Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 41, Number 2, 2015, Pages 63–69

PARA-NORDENIAN STRUCTURES ON THE COTANGENT BUNDLE WITH RESPECT TO THE CHEEGER-GROMOLL METRIC

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Abstract. In this paper we investigate para-Nordenian properties of the Cheeger-Gromoll metrics in the cotangent bundle..

1. Introduction

Cheeger-Gromoll metric was defined by Cheeger and Gromoll in [2] and the explicit formula for this metric was given by Musso and Tricerri in [9]. The Levi-Civita connection of ${}^{CG}g$ and its Riemannian curvature tensor are callculated by Sekizawa in [16] and corrected by Gudmundsson and Kappos in [5]. In [11] Salimov and Akbulut studied a paraholomorphic Cheeger-Gromoll metric on the tangent bundle. The similar metric in theoritical physics has been obtained by Tamm (Nobel Laureate in Physics for the year 1958, see [17]). The geometry of the tangent bundle equipped with Cheeger-Gromoll metric is well known and intensively studied (see for example [4], [5], [8], [12], [15]). In [1] Ağca and Salimov investigate curvature properties and geodesics on the cotangent bundle with respect to the Cheeger-Gromoll metric.

The tangent bundles of differentiable manifolds are very important in many areas of mathematics and physics. Cotangent bundle is dual of the tangent bundle. Because of this duality, some of the geometric results are similar to each other. The most significant difference between them is construction of lifts (see [20] for more details). In our previous paper [10], we have given the outline of the theory of para-Nordenian structures with respect to the Sasakian metric in the cotangent bundle. The present paper is concerned with para-Nordenian property of the Cheeger-Gromoll metric in the cotangent bundle.

Let (M^n, g) be an n-dimensional Riemannian manifold T^*M^n be the cotangent bundle of M^n and π the natural projection $T^*M^n \to M^n$. A system of local coordinates $(U, x^i), i = 1, ..., n$ in M^n induces on T^*M^n a system of local coordinates $(\pi^{-1}(U), x^i, x^{\bar{i}} = p_i)$, $\bar{i} := n + i = n + 1, ..., 2n$, where $x^{\bar{i}} = p_i$ are the components of the covector p in each cotangent space $T^*_x M^n$, $x \in U$ with respect to the natural coframe $\{dx^i\}$, $i = 1, ..., n$.

We denote by $\Im^r_s(M^n)$ the set of all tensor fields of type (r, s) on M^n and by

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2010 *Mathematics Subject Classification.* 53C07, 53C25, 53C55.

Key words and phrases. Cheeger-Gromoll metric, cotangent bundle, vertical and horizontal lift, almost paracomplex structure, para-Norden metric.

$\Im_s^r(T^*M^n)$ the corresponding set on the cotangent bundle T^*M^n. During this paper, manifolds, tensor fields and connections are always supposed to be differentiable of class C^∞.

The local expressions a vector and a covector (1-form) field $X \in \Im_0^1(M^n)$ and $\omega \in \Im_1^0(M^n)$ are $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ in $U \subset M^n$, respectively. Then the complete and horizontal lifts ${}^CX, {}^HX \in \Im_0^1(T^*M^n)$ of $X \in \Im_0^1(M^n)$ and the vertical lift ${}^V\omega \in \Im_0^1(T^*M^n)$ of $\omega \in \Im_1^0(M^n)$ are given, with respect to the natural frame $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{i}}}\}$, by

$$ {}^CX = X^i \frac{\partial}{\partial x^i} - \sum_i p_h \partial_i X^h \frac{\partial}{\partial x^{\bar{i}}}, \tag{1.1} $$

$$ {}^HX = X^i \frac{\partial}{\partial x^i} + \sum_i p_h \Gamma_{ij}^h X^j \frac{\partial}{\partial x^{\bar{i}}}, \tag{1.2} $$

$$ {}^V\omega = \sum_i \omega_i \frac{\partial}{\partial x^{\bar{i}}} \tag{1.3} $$

where Γ_{ij}^h are the components of the Levi-Civita connection ∇_g on M^n (see [20] for more details).

Let (M^n, g) be an n-dimensional Riemannian manifold and denote by $r^2 = g^{-1}(p,p) = g^{ij}p_ip_j$. Then the Cheeger-Gromoll metric ${}^{CG}g$ is defined on T^*M^n by the following three equations

$$ {}^{CG}g\big({}^HX, {}^HY\big) = {}^V\big(g(X,Y)\big) = g(X,Y) \circ \pi, \tag{1.4} $$

$$ {}^{CG}g\big({}^V\omega, {}^HY\big) = 0, \tag{1.5} $$

$$ {}^{CG}g\big({}^V\omega, {}^V\theta\big) = \frac{1}{1+r^2}\big(g^{-1}(\omega,\theta) + g^{-1}(\omega,p)g^{-1}(\theta,p)\big) \tag{1.6} $$

for any $X, Y \in \Im_0^1(M^n)$ and $\omega, \theta \in \Im_1^0(M^n)$ [1].

2. Levi-Civita connection of ${}^{CG}g$

In each local chart $U \subset M^n$, we put $X_{(i)} = \frac{\partial}{\partial x^i}, \quad \theta^{(i)} = dx^i, i = 1, ..., n$. By virtue of (1.2) and (1.3) we see that

$$ \tilde{e}_{(i)} = {}^HX_{(i)} = \frac{\partial}{\partial x^i} + \sum_h p_a \Gamma_{hi}^a \frac{\partial}{\partial x^{\bar{h}}}, \tag{2.1} $$

$$ \tilde{e}_{(\bar{i})} = {}^V\theta^{(i)} = \frac{\partial}{\partial x^{\bar{i}}}. \tag{2.2} $$

This set $\{\tilde{e}_{(\alpha)}\} = \{\tilde{e}_{(i)}, \tilde{e}_{(\bar{i})}\} = \{{}^HX_{(i)}, {}^V\theta^{(i)}\}$ is called the frame adapted to the connection ∇_g in $\pi^{-1}(U) \subset T^*M^n$.

Using (1.2), (1.3), (2.1) and (2.2), we have

$$ {}^HX = X^i \tilde{e}_{(i)}, \quad {}^HX = ({}^HX^\alpha) = \begin{pmatrix} X^i \\ 0 \end{pmatrix}, \tag{2.3} $$

$$ {}^V\omega = \sum_i \omega_i \tilde{e}_{(\bar{i})}, \quad {}^V\omega = ({}^V\omega^\alpha) = \begin{pmatrix} 0 \\ \omega_i \end{pmatrix} \tag{2.4} $$

with respect to the adapted frame $\{\tilde{e}_{(\alpha)}\}$, where X^i and ω_i being local components of $X \in \Im_0^1(M^n)$ and $\omega \in \Im_1^0(M^n)$, respectively (see [20] for more details).

From (2.2), (2.3) and (2.4), we have

$$\begin{aligned}
{}^{CG}g_{ij} = {}^{CG}g\big(\tilde{e}_{(i)}, \tilde{e}_{(j)}\big) &= {}^{V}\big(g(\partial_i, \partial_j)\big) = g_{ij}, \\
{}^{CG}g_{\bar{i}j} = {}^{CG}g\big(\tilde{e}_{(\bar{i})}, \tilde{e}_{(j)}\big) &= 0, \\
{}^{CG}g_{\bar{i}\bar{j}} = {}^{CG}g\big(\tilde{e}_{(\bar{i})}, \tilde{e}_{(\bar{j})}\big) &= \frac{1}{1+r^2}\big(g^{-1}(dx^i, dx^j) + g^{-1}(dx^i, p_k)g^{-1}(dx^j, p_l)\big) \\
&= \frac{1}{1+r^2}\big(g^{ij} + g^{ik}g^{lj}p_k p_l\big),
\end{aligned}$$

i.e. ${}^{CG}g$ has components

$${}^{CG}g = \begin{pmatrix} g_{ij} & 0 \\ 0 & \frac{1}{1+r^2}\big(g^{ij} + g^{ik}g^{lj}p_k p_l\big) \end{pmatrix} \tag{2.5}$$

with respect to the adapted frame $\{\tilde{e}_{(\alpha)}\}$ [1].

For the Levi-Civita connection of the Cheeger-Gromoll metric we have the following (see [1])

Theorem 2.1 *Let M^n be a Riemannian manifold with metric g and ${}^{CG}\nabla$ be the Levi-Civita connection of the cotangent bundle T^*M^n equipped with the Cheeger-Gromoll metric ${}^{CG}g$. Then ${}^{CG}\nabla$ satisfies*

$$\begin{aligned}
&i) \quad {}^{CG}\nabla_{{}^HX}{}^HY = {}^H(\nabla_X Y) + \frac{1}{2}{}^V(pR(X,Y), \\
&ii) \quad {}^{CG}\nabla_{{}^HX}{}^V\omega = {}^V(\nabla_X \omega) + \frac{1}{2\alpha}{}^H(p(g^{-1}\circ R(\ ,X)\tilde{\omega}), \\
&iii) \quad {}^{CG}\nabla_{{}^V\omega}{}^HY = \frac{1}{2\alpha}{}^H(p(g^{-1}\circ R(\ ,Y)\tilde{\omega}), \\
&iv) \quad {}^{CG}\nabla_{{}^V\omega}{}^V\theta = -\frac{1}{\alpha}\big({}^{CG}g({}^V\omega, \gamma\delta){}^V\theta + {}^{CG}g({}^V\theta, \gamma\delta){}^V\omega\big) + \frac{\alpha+1}{\alpha}{}^{CG}g({}^V\omega, {}^V\theta)\gamma\delta \\
&\qquad - \frac{1}{\alpha}{}^{CG}g({}^V\omega, \gamma\delta){}^{CG}g({}^V\theta, \gamma\delta)\gamma\delta
\end{aligned} \tag{2.6}$$

for all $X, Y \in \Im_0^1(M^n)$, $\omega, \theta \in \Im_1^0(M^n)$, where $\tilde{\omega} = g^{-1}\circ\omega \in \Im_0^1(M^n)$, $R(\ ,X)\tilde{\omega} \in \Im_1^1(M^n)$, $g^{-1}\circ R(\ ,X)\tilde{\omega} \in \Im_0^2(M^n)$, $\alpha = 1+r^2$, R and $\gamma\delta$ denotes respectively the curvature tensor of ∇ and the canonical vertical vector field on T^*M^n with expression $\gamma\delta = p_i e_{(\bar{i})}$.

3. Para-Nordenian structures on $(T^*M^n, {}^{CG}g)$

In [3], an almost paracomplex manifold is an almost product manifold (M^n, φ), $\varphi^2 = I$, such that the two eigenbundles T^+M^n and T^-M^n associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank. It is well known that the dimension of an almost paracomplex manifold is necessarily even.

The purity conditions for a tensor field $\omega \in \Im_q^0(M^{2n})$ with respect to the paracomplex structure φ given by

$$\omega(\varphi X_1, X_2, ..., X_q) = \omega(X_1, \varphi X_2, ..., X_q) = \omega(X_1, X_2, ..., \varphi X_q) \tag{3.1}$$

for any $X_1, X_2, ..., X_q \in \Im_0^1(M^{2n})$ [14].

In [19], an operator $\phi_\varphi : \Im_q^0(M^{2n}) \to \Im_{q+1}^0(M^{2n})$ joined with φ and applied to the pure tensor field ω, given by

$$\begin{aligned}(\phi_\varphi\omega)(Y, X_1, X_2, ..., X_q) &= (\varphi Y)(\omega(X_1, X_2, ..., X_q)) - Y(\omega(\varphi X_1, X_2, ..., X_q)) \\ &+\omega((L_{X_1}\varphi)Y, X_2, ..., X_q) + ... + \omega(X_1, X_2, ..., (L_{X_q}\varphi)Y). \end{aligned} \tag{3.2}$$

where L_X denotes the Lie derivative with respect to X.

If $\phi_\varphi\omega$ vanishes, then ω is said to be almost para-holomorphic with respect to the paracomplex algebra $R(j)$ (see [6], [14]).

Let M^{2n} be an almost paracomplex manifold with the structure φ. A Riemannian metric g is a para-Norden metric (B-metric) if

$$g(\varphi X, \varphi Y) = g(X, Y)$$

or equivalently

$$g(\varphi X, Y) = g(X, \varphi Y)$$

for any $X, Y \in \Im_0^1(M^{2n})$. If (M^{2n}, φ) is an almost paracomplex manifold with a para-Norden metric g, then (M^{2n}, φ, g) is an almost paracomplex Norden manifold [14, 18]. If φ is integrable, we say that (M^{2n}, φ, g) is a paracomplex Norden manifold [12].

In [14], Salimov and his collaborators showed that for an almost paracomplex manifold with Norden metric , the condition $\phi_\varphi g = 0$ is equivalent to $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection of g.

If the Riemannian metric g is a pure with respect to an almost complex structure φ, then a Riemannian manifold (M^{2n}, g) with φ, is said to be almost para-Nordenian. Its known that, the almost para-Nordenian manifold is para-Kahler $(\nabla_g\varphi = 0)$ if and only if g is paraholomorphic $(\phi_\varphi g = 0)$ (see [12]).

We investigate para-Nordenian properties with using an almost paracomplex structure on T^*M^n as follows:

$$\begin{cases} F^H X = -^H X, \\ F^V\omega = {}^V\omega \end{cases} \tag{3.3}$$

for any $X \in \Im_0^1(M^n)$ and $\omega \in \Im_1^0(M^n)$ which defined by Cruceanu and his collaborators in [3]. Then we obtain $F^2 = I$.

We put

$$S(\tilde{X}, \tilde{Y}) = ^{CG} g(F\tilde{X}, \tilde{Y}) - ^{CG} g(\tilde{X}, F\tilde{Y})$$

If $S(\tilde{X}, \tilde{Y}) = 0$ for all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form ${}^V\omega$, ${}^V\theta$ or ${}^HX$, ${}^HY$, then $S = 0$. Then from (3.3), (1.4), (1.5) and (1.6), we have

$$\begin{aligned}
&S({}^V\omega, {}^V\theta) = ^{CG} g(F^V\omega, {}^V\theta) - ^{CG} g({}^V\omega, F^V\theta) = 0, \\
&S({}^V\omega, {}^HX) = ^{CG} g({}^V\omega, {}^HX) - ^{CG} g({}^V\omega, -^HX) = 2^{CG}g({}^V\omega, {}^HX) = 0, \\
&S({}^HX, {}^V\theta) = ^{CG} g(-^HX, {}^V\theta) - ^{CG} g({}^HX, {}^V\theta) = -2^{CG}g({}^V\omega, {}^HX) = 0, \\
&S({}^HX, {}^HY) = ^{CG} g(-^HX, {}^HY) - ^{CG} g({}^HX, -^HY) = 0,
\end{aligned}$$

i.e. ${}^{CG}g$ is pure metric with respect to F. Then we have

Theorem 3.1 *Let (M^n, g) be a Riemannian manifold and $(T^*M^n, {}^{CG}g)$ be its cotangent bundle. Then $(T^*M^n, {}^{CG}g)$ with almost paracomplex structure F is an* *almost paracomplex Norden manifold.*

The Nijenhuis tensor is given by the following equation

$$N_{\varphi}(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + [X,Y]. \tag{3.4}$$

It is well known that the integrability of the almost paracomplex structure is equivalent to the vanishing of the Nijenhuis tensor.

In [20, p.238, p.277], the following formulas was given

$$\begin{array}{ll} i)^{V}\omega^{V}f = 0, & ii)^{H}X^{V}f =^{V}(Xf), \\ iii)[^{H}X,^{V}\omega] =^{V}(\nabla_X\omega), & iv)[^{V}\omega,^{V}\theta] = 0, \\ v)[^{H}X,^{H}Y] =^{H}[X,Y] + \gamma R(X,Y) =^{H}[X,Y] +^{V}(pR(X,Y)) & \end{array} \tag{3.5}$$

for any $X, Y \in \Im_0^1(M^n)$, $\omega, \theta \in \Im_1^0(M^n)$, where $pR(X,Y) = (p_i(R(X,Y)_j^i))$.

Then using (3.5), we have

$$N_F(^{H}X,^{H}Y) = 4^{V}(pR(X,Y))$$

and the other is zero. Therefore, we have the following result.

Theorem 3.2 *Let (M^n, g) be a Riemannian manifold and $(T^*M^n, {}^{CG}g)$ be its cotangent bundle. Then the almost paracomplex structure F is integrable if and only if M^n is locally flat.*

Now we investigate paraholomorphy property for the Cheeger-Gromoll metric with respect to the almost paracomplex structure F defined by (3.3). From (1.4), (1.5), (1.6), (3.1), (3.3) and (3.5) we have

$$\begin{aligned} (\phi_F{}^{CG}g)(^{H}X,^{H}Y,^{V}\omega) &= -2^{CG}g(^{V}(pR(X,Y)),^{V}\omega), \\ (\phi_F{}^{CG}g)(^{H}X,^{V}\omega,^{H}Y) &= 2^{CG}g(^{V}\omega,^{V}(pR(X,Y))), \end{aligned} \tag{3.6}$$

and the others is zero. Therefore we have

Theorem 3.3 *The triple $(T^*M^n, F, {}^{CG}g)$ is paraholomorphic Norden (para-Kahler-Norden) manifold if and only if M^n is flat.*

Remark. Let $J \in \Im_1^1(M^n)$. We define a tensor field ${}^{D}J$ of type (1,1) in T^*M^n by

$${}^{D}J^{H}X =^{H}(FX), \qquad {}^{D}J^{V}\omega = -^{V}(\omega \circ J) = -^{V}(\omega J) \tag{3.7}$$

for any $X \in \Im_0^1(M^n)$ and $\omega \in \Im_1^0(M^n)$. The diagonal lift ${}^{D}I$ of identity tensor field I of type (1,1) has the components

$${}^{D}I = \begin{pmatrix} \delta_j^i & 0 \\ 2p_a\Gamma_{ij}^a & -\delta_j^i \end{pmatrix} \tag{3.8}$$

with respect to the induced coordinates and satisfies ${}^{D}I^2 = I$ [20]. Thus ${}^{D}I$ is an almost paracomplex structure.

We note that the Cheeger-Gromoll metric ${}^{CG}g$ is pure with respect to the almost paracomplex structure ${}^{D}I$. Therefore, we have the following

Theorem 3.4 *Let (M^n, g) be a Riemannian manifold and $(T^*M^n, {}^{CG}g)$ be its cotangent bundle. The triple $(T^*M^n, {}^{D}I, {}^{CG}g)$ is an almost paracomplex Norden manifold.*

Using (1.4), (1.5), (1.6), (3.1), (3.3) and (3.5), we get

$$\begin{aligned} (\phi_{{}^{D}I}{}^{CG}g)(^{H}X,^{H}Y,^{V}\omega) &= 2^{CG}g(^{V}(pR(X,Y)),^{V}\omega), \\ (\phi_{{}^{D}I}{}^{CG}g)(^{H}X,^{V}\omega,^{H}Y) &= -2^{CG}g(^{V}\omega,^{V}(pR(X,Y))), \end{aligned}$$

and the other is zero. Thus, we have the following

Theorem 3.5 *Let (M^n, g) be a Riemannian manifold and $(T^*M^n, {}^{CG}g)$ be its cotangent bundle. The almost paracomplex Riemannian manifold $(T^*M^n, {}^DI, {}^{CG}g)$ is paraholomorphic Norden if and only if M^n is flat.*

If $L_X g = 0$ ($L_X\nabla_g = 0$), then a vector field $X \in \Im_0^1(M^n)$ is called Killing vector field (infinitesimal affine transformation) where L_X is the Lie derivative.

Given a vector field $\tilde{X} \in \Im_0^1(T^*M^n)$ and the almost para-Nordenian structure F, if $L_{\tilde{X}}F = 0$, then $\tilde{X}$ is said to be almost paraholomorphic [16].

It is well known that [20, p.277]

$$\begin{cases} [{}^CX, {}^HY] = {}^H[X,Y] + {}^V(p(L_X\nabla)Y), \\ [{}^CX, {}^V\omega] = {}^V(L_X\omega), \end{cases} \tag{3.9}$$

where $(L_X\nabla)Y = \nabla_Y\nabla X + R(X,Y)$ and $(L_X\nabla)(Y,Z) = L_X(\nabla_Y X) - \nabla_Y(L_X Z) - \nabla_{[X,Y]}Z$.

We now consider the Lie derivative of F with respect to the complete lift ${}^CX$. Using (3.3) and (3.8), we get

$$\begin{aligned} (L_{{}^CX}F)^V\theta &= L_{{}^CX}F^V\theta - F(L_{{}^CX}{}^V\theta) = L_{{}^CX}{}^V\theta - F({}^V(L_X\theta)) \\ &= {}^V(L_X\theta) - {}^V(L_X\theta) = 0, \end{aligned} \tag{3.10}$$

$$\begin{aligned} (L_{{}^CX}F)^HY &= L_{{}^CX}F^HY - F(L_{{}^CX}{}^HY) \\ &= -{}^H[X,Y] - {}^V(p(L_X\nabla)Y) - F\big({}^H[X,Y] + {}^V(p(L_X\nabla)Y)\big) \\ &= -2{}^V(p(L_X\nabla)Y). \end{aligned} \tag{3.11}$$

Thus, we have the following

Theorem 3.6 *If X is an infinitesimal afine transformation ($L_X\nabla = 0$) of M^n, then its complete lift ${}^CX \in \Im_0^1(T^*M^n)$ is an almost paraholomorphic vector field with respect to the almost paracomplex structure $(F, {}^{CG}g)$.*

Given a non-integrable almost paracomplex manifold (M^{2n}, φ, g) with a Norden metric. An almost paracomplex Norden manifold (M^{2n}, φ, g) is a quasi-para-Kahler-Norden manifold, if $\underset{X,Y,Z}{\sigma} g\big((\nabla_X\varphi)Y, Z\big) = 0$ where σ is the cyclic sum by three arguments [7]. In [13], the authors proved that $\underset{X,Y,Z}{\sigma} g\big((\nabla_X\varphi)Y, Z\big) = 0$ is equivalent

$$(\phi_\varphi g)(X,Y,Z) + (\phi_\varphi g)(Y,Z,X) + (\phi_\varphi g)(Z,X,Y) = 0.$$

We put

$$A(\tilde{X},\tilde{Y},\tilde{Z}) = (\phi_F{}^{CG}g)(\tilde{X},\tilde{Y},\tilde{Z}) + (\phi_F{}^{CG}g)(\tilde{Y},\tilde{Z},\tilde{X}) + (\phi_F{}^{CG}g)(\tilde{Z},\tilde{X},\tilde{Y}) = 0$$

for all $\tilde{X},\tilde{Y},\tilde{Z} \in \Im_0^1(T^*M^n)$. Using (3.6), we have $A(\tilde{X},\tilde{Y},\tilde{Z}) = 0$ for all $\tilde{X},\tilde{Y},\tilde{Z} \in \Im_0^1(T^*M^n)$. Hence we have

Theorem 3.7 *Let (M^n, g) be a Riemannian manifold and $(T^*M^n, {}^{CG}g)$ be its cotangent bundle. Then $(T^*M^n, {}^{CG}g)$ with almost paracomplex structure F is a quasi-para-Kahler-Norden manifold.*

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Received: September 5, 2015; Accepted: November 12, 2015