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SOME REMARKS ON NONLINEAR HYPERBOLIC EQUATION

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Abstract. Here a mixed problem for a nonlinear hyperbolic equation with Neumann boundary value condition is investigated, and a priori estimations for the possible solutions of the considered problem are obtained. These results demonstrate that any solution of this problem possess certain smoothness properties.

1. Introduction and Formulation of Problem

In this article we consider a mixed problem for a nonlinear hyperbolic equation and study the smoothness of a possible solution of the problem, in some sense. Here we got some new a priori estimations for a solution of the considered problem.

It is known that, up to now, the problem of the solvability of a nonlinear hyperbolic equation with nonlinearity of this type has not been solved when $\Omega \subset R^n$, $n \geq 2$. It should also be noted that it is not possible to use the a priori estimations, which can be obtained by the known methods, to prove the solvability in this case. Consequently, there are no obtained results on a solvability of a mixed problem for the equation of the following type

$$\frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} D_i \left[a_i(t,x) \left| D_i u \right|^{p-2} D_i u \right] = h(t,x), \quad (t,x) \in Q_T$$
$$Q_T \equiv (0,T) \times \Omega, \quad \Omega \subset R^n, \quad T > 0, \quad p > 2.$$

As known, the investigation of a mixed problem for the nonlinear hyperbolic equations of such type on the Sobolev type spaces when $\Omega \subset R^n$, $n \geq 2$ is connected with many difficulties (see, for example, the works of Leray, Courant, Friedrichs, Lax, F. John, Garding, Ladyzhenskaya, J.-L. Lions, H. Levine, Rozdestvenskii and also, [2, 5, 7 - 12, 14 - 16, 18, 19], etc.). Furthermore the possible solutions of this problem may possess a gradient catastrophe. Only in the case $n = 1$, it is achieved to prove solvability theorems for the problems of such type (and essentially with using the Riemann invariants).

However, recently certain classes of nonlinear hyperbolic equations were investigated and results on the solvability of the considered problems in a more

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generalized sense were obtained (see, for example, [14] its references) and also certain result about dense solvability was obtained ([20]). Furthermore there are such special class of the nonlinear hyperbolic equations, for which the solvabilities were studied under some additional conditions (see, [3 - 5, 14, 17, 18, 23] and its references), for example, under some geometrical conditions.

Here, we investigate a mixed problem for equations of certain class with the Neumann boundary-value conditions. In the beginning, a mixed problem for a nonlinear parabolic equation with similar nonlinearity and conditions as above is studied and the existence of the strongly solutions of this problem is proved. Further some a priori estimations for a possible solution of the considered problem is received in the hyperbolic case with use of the result on the parabolic problem studied above. These results demonstrate that any solution of our main problem possesses certain smoothness properties, which might help for the proof of some existence theorems.

Consider the problem (for simplisity hereafter we assume that $a_i(t,x) = 1$, $i = \overline{1,n}$)

$$\frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} D_i\left(|D_i u|^{p-2} D_i u\right) = h(t,x), \quad (t,x) \in Q_T, \quad p > 2, \tag{1.1}$$

$$u(0,x) = u_0(x), \quad \frac{\partial u}{\partial t} \mid_{t=0} = u_1(x), \quad x \in \Omega \subset R^n, \quad n \geq 2, \tag{1.2}$$

$$\frac{\partial u}{\partial \widehat{\nu}} \mid_{\Gamma} \equiv \sum_{i=1}^{n} |D_i u|^{p-2} D_i u \cos(\nu, x_i) = 0, \quad (x', t) \in \Gamma \equiv \partial\Omega \times [0,T], \tag{1.3}$$

here (and onward) $\Omega \subset R^n, n \geq 2$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$; $u_0(x)$, $u_1(x)$, $h(t,x)$ are functions such that $u_0, u_1 \in W_p^1(\Omega)$, $h \in L_p\left(0,T; W_p^1(\Omega)\right)$, ν denote the unit outward normal to $\partial\Omega$ (see, [1, 13]).

Introduce the class of the functions $u : Q_T \longrightarrow R$

$$V(Q_T) \equiv W_2^1(0,T; L_2(\Omega)) \cap L^\infty\left(0,T; W_p^1(\Omega)\right) \cap L_{p-1}\left(0,T; \widetilde{S}^1_{1,2(p-2),2}(\Omega)\right) \cap$$
$$\left\{ u(t,x) \;\middle|\; \frac{\partial^2 u}{\partial t^2}, \sum_{i=1}^{n}\left(|D_i u|^{p-2} D_i^2 u\right) \in L_1(0,T; L_2(\Omega)) \right\} \cap$$
$$\left\{ u(t,x) \;\middle|\; \sum_{i=1}^{n}\int_0^t |D_i u|^{p-2} D_i u d\tau \in W^1_\infty(0,T; L_q(\Omega)) \cap L^\infty\left(0,T; W_2^1(\Omega)\right) \right\}$$
$$\left\{ u(t,x) \;\middle|\; u(0,x) = u_0(x), \frac{\partial u}{\partial t} \mid_{t=0} = u_1(x), \frac{\partial u}{\partial \widehat{\nu}} \mid_{\Gamma} = 0 \right\} \tag{DS}$$

where

$$\widetilde{S}^1_{1,\alpha,\beta}(\Omega) \equiv \left\{ u(t,x) \;\middle|\; [u]^{\alpha+\beta}_{S^1_{1,\alpha,\beta}} = \|u\|^{\alpha+\beta}_{\alpha+\beta} + \sum_{i=1}^{n} \|D_i u\|^{\alpha+\beta}_{\alpha+\beta} + \right.$$
$$\left. \left\| \sum_{i,j=1}^{n} |D_i u|^{\frac{\alpha}{\beta}} D_j D_i u \right\|^{\beta}_{\beta} < \infty \right\}, \quad \alpha \geq 0, \ \beta \geq 1.$$

Thus, we will understand the solution of the problem in the following form:

Definition 1.1. A function $u(t,x) \in V(Q_T)$ is called solution of problem (1.1) - (1.3) if $u(t,x)$ satisfies the following equality

$$\left[\frac{\partial^2 u}{\partial t^2}, v\right] - \left[\sum_{i=1}^{n} D_i\left(|D_i u|^{p-2} D_i u\right), v\right] = [h, v]$$

for any $v \in W_q^1(0,T; L_2(\Omega)) \cap L^\infty(Q_T)$, where $[\circ,\circ] \equiv \int\limits_{Q_T} \circ \times \circ \; dxdt.$

Our aim in this article is to prove

Theorem 1.1. *Let $u_0, u_1 \in W_p^1(\Omega)$ and $h \in L_p\left(0,T; W_p^1(\Omega)\right)$, $p > 2$ then each solution of problem (1.1)-(1.3) belongs to a bounded subset of the space $V(Q_T)$ defined in (DS).*

For the investigation of the posed problem in the beginning we will study two problems, which are connected with considered problem. One of these problems immediately follows from problem (1.1)-(1.3) and have the form:

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n} D_i \int\limits_0^t \left(|D_i u|^{p-2} D_i u\right) d\tau = H(t,x) + u_1(x), \qquad (1.4)$$

where $H(t,x) = \int\limits_0^t h(\tau,x)\, d\tau.$

Consequently, if $u(t,x)$ is a solution of problem (1.1) - (1.3) then $u(t,x)$ is a such solution of equation (1.4) that the following conditions are fulfilled:

$$u(0,x) = u_0(x), \quad \frac{\partial u}{\partial \widehat{\nu}}|_\Gamma = 0. \qquad (1.5)$$

From here it follows that problems (1.1) - (1.3) and (1.4) - (1.5) are equivalent. And other problem is the nonlinear parabolic problem

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n} D_i\left(|D_i u|^{p-2} D_i u\right) = h(t,x), \quad (t,x) \in Q_T,\ p > 2,\ n \geq 2, \qquad (1.6)$$

$$u(0,x) = u_0(x), \quad x \in \Omega, \quad \frac{\partial u}{\partial \widehat{\nu}} \mid_\Gamma = 0, \qquad (1.7)$$

where $u_0 \in W_p^1(\Omega)$, $h \in L_2\left(0,T; W_2^1(\Omega)\right)$ and $p > 2$.

In the beginning the solvability of this problem is studied, for which the general result is used, therefore we begin with this result.

2. Some General Solvability Results

Let X, Y be locally convex vector topological spaces, $B \subseteq Y$ be a Banach space and $g : D(g) \subseteq X \longrightarrow Y$ be a mapping. Introduce the following subset of X

$$\mathcal{M}_{gB} \equiv \{x \in X \mid g(x) \in B,\ \operatorname{Im} g \cap B \neq \varnothing\}.$$

Definition 2.1. A subset $\mathcal{M} \subseteq X$ is called a *pn*−space (i.e. pseudonormed space) if $\mathcal{M}$ is a topological space and there is a function $[\cdot]_S : \mathcal{M} \longrightarrow R_+^1 \equiv [0,\infty)$ (which is called p−norm of $\mathcal{M}$) such that

qn) $[x]_{\mathcal{M}} \geq 0$, $\forall x \in \mathcal{M}$ and $0 \in \mathcal{M}$, $x = 0 \Longrightarrow [x]_{\mathcal{M}} = 0$;

pn) $[x_1]_{\mathcal{M}} \neq [x_2]_{\mathcal{M}} \Longrightarrow x_1 \neq x_2$, for $x_1, x_2 \in \mathcal{M}$, and $[x]_{\mathcal{M}} = 0 \Longrightarrow x = 0$;

The following conditions are often fulfilled in the spaces $\mathcal{M}_{gB}$.

N) There exist a convex function $\nu : R^1 \longrightarrow \overline{R^1_+}$ and number $K \in (0,\infty]$ such that $[\lambda x]_{\mathcal{M}} \leq \nu(\lambda)[x]_{\mathcal{M}}$ for any $x \in \mathcal{M}$ and $\lambda \in R^1$, $|\lambda| < K$, moreover $\lim\limits_{|\lambda| \longrightarrow \lambda_j} \frac{\nu(\lambda)}{|\lambda|} = c_j$, $j = 0,1$ where $\lambda_0 = 0$, $\lambda_1 = K$ and $c_0 = c_1 = 1$ or $c_0 = 0$, $c_1 = \infty$, i.e. if $K = \infty$ then $\lambda x \in \mathcal{M}$ for any $x \in S$ and $\lambda \in R^1$.

Let $g : D(g) \subseteq X \longrightarrow Y$ be such a mapping that $\mathcal{M}_{gB} \neq \varnothing$ and the following conditions are fulfilled

(g_1) $g : D(g) \longleftrightarrow \operatorname{Im} g$ is bijection and $g(0) = 0$;

(g_2) there is a function $\nu : R^1 \longrightarrow \overline{R^1_+}$ satisfying condition N such that

$$\|g(\lambda x)\|_B \leq \nu(\lambda)\|g(x)\|_B, \ \forall x \in \mathcal{M}_{gB}, \ \forall \lambda \in R^1;$$

If mapping g satisfies conditions (g_1) and (g_2), then $\mathcal{M}_{gB}$ is a $pn-$space with $p-$norm defined in the following way: there is a one-to-one function $\psi : R^1_+ \longrightarrow R^1_+$, $\psi(0) = 0$, $\psi, \psi^{-1} \in C^0$ such that $[x]_{\mathcal{M}_{gB}} \equiv \psi^{-1}(\|g(x)\|_B)$. In this case $\mathcal{M}_{gB}$ is a metric space with a metric: $d_{\mathcal{M}}(x_1; x_2) \equiv \|g(x_1) - g(x_2)\|_B$. Further, we consider just such type of $pn-$spaces.

Definition 2.2. The $pn-$space $\mathcal{M}_{gB}$ is called weakly complete if $g(\mathcal{M}_{gB})$ is weakly closed in B. The pn-space $\mathcal{M}_{gB}$ is "reflexive" if each bounded weakly closed subset of $\mathcal{M}_{gB}$ is weakly compact in $\mathcal{M}_{gB}$.

It is clear that if B is a reflexive Banach space and $\mathcal{M}_{gB}$ is a $pn-$space, then $\mathcal{M}_{gB}$ is "reflexive". Moreover, if B is a separable Banach space, then $\mathcal{M}_{gB}$ is separable (see, for example, [21, 22] and their references). Now, consider a nonlinear equation in the general form. Let X, Y be Banach spaces with dual spaces X^*, Y^* respectively, $\mathcal{M}_0 \subseteq X$ is a weakly complete $pn-$space, $f : D(f) \subseteq X \longrightarrow Y$ be a nonlinear operator. Consider the equation

$$f(x) = y, \quad y \in Y. \tag{2.1}$$

It is clear that (2.1) is equivalent to the following functional equation:

$$\langle f(x), y^* \rangle = \langle y, y^* \rangle, \quad \forall y^* \in Y^*. \tag{2.2}$$

Let $f : D(f) \subseteq X \longrightarrow Y$ be a nonlinear bounded operator and the following conditions hold

1) $f : \mathcal{M}_0 \subseteq D(f) \longrightarrow Y$ is a weakly compact (weakly "continuous") mapping, i.e. for any weakly convergence sequence $\{x_m\}_{m=1}^{\infty} \subset \mathcal{M}_0$ in $\mathcal{M}_0$ (i.e. $x_m \overset{\mathcal{M}_0}{\rightharpoonup} x_0 \in \mathcal{M}_0$) there is a subsequence $\{x_{m_k}\}_{k=1}^{\infty} \subseteq \{x_m\}_{m=1}^{\infty}$ such that $f(x_{m_k}) \overset{Y}{\rightharpoonup} f(x_0)$ weakly in Y (or for a general sequence if $\mathcal{M}_0$ is not a separable space) and $\mathcal{M}_0$ is a weakly complete $pn-$space;

2) there exist a mapping $g : X_0 \subseteq X \longrightarrow Y^*$ and a continuous function $\varphi : R^1_+ \longrightarrow R^1$ nondecreasing for $\tau \geq \tau_0 \geq 0$ & $\varphi(\tau_1) > 0$ for a number $\tau_1 > 0$, such that g generates a "coercive" pair with f in a generalized sense on the topological space $X_1 \subseteq X_0 \cap \mathcal{M}_0$, i.e.

$$\langle f(x), g(x) \rangle \geq \varphi([x]_{\mathcal{M}_0})[x]_{\mathcal{M}_0}, \quad \forall x \in X_1,$$

where X_1 is a topological space such that $\overline{X_1}^{X_0} \equiv X_0$ and $\overline{X_1}^{\mathcal{M}_0} \equiv \mathcal{M}_0$, and $\langle \cdot, \cdot \rangle$ is a dual form of the pair (Y, Y^*). Moreover one of the following conditions (α) or (β) holds:

(α) if $g \equiv L$ is a linear continuous operator, then $\mathcal{M}_0$ is a "reflexive" space (see [20 - 22]), $X_0 \equiv X_1 \subseteq \mathcal{M}_0$ is a separable topological vector space which is dense in $\mathcal{M}_0$ and $\ker L^* = \{0\}$.

(β) if g is a bounded operator (in general, nonlinear), then Y is a reflexive separable space, $g(X_1)$ contains an everywhere dense linear manifold of Y^* and g^{-1} is a weakly compact (weakly continuous) operator from Y^* to $\mathcal{M}_0$.

Theorem 2.1. *Let conditions 1 and 2 hold. Then equation (2.1) (or (2.2)) is solvable in $\mathcal{M}_0$ for any $y \in Y$ satisfying the following inequation: there exists $r > 0$ such that the inequality*

$$\varphi\left([x]_{\mathcal{M}_0}\right)[x]_{\mathcal{M}_0} \geq \langle y, g(x)\rangle, \ \textit{for} \quad \forall x \in X_1 \quad \textit{with} \quad [x]_{\mathcal{M}} \geq r. \tag{2.3}$$

holds.

Proof. Assume that conditions 1 and 2 (α) are fulfilled and $y \in Y$ such that (2.3) holds. We are going to use Galerkin's approximation method. Let $\left\{x^k\right\}_{k=1}^{\infty}$ be a complete system in the (separable) space $X_1 \equiv X_0$. Then we are looking for approximate solutions in the form $x_m = \sum_{k=1}^{m} c_{mk}x^k$, where c_{mk} are unknown coefficients, which can be determined from the system of algebraic equations

$$\Phi_k\left(c_m\right) := \left\langle f\left(x_m\right), g\left(x^k\right)\right\rangle - \left\langle y, g\left(x^k\right)\right\rangle = 0, \quad k = 1, 2, ..., m \tag{2.4}$$

where $c_m \equiv (c_{m1}, c_{m2}, ..., c_{mm})$.

We observe that the mapping $\Phi\left(c_m\right) := \left(\Phi_1\left(c_m\right), \Phi_2\left(c_m\right), ..., \Phi_m\left(c_m\right)\right)$ is continuous by virtue of condition 1. (2.3) implies the existence of such $r = r\left(\|y\|_Y\right) > 0$ that the "acute angle" condition is fulfilled for all x_m with $[x_m]_{\mathcal{M}_0} \geq r$, i.e. for any $c_m \in S_{r_1}^{R^m}(0) \subset R^m$, $r_1 \geq r$ the inequality

$$\sum_{k=1}^{m}\left\langle\Phi_k\left(c_m\right), c_{mk}\right\rangle \equiv \left\langle f\left(x_m\right), g\left(\sum_{k=1}^{m} c_{mk}x^k\right)\right\rangle - \left\langle y, g\left(\sum_{k=1}^{m} c_{mk}x^k\right)\right\rangle =$$

$$\left\langle f\left(x_m\right), g\left(x_m\right)\right\rangle - \left\langle y, g\left(x_m\right)\right\rangle \geq 0, \quad \forall c_m \in \mathbb{R}^m, \|c_m\|_{\mathbb{R}^m} = r_1.$$

holds. The solvability of system (2.4) for each $m = 1, 2, \ldots$ follows from a well-known "acute angle" lemma ([10, 11, 21, 22]), which is equivalent to the Brouwer's fixed-point theorem. Thus, $\{x_m \mid m \geq 1\}$ is the sequence of approximate solutions which are contained in a bounded subset of the space $\mathcal{M}_0$. Further arguments are analogous to those from [11, 12, 22] therefore we omit them. It remains to pass to the limit in (2.4) by m and use the weak convergency of a subsequence of the sequence $\{x_m \mid m \geq 1\}$, the weak compactness of the mapping f, and the completeness of the system $\left\{x^k\right\}_{k=1}^{\infty}$in the space X_1.

Hence we get the limit element $x_0 = w - \lim_{j \nearrow \infty} x_{m_j} \in S_0$ which is the solution of the equation

$$\left\langle f\left(x_0\right), g\left(x\right)\right\rangle = \left\langle y, g\left(x\right)\right\rangle, \quad \forall x \in X_0, \tag{2.5}$$

or of the equation

$$\left\langle g^* \circ f\left(x_0\right), x\right\rangle = \left\langle g^* \circ y, x\right\rangle, \quad \forall x \in X_0. \tag{2.5'}$$

In the second case, i.e. when conditions 1 and 2 (β) are fulfilled and $y \in Y$ such that (2.3) holds, we suppose that the approximate solutions are searched in the form

$$x_m = g^{-1}\left(\sum_{k=1}^{m} c_{mk} y_k^*\right) \equiv g^{-1}\left(y_{(m)}^*\right), \quad i.e.\ x_m = g^{-1}\left(y_{(m)}^*\right) \tag{2.6}$$

where $\{y_k^*\}_{k=1}^{\infty} \subset Y^*$ is a complete system in the (separable) space Y^* and belongs to $g(X_1)$. In this case the unknown coefficients c_{mk}, that might be determined from the system of algebraic equations

$$\widetilde{\Phi}_k(c_m) := \langle f(x_m), y_k^*\rangle - \langle y, y_k^*\rangle = 0, \quad k = 1, 2, ..., m \tag{2.7}$$

with $c_m \equiv (c_{m1}, c_{m2}, ..., c_{mm})$, from which under the our conditions we get

$$\langle f(x_m), y_k^*\rangle - \langle y, y_k^*\rangle = \left\langle f\left(g^{-1}\left(y_{(m)}^*\right)\right), y_k^*\right\rangle - \langle y, y_k^*\rangle = 0, \tag{2.7'}$$

for $k = 1, 2, ..., m$.

As above we observe that the mapping

$$\widetilde{\Phi}(c_m) := \left(\widetilde{\Phi}_1(c_m), \widetilde{\Phi}_2(c_m), ..., \widetilde{\Phi}_m(c_m)\right)$$

is continuous by virtue of conditions 1 and 2(β). (2.3) implies the existence of such $\widetilde{r} > 0$ that the "acute angle" condition is fulfilled for all $y_{(m)}^*$ with $\left\|y_{(m)}^*\right\|_{Y^*} \geq \widetilde{r}$, i.e. for any $c_m \in S_{\widetilde{r}_1}^{R^m}(0) \subset R^m$, $\widetilde{r}_1 \geq \widetilde{r}$ the inequality

$$\sum_{k=1}^{m} \left\langle \widetilde{\Phi}_k(c_m), c_{mk}\right\rangle \equiv \left\langle f(x_m), \sum_{k=1}^{m} c_{mk} y_k^*\right\rangle - \left\langle y, \sum_{k=1}^{m} c_{mk} y_k^*\right\rangle =$$

$$\left\langle f\left(g^{-1}\left(y_{(m)}^*\right)\right), y_{(m)}^*\right\rangle - \left\langle y, y_{(m)}^*\right\rangle = \langle f(x_m), g(x_m)\rangle - \langle y, g(x_m)\rangle \geq 0,$$

$$\forall c_m \in \mathbb{R}^m, \|c_m\|_{\mathbb{R}^m} = \widetilde{r}_1.$$

holds by virtue of the conditions. Consequently the solvability of the system (2.7) (or (2.7')) for each $m = 1, 2, \ldots$ follows from the "acute angle" lemma, as above. Thus, we obtain $\left\{y_{(m)}^* \mid m \geq 1\right\}$ is the sequence of the approximate solutions of system (2.7'), that is contained in a bounded subset of Y^*. From here it follows there is a subsequence $\left\{y_{(m_j)}^*\right\}_{j=1}^{\infty}$ of the sequence $\left\{y_{(m)}^* \mid m \geq 1\right\}$ such that it is weakly convergent in Y^*, and consequently the sequence $\left\{x_{m_j}\right\}_{j=1}^{\infty} \equiv \left\{g^{-1}\left(y_{(m_j)}^*\right)\right\}_{j=1}^{\infty}$ weakly converges in the space $\mathcal{M}_0$ by the condition 2(β) (maybe after passing to the subsequence of it). It remains to pass to the limit in (2.7') by j and use a weak convergency of a subsequence of the sequence $\left\{y_{(m)}^* \mid m \geq 1\right\}$, the weak compactness of mappings f and g^{-1}, and next the completeness of the system $\{y_k^*\}_{k=1}^{\infty}$in the space Y^*.

Hence we get the limit element $x_0 = w-\lim\limits_{j\nearrow\infty} x_{m_j} = w-\lim\limits_{j\nearrow\infty} g^{-1}\left(y_{(m_j)}^*\right) \in \mathcal{M}_0$ and it is the solution of the equation

$$\langle f(x_0), y^*\rangle = \langle y, y^*\rangle, \quad \forall y^* \in Y^*. \tag{2.8}$$

□

Remark 2.1. *It is obvious that if there exists a function* $\psi : R^1_+ \longrightarrow R^1_+$, $\psi \in C^0$ *such that* $\psi(\xi) = 0 \iff \xi = 0$ *and if the following inequality is fulfilled* $\|x_1 - x_2\|_X \le \psi(\|f(x_1) - f(x_2)\|_Y)$ *for all* $x_1, x_2 \in \mathcal{M}_0$ *then solution of equation (2.2) is unique.*

It should be noted the spaces of the $pn-$space type often arising from nonlinear problems with nonlinear main parts, for example,

1) the equation of the nonlinear filtration or diffusion that have the expression:

$$\frac{\partial u}{\partial t} - \nabla \cdot (g(u) \nabla u) + h(t, x, u) = 0, \quad u\big|_{\partial\Omega\times[0.T]} = 0,$$
$$u(0, x) = u_0(x), \quad x \in \Omega \subset \mathbb{R}^n, \quad n \ge 1$$

where $g : \mathbb{R} \longrightarrow \mathbb{R}_+$ is a convex function ($g(s) \equiv |s|^\rho$, $\rho > 0$) and $h(t, x, s)$ is a Caratheodory function, in this case it is needed to investigate

$$S_{1,\rho,2}(\Omega) \equiv \left\{ u \in L^1(\Omega) \,\middle|\, \int_\Omega g(u(x)) |\nabla u|^2 dx < \infty; \quad u\big|_{\partial\Omega} = 0 \right\};$$

2) the equation of the Prandtl-von Mises type equation that have the expression:

$$\frac{\partial u}{\partial t} - |u|^\rho \Delta u + h(t, x, u) = 0, \quad u\big|_{\partial\Omega\times[0.T]} = 0,$$
$$u(0, x) = u_0(x), \quad x \in \Omega \subset \mathbb{R}^n, \quad n \ge 1$$

where $\rho > 0$ and $h(t, x, s)$ is a Caratheodory function, in this case it is needed to investigate the spaces of the following spaces type $S_{1,\mu,q}(\Omega)$ $(\mu \ge 0, q \ge 1)$ and

$$S_{\Delta,\rho,2}(\Omega) \equiv \left\{ u \in L^1(\Omega) \,\middle|\, \int_\Omega |u(x)|^\rho |\Delta u|^2 dx < \infty; \quad u\big|_{\partial\Omega} = 0 \right\}$$

etc. Theorem and the spaces of such type were used earlier in many works see, for example, [11, 21].

Corollary 2.1. *Assume that the conditions of Theorem 2.1 are fulfilled and there is a continuous function* $\varphi_1 : R^1_+ \longrightarrow R^1_+$ *such that* $\|g(x)\|_{Y^*} \le \varphi_1([x]_{S_0})$ *for any* $x \in X_0$ *and* $\varphi(\tau) \nearrow +\infty$ *and* $\frac{\varphi(\tau)\tau}{\varphi_1(\tau)} \nearrow +\infty$ *as* $\tau \nearrow +\infty$. *Then equation (2.2) is solvable in* $\mathcal{M}_0$, *for any* $y \in Y$.

3. Solvability of Problem (1.6) - (1.7)

A solution of problem (1.6) - (1.7) we will understand in following sense.

Definition 3.1. A function $u(t, x)$ of the space $P^1_{1,(p-2)q,q,2}(Q_T)$ is called a solution of problem (1.6) - (1.7) if $u(t, x)$ satisfies the following equality

$$\left[\frac{\partial u}{\partial t}, v\right] - \left[\sum_{i=1}^n D_i \left(|D_i u|^{p-2} D_i u\right), v\right] = [h, v], \quad \forall v \in L_p(Q_T), \tag{3.1}$$

where

$$P^1_{1,(p-2)q,q,2}(Q_T) \equiv L_p\left(0, T; \overset{0}{S}{}^1_{1,(p-2)q,q}(\Omega)\right) \cap W^1_2(0, T; L_2(\Omega))$$

$$S^1_{1,\alpha,\beta}(\Omega) \equiv \left\{ u(t,x) \;\middle|\; [u]^{\alpha+\beta}_{S^1_{1,\alpha,\beta}} = \sum_{i=1}^{n} \|D_i u\|^{\alpha+\beta}_{\alpha+\beta} + \right.$$

$$\left. \sum_{i,j=1}^{n} \left\| |D_i u|^{\frac{\alpha}{\beta}} D_j D_i u \right\|^{\beta}_{\beta} < \infty \right\}, \quad \alpha \geq 0, \; \beta \geq 1.$$

and $[\cdot,\cdot]$ denotes dual form for the pair $(L_q(Q_T), L_p(Q_T))$ as in the section 1.

For the study of problem (1.6) - (1.7) we use Theorem 2.1 and Corollary 2.1 of the previous section. For applying these results to problem (1.6) - (1.7), we will choose the corresponding spaces and mappings:

$$\mathcal{M}_0 \equiv P^1_{1,(p-2)q,q,2}(Q_T) \equiv L_p\left(0,T; \overset{0}{S}{}^1_{1,(p-2)q,q}(\Omega)\right) \cap W^1_2(0,T; L_2(\Omega)),$$

$$\Phi(u) \equiv -\sum_{i=1}^{n} D_i\left(|D_i u|^{p-2} D_i u\right), \quad \gamma_0 u \equiv u(0,x),$$

$$f(\cdot) \equiv \left\{ \frac{\partial \cdot}{\partial t} + \Phi(\cdot); \; \gamma_0 \cdot \right\}, \quad g(\cdot) \equiv \left\{ \frac{\partial \cdot}{\partial t} - \Delta \cdot; \quad \gamma_0 \cdot \right\},$$

$$X_0 \equiv W^1_p(0,T; L_p(\Omega)) \cap \widetilde{X};$$

$$X_1 \equiv X_0 \cap \left\{ u(t,x) \;\middle|\; \frac{\partial u}{\partial \widehat{\nu}} \mid_{\Gamma} = 0 \right\};$$

$$Y \equiv L_q(Q), \; q = p', \widetilde{X} \equiv L_p\left(0,T; W^2_p(\Omega)\right) \cap \left\{ u(t,x) \;\middle|\; \frac{\partial u}{\partial \nu} \mid_{\Gamma} = 0 \right\}$$

here

$$\overset{0}{S}{}^1_{1,(p-2)q,q}(\Omega) \equiv S^1_{1,(p-2)q,q}(\Omega) \cap \left\{ u(t,x) \;\middle|\; \frac{\partial u}{\partial \widehat{\nu}} \mid_{\partial\Omega} = 0 \right\}$$

Thus, as we can see from the above denotations, mapping f is defined by problem (1.6)-(1.7) and mapping g is defined by the following problem

$$\frac{\partial u}{\partial t} - \Delta u = v(t,x), \quad (t,x) \in Q_T, \tag{3.2}$$

$$\gamma_0 u \equiv u(0,x) = u_0(x), \; \frac{\partial u}{\partial \nu} \mid_{\Gamma} = 0. \tag{3.3}$$

As known (see, [1, 5, 6, 12]), problem (3.2)-(3.3) is solvable in the space

$$X_0 \equiv W^1_p(0,T; L_p(\Omega)) \cap L_p\left(0,T; W^2_p(\Omega)\right) \cap \left\{ u(t,x) \;\middle|\; \frac{\partial u}{\partial \nu} \mid_{\Gamma} = 0 \right\}$$

for any $v \in L_p(Q_T)$, $u_0 \in W^1_p(\Omega)$.

Now we will demonstrate that all conditions of Theorem 2.1 and also of Corollary 2.1 are fulfilled.

Proposition 3.1. *Mappings f and g , defined above, generate a "coercive" pair on X_1 in the generalized sense, and moreover the statement of Corollary 2.1 is valid.*

Proof. Let $u \in X_1$, i.e.

$$u \in X_0 \cap \left\{ u(t,x) \left| \frac{\partial u}{\partial \widehat{\nu}} \mid \Gamma = 0 \right. \right\}.$$

Consider the dual form $\langle f(u), g(u) \rangle$ for any $u \in X_1$. More exactly, it is enough to consider the dual form in the form

$$\int_0^t \int_\Omega f(u)\ g(u)\ dx d\tau \equiv [f(u),\ g(u)]_t \tag{*}$$

Hence, if we consider the above expression then after certain action and in view of the boundary conditions, we get

$$[f(u),\ g(u)]_t \equiv \left[\frac{\partial u}{\partial t},\ \frac{\partial u}{\partial t}\right]_t + \left[\Phi(u),\ \frac{\partial u}{\partial t}\right]_t -$$

$$\left[\frac{\partial u}{\partial t},\ \Delta u\right]_t - [\Phi(u),\ \Delta u]_t + \int_\Omega u_0\ u_0\ dx =$$

$$= \int_0^t \left\| \frac{\partial u}{\partial t} \right\|_2^2 d\tau + \sum_{i=1}^n \left[\frac{1}{p} \|D_i u\|_p^p (t) + \frac{1}{2} \|D_i u\|_2^2 (t) \right] +$$

$$\|u_0\|_2^2 + (p-1) \sum_{i,j=1}^n \int_0^t \left\| |D_i u|^{\frac{p-2}{2}} D_i D_j u \right\|_2^2 -$$

$$\sum_{i=1}^n \left[\frac{1}{p} \|D_i u_0\|_p^p + \frac{1}{2} \|D_i u_0\|_2^2 \right] \tag{3.4}$$

here and in (3.5) we denote $\|\cdot\|_{p_1} \equiv \|\cdot\|_{L_{p_1}(\Omega)}$, $p_1 \geq 1$.

From here it follows,

$$[f(u),\ g(u)] \geq c \left(\left\| \frac{\partial u}{\partial t} \right\|_{L_2(Q)}^2 + \sum_{i=1}^n \left\| |D_i u|^{\frac{p-2}{2}} D_i u \right\|_{L_2(Q)}^2 \right) -$$

$$c_1 \|u_0\|_{W_p^1}^p - c_2 \geq \widetilde{c} \left(\left\| \frac{\partial u}{\partial t} \right\|_{L_2(Q)}^2 + [u]_{L_p\left(S_{1,(p-2)q,q}\right)}^p \right) -$$

$$c_1 \|u_0\|_{W_p^1}^p - c_2 \geq \widetilde{c}\,[u]_{\mathbf{P}^1_{1,(p-2)q,q,2}(Q)}^2 - c_1 \|u_0\|_{W_p^1}^p - \widetilde{c}_2,$$

which demonstrates fulfillment of the statement of Corollary 2.1[1]. Consequently, Proposition 3.1 is true. □

Further for the right part of the dual form, we obtain under the conditions of Proposition 3.1 (using same way as in the above proof)

$$\left| \int_0^t \int_\Omega h \left(\frac{\partial u}{\partial t} - \Delta u \right)\ dx d\tau \right| \leq C(\varepsilon) \int_0^t \|h\|_2^2\ d\tau +$$

[1]From definitions of these spaces is easy to see that $S^1_{1,p-2,2}(\Omega) \subset S_{1,(p-2)q,q}(\Omega)$

$$\varepsilon\int_0^t \left\|\frac{\partial u}{\partial t}\right\|_2^2 \ d\tau + C\left(\varepsilon_1\right)\int_0^t \|h\|_{W_q^1}^q \ d\tau + \varepsilon_1\sum_{i=1}^n\int_0^t \|D_i u\|_p^p \ d\tau. \qquad (3.5)$$

It is not difficult to see mapping g defined by problem (3.2)-(3.3) satisfies of the conditions of Theorem 2.1, i.e. $g\left(X_1\right)$ contains an everywhere dense linear manifold of $L_p\left(Q_T\right)$ and g^{-1} is weakly compact operator from $L_p\left(Q_T\right)$ to $\mathcal{M}_0 \equiv L_p\left(0,T;\overset{0}{S}{}^1_{1,(p-2)q,q}\left(\Omega\right)\right) \cap W_2^1\left(0,T;L_2\left(\Omega\right)\right)$.

Thus we have that all conditions of Theorem 2.1 and Corollary 2.1 are fulfilled for the mappings and spaces corresponding to the studied problem. Consequently, using Theorem 2.1 and Corollary 2.1 we obtain the solvability of problem (1.6)-(1.7) in the space $P^1_{1,(p-2)q,q,2}\left(Q_T\right)$ for any $h \in L_2\left(0,T;W_2^1\left(\Omega\right)\right)$ and $u_0 \in W_p^1\left(\Omega\right)$.

Furthermore, from here it follows that the solution of this problem possesses the complementary smoothness, i.e. $\sum_{i=1}^n D_i\left(\left|D_i u\right|^{p-2} D_i u\right) \in L_2\left(Q\right)$ as far as we have $\frac{\partial u}{\partial t} \in L_2\left(Q_T\right)$ and $h \in L_2\left(0,T;W_2^1\left(\Omega\right)\right)$ by virtue of the conditions of the considered problem.

So the following result is proved.

Theorem 3.1. *Let $u_0 \in W_p^1\left(\Omega\right)$, $h \in L_2\left(0,T;W_2^1\left(\Omega\right)\right)$ and $p > 2$, then problem (1.6) - (1.7) is solvable in $P\left(Q_T\right)$ where*

$$P\left(Q_T\right) \equiv L_p\left(0,T;\overset{0}{\widetilde{S}}{}^1_{1,p-2,2}\left(\Omega\right)\right) \cap W_2^1\left(0,T;L_2\left(\Omega\right)\right) \cap P^1_{1,(p-2)q,q,2}\left(Q_T\right).$$

4. A priori Estimations for Solutions of Problem (1.4) - (1.5)

Now, we can investigate the main problem of this article, which is posed for problem (1.4) - (1.5). We introduce denotations of the mappings A and f that are generated by problems (1.4)-(1.5) and (1.6)-(1.7), respectively.

Theorem 4.1. *Let $u_0, u_1 \in W_p^1\left(\Omega\right)$, $h \in L_p\left(0,T;W_p^1\left(\Omega\right)\right)$ and $p > 2$. Then any solution $u\left(t,x\right)$ of problem (1.4) -(1.5) belongs to the bounded subset of the function class $\widetilde{P}\left(Q_T\right)$ defined in the form*

$$u \in L^\infty\left(0,T;W_p^1\left(\Omega\right)\right); \quad \frac{\partial u}{\partial t} \in L^\infty\left(0,T;L_2\left(\Omega\right)\right);$$

$$\sum_{i=1}^n\int_0^t \left|D_i u\right|^{p-2} D_i u d\tau \in W_\infty^1\left(0,T;L_q\left(\Omega\right)\right) \cap L^\infty\left(0,T;W_2^1\left(\Omega\right)\right) \qquad (4.1)$$

that satisfies the conditions determined by the dates of problem (1.4)-(1.5).

Proof. Consider the dual form $\left\langle A\left(u\right), f\left(u\right)\right\rangle$ for any $u \in P\left(Q_T\right)$ that is defined by virtue of Theorem 3.1. We behave as in proof of Proposition 3.1 and consider only the integral with respect to x. Then we have after certain known acts

$$\int_\Omega \frac{\partial u}{\partial t}\frac{\partial u}{\partial t}\,dx + \int_\Omega \left(\int_0^t\sum_{i=1}^n D_i\left(\left|D_i u\right|^{p-2} D_i u\right) d\tau\right)\left(\sum_{j=1}^n D_j\left(\left|D_j u\right|^{p-2} D_j u\right)\right) dx -$$

$$\int\limits_{\Omega}\left(\int\limits_0^t\sum_{i=1}^n D_i\left(|D_iu|^{p-2}D_iu\right)d\tau\right)\frac{\partial u}{\partial t}dx-\int\limits_{\Omega}\frac{\partial u}{\partial t}\left(\sum_{i=1}^n D_i\left(|D_iu|^{p-2}D_iu\right)\right)dx\geq$$

$$\frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|_{L_2(\Omega)}^2+\frac{1}{2}\frac{\partial}{\partial t}\left\|\int\limits_0^t\sum_{i=1}^n D_i\left(|D_iu|^{p-2}D_iu\right)d\tau\right\|_{L_2(\Omega)}^2+$$

$$\frac{1}{p}\frac{\partial}{\partial t}\sum_{i=1}^n\|D_iu\|_{L_p(\Omega)}^p-\frac{1}{2}\left\|\int\limits_0^t\sum_{i=1}^n D_i\left(|D_iu|^{p-2}D_iu\right)d\tau\right\|_{L_2(\Omega)}^2(t) \tag{4.2}$$

Now, consider the right part of the dual form, i.e. $\langle H,f\rangle$, for the determination of the bounded subset, to which the solutions of the problem belongs (and for the receiving of the a priori estimations). Then we get

$$\left|\int\limits_{\Omega}H\ \frac{\partial u}{\partial t}\ dx-\int\limits_{\Omega}H\sum_{j=1}^n D_j\left(|D_ju|^{p-2}D_ju\right)\ dx\right|\leq C(\varepsilon)\|H\|_{L_2(Q)}^2+$$

$$\varepsilon\left\|\frac{\partial u}{\partial t}\right\|_{L_2(\Omega)}^2(t)+C(\varepsilon_1)\|H\|_{L_p\left(W_p^1\right)}^p+\varepsilon_1\sum_{j=1}^n\|D_ju\|_{L_p(\Omega)}^p(t). \tag{4.3}$$

From (4.2) and (4.3) it follows

$$0=\int\limits_{\Omega}(A(u)-H)\ f(u)\ dx\geq\frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|_{L_2(\Omega)}^2+\frac{1}{p}\frac{\partial}{\partial t}\sum_{j=1}^n\|D_ju\|_{L_p(\Omega)}^p+$$

$$\frac{1}{2}\frac{\partial}{\partial t}\left\|\int\limits_0^t\sum_{i=1}^n D_i\left(|D_iu|^{p-2}D_iu\right)d\tau\right\|_{L_2(\Omega)}^2-\frac{1}{2}\left\|\int\limits_0^t\sum_{i=1}^n D_i\left(|D_iu|^{p-2}D_iu\right)d\tau\right\|_{L_2(\Omega)}^2-$$

$$\varepsilon_1\sum_{i=1}^n\|D_iu\|_{L_p(\Omega)}^p-C(\varepsilon)\|H\|_{L_2(Q)}^2-\varepsilon\left\|\frac{\partial u}{\partial t}\right\|_{L_2(\Omega)}^2-C(\varepsilon_1)\|H\|_{L_p\left(W_p^1\right)}^p$$

or if we choose small parameters $\varepsilon>0$ and $\varepsilon_1>0$ such as needed, then we have

$$c\left\|\frac{\partial u}{\partial t}\right\|_{L_2(\Omega)}^2+\frac{1}{p}\frac{\partial}{\partial t}\sum_{i=1}^n\|D_iu\|_{L_p(\Omega)}^p+$$

$$\frac{1}{2}\frac{\partial}{\partial t}\left\|\int\limits_0^t\sum_{i=1}^n D_i\left(|D_iu|^{p-2}D_iu\right)d\tau\right\|_{L_2(\Omega)}^2\leq C\left(\|H\|_{L_2(Q)},\|H\|_{L_p\left(W_p^1\right)}\right)+$$

$$\frac{1}{p}\sum_{i=1}^n\|D_iu\|_{L_p(\Omega)}^p+\frac{1}{2}\left\|\int\limits_0^t\sum_{i=1}^n D_i\left(|D_iu|^{p-2}D_iu\right)d\tau\right\|_{L_2(\Omega)}^2. \tag{4.4}$$

Inequality (4.4) show that we can use Gronwall lemma. Consequently using Gronwall lemma we get

$$\sum_{i=1}^{n}\left\|D_iu\right\|_{L_p(\Omega)}^{p}(t)+\left\|\int\limits_0^t\sum_{i=1}^{n}D_i\left(\left|D_iu\right|^{p-2}D_iu\right)d\tau\right\|_{L_2(\Omega)}^2(t)\leq$$

$$C\left(\left\|H\right\|_{L_2(Q)},\left\|H\right\|_{L_p\left(W_p^1\right)},\left\|u_0\right\|_{W_p^1(\Omega)}\right)\tag{4.5}$$

holds for a.e. $t\in[0,T]$.

Thus we have for any solution of problem (1.4) -(1.5) the following estimations

$$\left\|u\right\|_{W_p^1(\Omega)}(t)\leq C\left(\left\|H\right\|_{L_p\left(W_p^1\right)},\left\|u_0\right\|_{W_p^1(\Omega)}\right),$$

$$\left\|\int\limits_0^t\sum_{i=1}^{n}D_i\left(\left|D_iu\right|^{p-2}D_iu\right)d\tau\right\|_{L_2(\Omega)}(t)\leq C\left(\left\|H\right\|_{L_p\left(W_p^1\right)},\left\|u_0\right\|_{W_p^1(\Omega)}\right),$$

$$\left\|\frac{\partial u}{\partial t}\right\|_{L_2(\Omega)}(t)\leq C\left(\left\|H\right\|_{L_p\left(W_p^1\right)},\left\|u_0\right\|_{W_p^1(\Omega)}\right)$$

hold for a.e. $t\in[0,T]$ by virtue of inequalities (4.2) - (4.5). In other words we have that any solution of problem (1.4) -(1.5) belongs to the bounded subset of the following class

$$u\in L^\infty\left(0,T;W_p^1(\Omega)\right);\quad\frac{\partial u}{\partial t}\in L^\infty\left(0,T;L_2(\Omega)\right);$$

$$\frac{\partial}{\partial t}\left(\sum_{i=1}^{n}\int\limits_0^t\left|D_iu\right|^{p-2}D_iud\tau\right)\in L^\infty\left(0,T;L_q(\Omega)\right)$$

$$\int\limits_0^t\sum_{i=1}^{n}D_i\left(\left|D_iu\right|^{p-2}D_iu\right)d\tau\in L^\infty\left(0,T;L_2(\Omega)\right),\tag{4.6}$$

for each given $u_0,u_1\in W_p^1(\Omega)$, $h\in L_p\left(0,T;W_p^1(\Omega)\right)$.

From here it follows that all solutions of this problem belong to a bounded subset of space $P(Q_T)$, which is defined by (4.1).

Indeed, firstly it is easy to see that the following inequality holds

$$\left\|\sum_{i=1}^{n}\int\limits_0^t\left|D_iu\right|^{p-2}D_iud\tau\right\|_{L_q(\Omega)}^q\leq C\sum_{i=1}^{n}\left\|\left|D_iu\right|^{p-2}D_iu\right\|_{L_q(\Omega)}^q\leq$$

$$C\left(T,mes\ \Omega\right)\left\|u\right\|_{W_p^1(\Omega)}^q(t)\Longrightarrow\int\limits_0^t\sum_{i=1}^{n}\left|D_iu\right|^{p-2}D_iud\tau\in L^\infty\left(0,T;L_q(\Omega)\right),$$

and secondary, the following equalities are correct

$$\int\limits_\Omega\left(\int\limits_0^t\sum_{i=1}^{n}D_i\left(\left|D_iu\right|^{p-2}D_iu\right)d\tau\right)^2dx\equiv\left\|\int\limits_0^t\sum_{i=1}^{n}D_i\left(\left|D_iu\right|^{p-2}D_iu\right)d\tau\right\|_2^2\equiv$$

$$\left\langle \int_0^t \sum_{i=1}^n D_i \left(|D_i u|^{p-2} D_i u \right) d\tau, \int_0^t \sum_{j=1}^n D_j \left(|D_j u|^{p-2} D_j u \right) d\tau \right\rangle =$$

$$\sum_{i,j=1}^n \left\langle \int_0^t D_j \left(|D_i u|^{p-2} D_i u \right) d\tau, \int_0^t D_i \left(|D_j u|^{p-2} D_j u \right) d\tau \right\rangle =$$

$$\sum_{i,j=1}^n \left\langle D_j \int_0^t |D_i u|^{p-2} D_i u d\tau, D_i \int_0^t |D_j u|^{p-2} D_j u d\tau \right\rangle ,$$

and also

$$\sum_{j=1}^n \left\| D_j \int_0^t \sum_{i=1}^n \left(|D_i u|^{p-2} D_i u \right) d\tau \right\|_2^2 =$$

$$\sum_{j=1}^n \left\langle D_j \int_0^t \sum_{i=1}^n |D_i u|^{p-2} D_i u d\tau, D_j \int_0^t \sum_{i=1}^n |D_i u|^{p-2} D_i u d\tau \right\rangle .$$

These demonstrate that the function

$$v(t,x) \equiv \int_0^t \sum_{i=1}^n |D_i u|^{p-2} D_i u \ d\tau$$

belongs to a bounded subset of the space

$$L^\infty(0,T; L_q(\Omega)) \cap \{ v(t,x) \mid Dv \in L^\infty(0,T; L_2(\Omega)) \} .$$

Therefore, in order to prove the correctness of (4.1), it remains to use the following inequality, i.e. the Nirenberg-Gagliardo-Sobolev inequality ([6, 13])

$$\left\| D^\beta v \right\|_{p_2} \leq C \left(\sum_{|\alpha|=m} \| D^\alpha v \|_{p_0}^\theta \right) \| v \|_{p_1}^{1-\theta}, \quad 0 \leq |\beta| = l \leq m-1, \qquad (4.7)$$

which holds for each $v \in W_{p_0}^m(\Omega)$, $\Omega \subset R^n$, $n \geq 1$, $C \equiv C(p_0, p_1, p_2, l, s)$ and θ such that $\frac{1}{p_2} - \frac{l}{n} = (1-\theta)\frac{1}{p_1} + \theta\left(\frac{1}{p_0} - \frac{m}{n}\right)$. Really, in inequality (4.7) for us it is enough to choose $p_2 = 2$, $l = 0$, $p_1 = q$, $p_0 = 2$ then we get

$$\frac{1}{2} = (1-\theta)\frac{p-1}{p} + \theta\left(\frac{1}{2} - \frac{1}{n}\right) \Longrightarrow \theta\left(\frac{1}{2} - \frac{1}{n} - \frac{p-1}{p}\right) = \frac{1}{2} - \frac{p-1}{p} \Longrightarrow$$

$\theta = \frac{n(p-2)}{n(p-2)+2p}$ for $p > 2$, and so (4.1) is correct. □

Now we can prove of Theorem 1.1.

Proof. (**of Theorem 1.1**) From (4.1) it follows

$$\sum_{i=1}^n \int_0^t |D_i u|^{p-2} D_i u d\tau \in L^\infty\left(0,T; \overset{0}{W}{}^1_2(\Omega)\right) \cap W^1_\infty(0,T; L_q(\Omega)),$$

moreover

$$\int_0^t \sum_{i,j=1}^n D_j \left(|D_i u|^{p-2} D_i u \right) d\tau \in L^\infty \left(0, T; L_2\left(\Omega\right)\right),$$

and is bounded in this space. Then taking into account the property of the Lebesgue integrals we obtain

$$\int_0^t \left\{ \int_\Omega \left[\sum_{i,j=1}^n D_j \left(|D_i u|^{p-2} D_i u \right) \right]^2 dx \right\}^{\frac{1}{2}} d\tau \leq C, \quad C \neq C\left(t\right)$$

from which we get

$$\sum_{i,j=1}^n D_j \left(|D_i u|^{p-2} D_i u \right) \in L_1 \left(0, T; L_2\left(\Omega\right)\right),$$

and so

$$\sum_{i=1}^n D_i \left(|D_i u|^{p-2} D_i u \right) \in L_1 \left(0, T; L_2\left(\Omega\right)\right) \tag{4.8}$$

in which is bounded.

If we consider equation (1.1), and take into account that it is solvable in the generalized sense and $\frac{\partial u}{\partial t} \in W^1_\infty \left(0, T; L_2\left(\Omega\right)\right)$ (by (4.1)) then from Definition 1.1 it follows that

$$\left[\frac{\partial^2 u}{\partial t^2}, v \right] - \left[\sum_{i=1}^n D_i \left(|D_i u|^{p-2} D_i u \right), v \right] = [h, v]$$

holds for any $v \in W^1_{\widetilde{p}} \left(0, T; L_2\left(\Omega\right)\right)$, $\widetilde{p} > 1$.

Hence

$$\left[\frac{\partial^2 u}{\partial t^2}, v \right] = \left[\sum_{i=1}^n D_i \left(|D_i u|^{p-2} D_i u \right) + h, v \right] \tag{4.9}$$

holds for any $v \in L^\infty \left(Q_T\right)$.

Thus we obtain $\frac{\partial^2 u}{\partial t^2} \in L_1 \left(0, T; L_2\left(\Omega\right)\right)$ by virtue of (4.1), (4.8) and as

$$\sum_{i=1}^n D_i \left(|D_i u|^{p-2} D_i u \right) + h \in L_1 \left(0, T; L_2\left(\Omega\right)\right).$$

□

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