempty.

Remark 2.1. From the definition of the sets S_k^-, S_k^+ and S_k, it follows directly that, they are disjoint and open in E. Furthermore, if $w\in\partial S_k\,(\partial S_k^\nu,\ \nu=+$ or $-)$, then there exists a point $\tau\in[0,\pi]$ such that $|w(\tau)|=0$, i.e. $u(\tau)=v(\tau)=0$.

Lemma 2.1. *If $(\lambda,w)\in\mathbb{R}\times E$ is a solution of problem (1.1)-(1.3) and $w\in\partial S_k^\nu,\ \nu=+$ or $-$, then $w\equiv 0$.*

Proof. Let (λ,w) is a solution of problem (1.1)-(1.3) and $w\in\partial S_k^\nu$. Then, by Remark 2.1, there exists $\zeta\in(0,\pi)$ such that $u(\zeta)=v(\zeta)=0$. Taking into account conditions (1.4) and (1.5) from (1.1) we obtain that in some neighborhood of ζ the following inequality holds:

$$|w'(x)|\ \le\ c_0\,|w(x)|, \tag{2.4}$$

where c_0 is a positive constant and $|\cdot|$ denotes a norm in $\mathbb{R}^2$.

Integrating both sides of the inequality (2.4) from ζ to x, we obtain

$$\left|\int\limits_\zeta^x|w'(t)|dt\right|\ \le\ c_0\left|\int\limits_\zeta^x|w\,(t)|dt\right|.$$

Consequently, by virtue of this inequality and equality $|w(\zeta)| = 0$, we have

$$|w(x)| \; = \; \left| \int\limits_{\zeta}^{x} w'(t)\, dt \right| \; \leq \; c_0 \left| \int\limits_{\zeta}^{x} |w\,(t)| dt \right|. \tag{2.5}$$

Using Gronwall's inequality, we conclude from (2.5) that $|w(x)| = 0$ in a neighborhood of ζ. This shows that the functions $u(x)$ and $v(x)$ is equal to zero in a neighborhood of ζ. Continuing the specified process, we obtain $w(x) \equiv 0$ on $[0, \pi]$. The proof of Lemma 2.1 is complete.

Assume that $\lambda = 0$ is not an eigenvalue of (1.1)-(1.3). Then the problem (1.1)-(1.3) can be converted to the equivalent integral equation

$$w(x) = \lambda \int\limits_0^{\pi} K(x,t) w(t) dt + \int\limits_0^{\pi} K(x,t) h(t, w(t), \lambda) dt, \tag{2.6}$$

where $K(x,t) = K(x,t,0)$ is the appropriate Green's matrix (see [19, Chapter 1, formula (13.8)]).

Define $L : E \to E$ by

$$Lw(x) = \int\limits_0^{\pi} K(x,t) w(t) dt, \tag{2.7}$$

$F : \mathbb{R} \times E \to E$ by

$$F(\lambda, w(x)) = \int\limits_0^{\pi} K(x,t) f(t, w(t), \lambda) dt, \tag{2.8}$$

$F : \mathbb{R} \times E \to E$ by

$$G(\lambda, w(x)) = \int\limits_0^{\pi} K(x,t) g(t, w(t), \lambda) dt. \tag{2.9}$$

The operators F and G can be represented as a compositions of a Fredholm operator L and the superposition operators $\mathbf{f}(\lambda, w(x)) = f(x, w(x), \lambda)$ and $\mathbf{g}(\lambda, w(x)) = g(x, w(x), \lambda)$ respectively. By [11, Ch.1, formula (13.8)] L can be regarded as a compact operator in E. Since $f(x, w, \lambda) \in C\left([0,\pi] \times \mathbb{R}^2 \times \mathbb{R}\,;\, \mathbb{R}^2\right)$ and $g(x, w, \lambda) \in C\left([0,\pi] \times \mathbb{R}^2 \times \mathbb{R}\,;\, \mathbb{R}^2\right)$, then the operators $\mathbf{f}$ and $\mathbf{g}$ maps $\mathbb{R} \times E$ to $C\left([0,\pi];\, \mathbb{R}^2\right)$. Hence the operators F and G are completely continuous. Furthermore, by virtue of (1.5) we have

$$G(\lambda, w) = o(||w||) \quad \text{as} \quad ||w|| \to 0, \tag{2.10}$$

uniformly with respect to $\lambda \in \Lambda$.

On the base (2.6)-(2.9) problem (1.1)-(1.3) can be written in the following equivalent form

$$w = \lambda L w + F(\lambda, w) + G(\lambda, w), \tag{2.11}$$

and therefore, it is enough to investigate the structure of the set of solutions of (1.1)-(1.3) in $\mathbb{R} \times E$.

We suppose that

$$f \equiv 0 \tag{2.12}$$

(in effect, we suppose that the nonlinearity h itself satisfies (1.4)). Then, by (2.11), problem (1.1)-(1.3) is equivalent to the following problem

$$w = \lambda L w + G(\lambda, w). \tag{2.13}$$

Note that problem (2.13) is of the form (0.1) of [18]. The linearization of this problem at $w = 0$ is the spectral problem

$$w = \lambda L w. \tag{2.14}$$

Obviously, the problem (2.14) is equivalent to the spectral problem (2.1).

We denote by Y the closure in $\mathbb{R} \times E$ of the set of nontrivial solutions of (2.11) (i.e. of (1.1)-(1.3)).

In the following, we will denote by $w_k^(x)+ = \left(u_k^+(x), v_k^+(x)\right)^t$, $k \in \mathbb{Z}$, the unique eigenvector-function of linear problem (2.1) associated to eigenvalue λ_k such that $\lim\limits_{x \to 0+} \operatorname{sgn} u_k^+(x) = 1$ and $||\tilde{w}_k^+|| = 1$.

The linear existence theory for the problem (2.1) (or problem (2.14)) can be stated as: for each integer k and each $\nu = +$ or $-$, there exists a half line of solutions of problem (2.1) in $\mathbb{R} \times S_k^\nu$ of the form $(\lambda_k, \gamma\, w_k^+)$, $\gamma \in \mathbb{R}^\nu$. This half line joins $(\lambda_k, 0)$ to infinity in E. (Here $\mathbb{R}^\nu = \{\varsigma \in \mathbb{R} : 0 \le \varsigma\nu \le +\infty\}$, $\nu = +$ or $-$).

An analogous result holds for problem (2.13).

Theorem 2.2. *Suppose that* (2.12) *holds. Then for every integer k and each $\nu = +$ or $-$, there exists a continuum of solutions C_k^ν of problem* (1.1)-(1.3) (*or problem* (2.13)) *in* $(\mathbb{R} \times S_k^\nu) \cup \{(\lambda_k, 0)\}$ *which meets* $(\lambda_k, 0)$ *and* ∞ *in* $\mathbb{R} \times E$.

The proof of this theorem is similar to that of Theorem 2.3 of [18] (see also [9]), using the above arguments, relation (2.10) and Lemma 2.1.

3. Global bifurcation of solutions of problem (1.1)-(1.3) in the case $g \equiv 0$

We suppose that

$$g \equiv 0 \tag{3.1}$$

(in effect, we suppose that the nonlinearity h itself satisfies (1.4)). Then the problem (1.1) -(1.3) takes the form

$$\begin{aligned} &Bw(x) = \lambda w(x) + f(x, w, \lambda),\ 0 < x < \pi, \\ &w \in \text{B.C.}. \end{aligned} \tag{3.2}$$

Together with (3.2), we consider the following approximation problem

$$\begin{aligned} &Bw(x) = \lambda w(x) + f(x, |w|^\varepsilon w, \lambda),\ 0 < x < \pi, \\ &w \in \text{B.C.}, \end{aligned} \tag{3.3}$$

where $\varepsilon \in (0, 1]$. By (1.6) the problem (3.3) is equivalent to the following system

$$\begin{aligned} v' &= \quad \lambda u + \bar{f}(x, |w|^{\varepsilon} u, |w|^{\varepsilon} v, \lambda), \\ u' &= -\lambda v + \bar{\bar{f}}(x, |w|^{\varepsilon} u, |w|^{\varepsilon} v, \lambda), \\ w &= (u,\, v)^t \in \text{B.C.}\,. \end{aligned} \tag{3.4}$$

Lemma 3.1. *For every integer k and each $\nu = +$ or $-$, and for any $0 < \varkappa < 1$ there exists solution $(\lambda_\varkappa, w_\varkappa)$ of problem (3.2) such that $\lambda_\varkappa \in J_k$, $w_\varkappa \in S_k^\nu$ and $||w_\varkappa|| = \varkappa$, where $J_k = [\lambda_k - (K+M), \lambda_k + (K+M)]$.*

Proof. By virtue of condition (1.4) we have

$$f(x, |w|^{\varepsilon} w, \lambda) = o(|w|) \ \text{ as } \ |w| \to 0, \tag{3.5}$$

uniformly with respect to $x \in [0, \pi]$ and $\lambda \in \Lambda$, for every compact interval $\Lambda \subset \mathbb{R}$. Then, by Theorem 2.2, for every integer k and each $\nu = +$ *or* $-$, there exists an unbounded continuum $C_{k,\varepsilon}^\nu$ of solutions of (3.3) (or (3.4)), such that

$$(\lambda_k, 0) \in C_{n,\varepsilon}^\nu \subset (\mathbb{R} \times S_k^\nu) \cup \{(\lambda_k, 0)\}\,.$$

Hence, for every $\varepsilon \in (0,1]$ there exists a solution $(\lambda_\varepsilon, w_\varepsilon) \in \mathbb{R} \times S_k^\nu$ of problem (3.3) such that $||w_\varepsilon|| \le 1$. Then we have $|w_\varepsilon(x)| \le 1$. We define the functions $\varphi_\varepsilon(x)$ and $\psi_\varepsilon(x)$ as follows:

$$\begin{aligned} \varphi_\varepsilon(x) &= \begin{cases} \frac{\bar{f}(x, |w_\varepsilon(x)|^{\varepsilon} u_\varepsilon(x), |w(x)|^{\varepsilon} v_\varepsilon(x), \lambda_\varepsilon)}{u_\varepsilon(x)}, & if \ \ u_\varepsilon(x) \ne 0, \\ 0, \ if \ \ u_\varepsilon(x) = 0, \end{cases} \\ \psi_\varepsilon(x) &= \begin{cases} -\frac{\bar{\bar{f}}(x, |w_\varepsilon(x)|^{\varepsilon} u_\varepsilon(x), |w(x)|^{\varepsilon} v_\varepsilon(x), \lambda_\varepsilon)}{v_\varepsilon(x)}, & if \ \ v_\varepsilon(x) \ne 0, \\ 0, \ if \ \ v_\varepsilon(x) = 0. \end{cases} \end{aligned} \tag{3.6}$$

From (3.4) and (3.6) it is seen that $(\lambda_\varepsilon, w_\varepsilon(x)) = \left(\lambda_\varepsilon, (u_\varepsilon(x),\, v_\varepsilon(x))^t\right)$ is a solution of linear eigenvalue problem

$$\begin{aligned} v' &= \quad \lambda u + \varphi_\varepsilon(x)\, u, \\ u' &= -\lambda v - \psi_\varepsilon(x)\, v, \\ w &= (u, v)^t \in \text{B.C.}\,. \end{aligned} \tag{3.7}$$

Taking into account (1.4), from (3.6) we obtain

$$\begin{aligned} |\varphi_\varepsilon(x)| \ &\le \ K|w(x)|^{\varepsilon} \le K, \ \ x \in [0, \pi], \\ |\psi_\varepsilon(x)| \ &\le \ M|w(x)|^{\varepsilon} \le M, \ \ x \in [0, \pi]. \end{aligned} \tag{3.8}$$

Remark 3.1. Following the arguments conducted in [2, Lemma 4.3] and using Theorem 2.2 we are convinced that Theorem 2.1 is true also for the linear problem (3.7).

Multiplying the first equation in (3.7) on $u_\varepsilon(x)$, the second equation on $v_\varepsilon(x)$ and subtracting from the first equation the second one, we find

$$v_\varepsilon'(x)\, u_\varepsilon(x) - \ u_\varepsilon'(x)\, v_\varepsilon(x) = \\ \lambda_\varepsilon(u_\varepsilon^2(x) + v_\varepsilon^2(x)) + \varphi_\varepsilon(x)\, u_\varepsilon^2(x) + \psi_\varepsilon(x)\, v_\varepsilon^2(x),$$

from which it follows by the uniqueness of the initial problem that

$$\frac{\left(\frac{v_\varepsilon(x)}{u_\varepsilon(x)}\right)'}{1+\left(\frac{v_\varepsilon(x)}{u_\varepsilon(x)}\right)^2} = \lambda_\varepsilon + \varphi_\varepsilon(x)\,\frac{u_\varepsilon^2(x)}{u_\varepsilon^2(x)+v_\varepsilon^2(x)} + \psi_\varepsilon(x)\,\frac{v_\varepsilon^2(x)}{u_\varepsilon^2(x)+v_\varepsilon^2(x)}\ .$$

Integrating both sides of this equality from 0 to π, and taking into account conditions (1.2)-(1.3), we obtain

$$-\beta + k\pi\ + \alpha\ = \lambda_\varepsilon \pi + \int\limits_0^\pi \left\{\varphi_\varepsilon(x)\,\frac{u_\varepsilon^2(x)}{u_\varepsilon^2(x)+v_\varepsilon^2(x)} + \psi_\varepsilon(x)\,\frac{v_\varepsilon^2(x)}{u_\varepsilon^2(x)+v_\varepsilon^2(x)}\right\} dx.$$

Note that, for eigenvalues λ_k of problem (2.1) following equality holds

$$\lambda_k \pi = -\beta + k\pi\ + \alpha\,.$$

Hence, from the last two relations, we obtain

$$\lambda_k \pi = \lambda_\varepsilon \pi + \int\limits_0^\pi \left\{\varphi_\varepsilon(x)\,\frac{u_\varepsilon^2(x)}{u_\varepsilon^2(x)+v_\varepsilon^2(x)} + \psi_\varepsilon(x)\,\frac{v_\varepsilon^2(x)}{u_\varepsilon^2(x)+v_\varepsilon^2(x)}\right\} dx,$$

from where it follows by (3.8) that

$$|\lambda_\varepsilon - \lambda_k|\ <\ K + M,$$

i.e., $\lambda_\varepsilon \in J_k$.

Let $\{\varepsilon_n\}_{n=1}^\infty$, $0\,<\,\varepsilon_n\,<\,1$, be a sequence converging to 0. Since $C_{k,\varepsilon}^\nu$ is unbounded continuum of the set of solutions of (3.4) containing the point $(\lambda_k, 0)$, then for every ε_n and for any $\varkappa \in (0,1)$ there exists a solution $(\lambda_{\varepsilon_n}, w_{\varepsilon_n}(x))$ of this problem such that $\lambda_{\varepsilon_n} \in J_k$, $w_{\varepsilon_n} \in S_k^\nu$ and $||w_{\varepsilon_n}|| = \varkappa$. Since w_{ε_n} is bounded in $C\left([0,\pi];\mathbb{R}^2\right)$ and f is continuous in $C\left([0,\pi]\times\mathbb{R}\times\mathbb{R}^2;\mathbb{R}^2\right)$, then from (3.3) (or (3.4)) implies that w_{ε_n} is bounded in $C^1\left([0,\pi];\mathbb{R}^2\right)$. Therefore, by the Arzela-Ascoli theorem, we may assume that $w_{\varepsilon_n} \to w$, $n\to\infty$, in $C\left([0,\pi];\mathbb{R}^2\right)$, $||w|| = \varkappa$. For all n, $w_{\varepsilon_n} \in S_k^\nu$, hence w lies in the closure of S_k^ν. Since $||w|| = \varkappa$, then by virtue of Lemma 2.1 we have $w \in S_k^\nu$. The proof of Lemma 3.1 is complete.

We say that the point $(\lambda,\,0)$ is a bifurcation point of problem (1.1) -(1.3) by the set $\mathbb{R}\times S_k^\nu$, $k\in\mathbb{Z}$, $\nu = +\,\text{or}\,-$, if in every small neighborhood of this point there is solution to this problem which contained in $\mathbb{R}\times S_k^\nu$.

Corollary 3.1. *The set of bifurcation points of problem* (3.2) *is nonempty, and if* $(\lambda, 0)$ *is a bifurcation point of* (3.2) *by the set* $\mathbb{R}\times S_k^\nu$*, then* $\lambda \in J_k$.

Interval J_k, $k\in\mathbb{Z}$, is called the bifurcation interval of problem (3.2) by the set $\mathbb{R}\times S_k^\nu$, $\nu = +$ or $-$.

For each $k\in\mathbb{Z}$ and $\nu = +\,\text{or}\,-$, we define the set $\widetilde{D}_k^\nu \subset Y$ to be the union of all the components $D_{k,\lambda}^\nu$ of Y which bifurcating from the bifurcation points $(\lambda, 0)$ of (3.2) by the set $\mathbb{R}\times S_k^\nu$. By Lemma 3.1 and Corollary 3.1 the set $\widetilde{D}_k^\nu$ is nonempty.

Let $D_k^\nu = \widetilde{D}_k^\nu \cup (J_k\times 0)$. Note that the set D_k^ν is connected in $\mathbb{R}\times E$, but $\widetilde{D}_k^\nu$ may not be connected in $\mathbb{R}\times E$.

Theorem 3.1. *For every $k \in \mathbb{Z}$ and each $\nu = +$ or $-$, the connected component D_k^ν of Y lies in $(\mathbb{R} \times S_k^\nu) \cup (J_k \times 0)$ and is unbounded in $\mathbb{R} \times E$.*

Proof of theorem 3.1. By Lemma 3.1, Corollary 3.1 and an argument similar to that of [12, Theorem 2.1], we can obtain the desired conclusion.

Assume that the function $f(x, w, \lambda)$ satisfies the condition (1.4) for all $x \in [0, \pi]$ and $(w, \lambda) \in \mathbb{R}^2 \times \mathbb{R}$. Thus we have the following result.

Lemma 3.2. *Let $(\hat{\lambda}, \hat{w}) = \left(\lambda,\ (\hat{u}\,,\ \hat{v})^t\right) \in \mathbb{R} \times E$ be a solution of problem (3.2). Then $\hat{w} \in \bigcup\limits_{n=-\infty}^{\infty} S_n$, and if $\hat{w} \in S_k$, then $\hat{\lambda} \in J_k$.*

Proof. Suppose that $(\hat{\lambda}, \hat{w}) \in \mathbb{R} \times E$ is a solution of problem (3.2). Let

$$
\begin{array}{l}
\varphi(x) = \begin{cases} \frac{\bar{f}(x,\hat{u}(x),\hat{v}(x),\hat{\lambda})}{\hat{u}\,(x)}, & \text{if } \hat{u}(x) \neq 0, \\ 0, \ \ if \ \ \hat{u}(x) = 0. \end{cases} \\
\psi(x) = \begin{cases} \frac{\bar{\bar{f}}(x,\hat{u}(x),\hat{v}(x),\hat{\lambda})}{\hat{v}\,(x)}, & \text{if } \hat{v}(x) \neq 0, \\ 0, \ \ if \ \ \hat{v}(x) = 0. \end{cases}
\end{array}
\tag{3.9}
$$

Then $(\hat{\lambda}, \hat{w})$ is a solution of the following spectral problem

$$
\begin{array}{l}
v' = \ \ \lambda u + \varphi(x)\, u, \\
u' = -\lambda v + \psi(x)\, v, \\
w = (u, v)^t \in \text{B.C.}.
\end{array}
\tag{3.10}
$$

Then, by Remark 3.1, we have $\hat{w} \in \bigcup\limits_{n=-\infty}^{\infty} S_n$.

Let $\hat{w} \in S_k$ for some $k \in \mathbb{Z}$. According to Remark 3.1 $\hat{\lambda}$ is a kth eigenvalue of problem (3.2). Then, from the proof of Lemma 3.1 it follows that $\hat{\lambda} \in J_k$. The proof of Lemma 3.2 is complete.

By virtue of Lemma 3.2 from Theorem 3.1 we obtain the following result.

Theorem 3.2. *Let the function $f(x, w, \lambda)$ satisfies the condition (1.4) for all $(x, w, \lambda) \in [0, \pi] \times \mathbb{R}^2 \times \mathbb{R}$. Then for every $k \in \mathbb{Z}$ and each $\nu = +$ or $-$, the connected component D_k^ν of Y, lies in $J_k \times S_k^\nu$ and is unbounded in $\mathbb{R} \times E$.*

4. Global bifurcation of solutions of problem (1.1)-(1.3)

Lemma 4.1. *For each $k \in \mathbb{Z}$, $\nu = +$ or $-$, and for sufficiently small $\tau > 0$ there exists a solution (λ_τ, w_τ) of problem (1.1)-(1.3) such that $w_\tau \in S_k^\nu$ and $||w_\tau|| = \tau$.*

Proof. Alongside with the problem (1.1)-(1.3) we shall consider the following approximate problem

$$
\begin{array}{l}
\ell w(x) = \lambda w(x) + f(x, |w|^\varepsilon w, \lambda) + g(x, w, \lambda), \ 0 < x < \pi, \\
w \in \text{B.C.},
\end{array}
\tag{4.1}
$$

where $\varepsilon \in (0, 1]$.

By (1.4) the function $f(x,|w|^{\varepsilon}w,\lambda)$ satisfies the condition (3.5). Then, by Theorem 2.2, for every integer k and each $\nu = +$ *or* $-$, there exists an unbounded continuum $T^{\nu}_{k,\varepsilon}$ of solutions of (4.1) such that

$$(\lambda_k,0)\in T^{\nu}_{k,\varepsilon}\subset(\mathbb{R}\times S^{\nu}_k)\cup\{(\lambda_k,0)\}\,.$$

Hence it follows that for any $\varepsilon\in(0,1]$ there exists a solution $(\lambda_{\tau,\,\varepsilon},w_{\tau,\,\varepsilon})$ of problem (4.1) such that $w_{\tau,\,\varepsilon}\in S^{\nu}_k$ and $||w_{\tau,\,\varepsilon}||=\tau$. It is obvious that $(\lambda_{\tau,\,\varepsilon},w_{\tau,\,\varepsilon})$ is a solution of the nonlinear problem

$$\begin{array}{l} Bw(x)=\lambda w(x)+P_{\varepsilon}(x)w(x)+g(x,w(x),\lambda),\ 0<x<\pi,\\ w\in \text{B.C.}. \end{array}\tag{4.2}$$

where

$$P(x)=\begin{pmatrix}\varphi_{\varepsilon}(x) & 0\\ 0 & \psi_{\varepsilon}(x)\end{pmatrix}$$

and the functions $\varphi_{\varepsilon}(x)$ and $\psi_{\varepsilon}(x)$ are determined of right hand sides of (3.6) with $(\lambda_{\tau,\,\varepsilon},w_{\tau,\,\varepsilon})$ instead of $(\lambda_{\varepsilon},w_{\varepsilon})$.

Taking into account condition (1.4) we have

$$\begin{array}{l} |\varphi_{\varepsilon}(x)|\,\le\,K,\ x\in[0,\pi],\\ |\psi_{\varepsilon}(x)|\,\le\,M,\ x\in[0,\pi]. \end{array}$$

Therefore, from the proof of Lemma 3.1 it follows that the k-th eigenvalue $\lambda_{k,\,\varepsilon}$ of the linear problem

$$\begin{array}{l} Bw(x)=\lambda w(x)+P_{\varepsilon}(x)w(x),\ 0<x<\pi,\\ w\in \text{B.C.}. \end{array}\tag{4.3}$$

is contained in J_k. By [10, Ch. 4, § 2, Theorem 2.1], Theorem 2.1 and Remark 3.1 $(\lambda_{k,\,\varepsilon},0)$ is a only bifurcation point of problem (4.2) by the set $\mathbb{R}\times S^{\nu}_k$, and this point corresponds to a continuous branch of nontrivial solutions. Consequently, each sufficiently small $\tau>0$ responds arbitrarily small $\rho_{\tau,\varepsilon}$ such that

$$\lambda_{\tau,\,\varepsilon}\in(\lambda_{k,\,\varepsilon}-\rho_{\tau,\,\varepsilon},\lambda_{k,\,\varepsilon}+\rho_{\tau,\,\varepsilon})\subset[\lambda_k-(K+M)-\rho_0,\lambda_k+(K+M)+\rho_0],\tag{4.4}$$

where $\rho_0=\sup\limits_{\varepsilon,\tau}\rho_{\tau\varepsilon}>0$.

Since the set $\{w_{\tau,\,\varepsilon}\in E\ :\ 0<\varepsilon\le1\}$ is bounded in $C([0,\pi];\mathbb{R}^2)$, the functions f and g are continuous in $[0,\pi]\times\mathbb{R}^2\times\mathbb{R}$ and $\{\lambda_{\tau,\,\varepsilon}\in R\ :\ 0<\varepsilon\le1\}$ is bounded in $\mathbb{R}$, then by (4.2) the set $\{w_{\tau,\,\varepsilon}\in E\ :\ 0<\varepsilon\le1\}$ is also bounded in $C^1([0,\pi];\mathbb{R}^2)$. Then, by the Arzela Ascoli theorem this set is compact in E.

Let $\{\varepsilon_n\}_{n=1}^{\infty}$, $0<\varepsilon_n<1$, be a sequence converging to 0, and such that $(\lambda_{\tau,\,\varepsilon_n},w_{\tau,\,\varepsilon_n})\to(\lambda_{\tau},w_{\tau})$ in $\mathbb{R}\times E$. Passing to the limit as $n\to\infty$ in (4.2) we obtain that $(\lambda_{\tau},w_{\tau})$ is a solution of the nonlinear problem (1.1)-(1.3). Since $||w_{\tau}||=\tau$ then by Lemma 2.1 we have $w_{\tau}\in S^{\nu}_k$. The proof of Lemma 4.1 is complete.

Corollary 4.1. *The set of bifurcation points of problem* (1.1)-(1.3) *by the set* $\mathbb{R}\times S^{\nu}_k$ *is nonempty.*

Lemma 4.2. *Let* $\varepsilon_n,\ 0\le\varepsilon_n\le1,\ n=1,\,2,\,\ldots,$ *be a sequence converging to* 0. *If* $(\lambda_{\varepsilon_n},w_{\varepsilon_n})$ *is a solution of problem* (4.1) *corresponding to* $\varepsilon=\varepsilon_n$, *and sequence* $\{(\lambda_{\varepsilon_n},\ w_{\varepsilon_n})\}_{n=1}^{\infty}$ *converges to* $(\xi,\ 0)$ *in* $\mathbb{R}\times E$, *then* $\xi\in J_k$.

Proof. Assume the contrary, i.e. let $\xi \notin J_k$. We denote $\sigma = \text{dist}\{\xi, J_k\}$. Since $\lambda_{\varepsilon_n} \to \xi$, then there exists $n_\sigma \in \mathbb{N}$ such that for all $n > n_\sigma$ we have the inequality $|\lambda_{\varepsilon_n} - \xi| < \sigma/2$. Hence, $\text{dist}\{\lambda_{\varepsilon_n}, J_k\} > \sigma/2$ at $n > n_\sigma$.

Note that $(\lambda_{\varepsilon_n}, w_{\varepsilon_n})$ is a solution of nonlinear problem (4.2) for $\varepsilon = \varepsilon_n$. Since $(\lambda_{k,n}, 0)$ is a only bifurcation point of problem (4.2) by the set $\mathbb{R} \times S_k^\nu$, then every sufficiently large $n > n_\sigma$ corresponds to a arbitrarily small $\rho_n > 0$ that $\rho_n < \sigma/2$ and $\lambda_{\varepsilon_n} \in (\lambda_{k,n} - \rho_n, \lambda_{k,n} + \rho_n)$, where $\lambda_{k,n}$ is the k-th eigenvalue of the linear problem (4.3) for $\varepsilon = \varepsilon_n$. Consequently, $\lambda_{\varepsilon_n} \in (\lambda_{k,n} - \sigma/2, \lambda_{k,n} + \sigma/2)$. From the proof of Lemma 3.1 we have $\lambda_{k,n} \in J_k$, whence it follows inequality $\text{dist}\{\lambda_{\varepsilon_n}, J_k\} < \sigma/2$, which contradicts $\text{dist}\{\lambda_{\varepsilon_n}, J_k\} > \sigma/2$. The proof of Lemma 4.2 is complete.

Corollary 4.2. *If* $(\lambda, 0)$ *is a bifurcation point of problem* (1.1)-(1.3) *by the set* S_k^ν, *then* $\lambda \in J_k$.

For each $k \in \mathbb{Z}$ and $\nu = +\,\text{or}\,-$, we define the set $\widetilde{T}_k^\nu \subset Y$ to be the union of all the components $T_{k,\lambda}^\nu$ of Y which bifurcating from the bifurcation points $(\lambda, 0)$ of (1.1)-(1.3) by the set $\mathbb{R} \times S_k^\nu$. Let $T_k^\nu = \widetilde{T}_k^\nu \cup (J_k \times 0)$.

Theorem 4.1. *For every* $k \in \mathbb{Z}$ *and each* $\nu = +$ *or* $-$ *, the connected component* T_k^ν *of* Y *lies in* $(\mathbb{R} \times S_k^\nu) \cup (J_k \times 0)$ *and is unbounded in* $\mathbb{R} \times E$.

The proof of Theorem 4.1 is similar to that of [12; Theorem 2.1] using Lemmas 4.1, 4.2 and Corollaries 4.1, 4.2.

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Humay Sh. Rzayeva
Ganja State University, Ganja, AZ2000, Azerbaijan.
E-mail address: humay_rzayeva@bk.ru

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