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ON THE EQUIVALENT BASES OF COSINES IN GENERALIZED LEBESGUE SPACES

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Abstract. In this paper the perturbed system of cosines is considered. The concept of $\mathscr{K}^*$-close systems generated by a space of coefficients $\mathscr{K}$ is introduced. Under certain conditions on the considered systems and on the space $\mathscr{K}$ the equivalence of bases of these systems in Lebesgue spaces with a variable summability exponent is established. This result generalizes the previous results in this direction.

1. Introduction

Perturbed systems of exponents, cosines and sines are playing an important role in the theory of spectral theory of differential operators, in the theory of optimal control, in approximation theory, and so on. Therefore, the numerous works are devoted to the study of frame properties, also basis properties (completeness, minimality, basicity and etc.). More details can be found in [1, 2, 3, 4, 5, 7, 8, 9, 14, 16, 18].

Since recently, there arose a great interest to the study of various problems, related to some research fields of mechanics and theoretical physics, in Lebesgue spaces with a variable summability exponent. For more information in this direction see monograph [6] and the papers [11, 17].

In this paper we consider the perturbed system of cosines. The stability of basicity of this system is studied in Lebesgue space with a variable summability exponent.

2. Needful information

Banach space will be referred to as B-space. Banach space of the sequences of scalars on the field K we will call K-space.

Let us present some facts from the theory of Lebesgue spaces with a variable summability exponent.

Let $p : [-\pi, \pi] \to [1, +\infty)$ be some Lebesgue measurable function. By $\mathscr{L}_0$ we denote the class of all functions measurable on $[-\pi, \pi]$ with respect to Lebesgue measure. Denote

$$I_p(f) \stackrel{def}{\equiv} \int_{-\pi}^{\pi} |f(t)|^{p(t)}\, dt.$$

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Let

$$\mathscr{L} \equiv \{f \in \mathscr{L}_0 : I_p(f) < +\infty\}.$$

With respect to the usual linear operations of addition and multiplication by a number $\mathscr{L}$ is a linear space as $p^+ = \sup\underset{[-\pi,\pi]}{vrai}\, p(t) < +\infty$. With respect to the norm

$$\|f\|_{p(\cdot)} \overset{def}{\equiv} \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \leq 1\right\},$$

$\mathscr{L}$ is a Banach space, and we denote it by $L_{p(\cdot)}$. Let

$$WL_0 \overset{def}{\equiv} \{p : \exists C > 0, \quad \forall t_1, t_2 \in [0,\pi] : |t_1 - t_2| \leq \tfrac{1}{2} \Rightarrow$$
$$\Rightarrow |p(t_1) - p(t_2)| \leq \tfrac{C}{-\ln|t_1 - t_2|}\right\}.$$

Throughout this paper, $q(\cdot)$ will denote the conjugate of a function $p(\cdot)$: $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$. Denote

$$p^- = \inf\underset{[-\pi,\pi]}{vrai}\, p(t).$$

The following generalized Hölder inequality is true

$$\int_{-\pi}^{\pi} |f(t)\, g(t)|\, dt \leq c\left(p^-; p^+\right) \|f\|_{p(\cdot)} \|g\|_{q(\cdot)},$$

where $c(p^-; p^+) = 1 + \frac{1}{p^-} - \frac{1}{p^+}$.

Directly from the definition we get the property, which will be used in the sequel.

Property A. *If $|f(t)| \leq |g(t)|$ a.e. on $(-\pi,\pi)$, then $\|f\|_{p(\cdot)} \leq \|g\|_{p(\cdot)}$.*

More details about the space $L_{p(\cdot)}$ can be found in [11, 12, 17] and in monograph [6].

We will also use the concept of the space of coefficients. We define it as follows. Let $\vec{x} \equiv \{x_n\}_{n\in N} \subset X$ be a non-degenerate system in a B-space X, i.e. $x_n \neq 0$, $\forall n \in N$. Define

$$\mathscr{K}_{\vec{x}} \equiv \left\{\{\lambda_n\}_{n\in N} : \textit{ the series } \sum_{n=1}^{\infty} \lambda_n x_n \textit{ is convergent in } X\right\}.$$

Introduce the norm in $\mathscr{K}_{\vec{x}}$:

$\left\|\vec{\lambda}\right\|_{\mathscr{K}_{\vec{x}}} = \sup\limits_m \|\sum_{n=1}^m \lambda_n x_n\|$, where $\vec{\lambda} = \{\lambda_n\}_{n\in N}$.

With respect to the usual operations of addition and multiplication by a complex number, $\mathscr{K}_{\vec{x}}$ is a K-space.

We also need some concepts and facts about the theory of the close bases.

Definition 2.1. Let X be some B-space. System $\{\varphi_n\}_{n\in N} \subset X$ is called ω-linearly independent in X (or simply ω- linearly independent), if from $\sum_n c_n \varphi_n = 0$ follows that $c_n = 0$, $\forall n \in N$.

The following theorem is true.

Theorem 2.1. *Let X be B-space with a basis $\{\varphi_n\}_{n\in N}$ and $F : X \to X$ be a Fredholm operator. Then for systems $\{\psi_n\}_{n\in N}$, the following properties are equivalent in X, where $\psi_n = F\varphi_n$, $\forall n \in N$:*

a) $\{\psi_n\}_{n\in N}$ is complete in X;

b) $\{\psi_n\}_{n\in N}$ *is minimal in* X*;*
v) $\{\psi_n\}_{n\in N}$ *is* ω*-linearly independent in* X*;*
q) $\{\psi_n\}_{n\in N}$ *forms a basis for* X *isomorphic to* $\{\varphi_n\}_{n\in N}$.

This theorem has the following

Corollary 2.1. *Let* $\{x_n\}_{n\in N}$ *form a basis for* X *and* $card\{n : y_n \neq x_n\} < +\infty$. *Then with respect to the system* $\{y_n\}_{n\in N}$ *the assertion of Theorem 2.1 is true.*

For more information about this and other facts we refer the reader to [1, 5, 8, 16, 18].

Accept the following

Definition 2.2. System $\{\varphi_n\}_{n\in Z_+} \subset L_{p(\cdot)}$ is called a K-Hilbertian system, if $\exists\delta > 0$:

$$\delta \left\|\{c_n\}\right\|_{\mathscr{K}} \leq \left\|\sum c_n\varphi_n\right\|_{L_{p(\cdot)}},$$

for any finite collection $\{c_n\}$.

3. Main results

Before the formulation of our main results let us present some results on systems of exponent and cosines in space $L_{p(\cdot)}$.

An analog of Levinson's theorem [12] on the completeness of the systems of exponents and cosines is also true in this case.

Theorem 3.1. *Let* $1 < p^- \leq p^+ < +\infty$. *If from complete system of functions* $\left\{e^{i\lambda_k x}\right\}$ *in* $L_{p(\cdot)}(-\pi, \pi)$ *throw out* n *arbitrary functions and add instead of them other functions* $e^{i\mu_j x}$, $j = \overline{1, n}$, *where* μ_k, $k = \overline{1, n}$*−are arbitrary pairwise different complex numbers, which are not equal to any of the numbers* λ_k, *then the obtained system will also be complete in* $L_{p(\cdot)}(-\pi, \pi)$.

The proof of this theorem is completely similar to the case of $L_p(-\pi, \pi)$ (i.e. $p(x) \equiv p = const$). Since, under the conditions of the theorem holds $\left(L_{p(\cdot)}(-\pi, \pi)\right)^* = L_{q(\cdot)}(-\pi, \pi)$, $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$.

The following theorem is also true.

Theorem 3.2. *Let* $\{\lambda_n\}_{n\in N} \subset C$ *be an arbitrary sequence of different numbers and* $1 < p^- \leq p^+ < +\infty$. *System* $\{\cos\lambda_n x\}_{n\in N}$ *is complete in* $L_{p(\cdot)}(0, \pi)$ *if and only if the system* $\left\{e^{\pm i\lambda_n x}\right\}_{n\in N}$ *is complete in* $L_{p(\cdot)}(-\pi, \pi)$. *If for some* $k_0 : \lambda_{k_0} = 0$, *then instead of the functions* $e^{i\lambda_{k_0}x}$ *and* $e^{-i\lambda_{k_0}x}$ *should take the functions* 1 *and* x, *respectively.*

These theorems follows the following corollary.

Corollary 3.1. *Let* $1 < p^- \leq p^+ < +\infty$. *If from complete system of functions* $\{\cos\lambda_k x\}$ *in* $L_{p(\cdot)}(0, \pi)$ *throw out* n *arbitrary functions and add instead of them* n *other functions* $\{\cos\mu_j x\}$, $j = \overline{1, n}$, *where* μ_k, $k = \overline{1, n}$*− are arbitrary complex numbers, such that* $\mu_i \neq \pm\mu_j$, *for* $i \neq j$, *and are not equal to any of the numbers* $\pm\lambda_k$, *then the obtained system will also be complete in* $L_{p(\cdot)}(0, \pi)$.

We now turn to the study of our main results. Let $\mathscr{K}$ be some K-*space* with a norm $\|\cdot\|_{\mathscr{K}}$, by l_0 we denote the linear space of all finite sequences, i.e. let

$$l_0 \equiv \left\{ \{\lambda_n\}_{n\in N} : \exists n_0 \in N \Rightarrow \lambda_k = 0, \, \forall k \geq n_0 \right\}.$$

Assume that the conjugate to $\mathscr{K}$ space $\mathscr{K}^*$ is also K-space and an arbitrary functional $\vec{\nu} = \{\nu_n\}_{n\in N} \in \mathscr{K}^*$ acts by the formula

$$\vec{\nu}\left(\vec{\lambda}\right) = \sum_{n=1}^{\infty} \lambda_n \vec{\nu}_n, \forall \vec{\lambda} = \{\lambda_n\}_{n\in N} \in \mathscr{K}.$$

Thus

$$\left| \sum_{n=1}^{\infty} \lambda_n \vec{\nu}_n \right| \leq \left\| \{\lambda_n\}_{n\in N} \right\|_{\mathscr{K}} \left\| \{\nu_n\}_{n\in N} \right\|_{\mathscr{K}^*}, \forall \vec{\lambda} \in \mathscr{K}, \forall \vec{\nu} \in \mathscr{K}^*.$$

We will assume that the space $\mathscr{K}$ satisfies the following conditions.

i) $\left\| \{\lambda_k\}_{k\in N} \right\|_{\mathscr{K}} = \left\| \left\{ \lambda_{\pi(k)} \right\}_{k\in N} \right\|_{\mathscr{K}}$, for any permutation $\pi : N \to N$.

ii)$l_0 \subset \mathscr{K} \wedge l_0 \subset \mathscr{K}^*$.

iii) For $\forall \vec{\nu} = \{\nu_n\}_{n\in N} \in \mathscr{K}^*$ *it holds*

$$\|\{0; \ldots; \nu_n; \nu_{n+1}; \ldots\}\|_{\mathscr{K}^*} \to 0, n \to \infty.$$

iv) From $|\alpha_n| \leq |\beta_n|$, $\forall n \in Z_+ \Rightarrow \left\| \{\alpha_n\}_{n\in Z_+} \right\|_{\mathscr{K}^*} \leq c \left\| \{\beta_n\}_{n\in Z_+} \right\|_{\mathscr{K}^*}$, where $c > 0$, is some constant.

Let us assume that the system of cosines $\{\cos \lambda_n x\}_{n\in Z_+}$ forms a $\mathscr{K}$-Hilbert basis for $L_{p(\cdot)}(0, \pi)$. Let $\{\varphi_n(\cdot)\}_{n\in Z_+} \subset L_{p(\cdot)}$ be some system.
We have

$$\left\| \sum_{n=0}^{\infty} c_n \left(\cos \lambda_n x - \varphi_n(x) \right) \right\|_{p(\cdot)} \leq \sum_{n=0}^{\infty} |c_n| \left\| \cos \lambda_n x - \varphi_n(x) \right\|_{p(\cdot)}. \tag{3.1}$$

Let $\{c_n\}_{n\in Z_+} \in l_0$. Then from (3.1) it follows

$$\left\| \sum_n c_n \left(\cos \lambda_n x - \varphi_n(x) \right) \right\|_{p(\cdot)} \leq \|\{c_n\}\|_{\mathscr{K}} \left\| \{\|\cos \lambda_m x - \varphi_n(x)\|\} \right\|_{\mathscr{K}^*}. \tag{3.2}$$

Since, the system $\{\cos \lambda_n x\}_{n\in Z_+}$ forms a $\mathscr{K}$-Hilbert basis for $L_{p(\cdot)}(0, \pi)$, then from (3.2) we have

$$\left\| \sum_n c_n \left(\cos \lambda_n x - \varphi_n(x) \right) \right\|_{p(\cdot)} \leq$$

$$\leq M \left\| \{\|\cos \lambda_m x - \varphi_n(x)\|\} \right\|_{\mathscr{K}^*} \left\| \sum_n c_n \cos \lambda_n x \right\|_{p(\cdot)}. \tag{3.3}$$

Consider the system $\{\cos \mu_n x\}$. Assume

$$\varphi_n(x) = \begin{cases} \cos \lambda_n x, & n = \overline{0, n_\varepsilon - 1}; \\ \cos \mu_n x, & n \geq n_\varepsilon, \ n \in Z_+. \end{cases}$$

Put

$$\nu_n = |\lambda_n - \mu_n|, \ n \in Z_+.$$

Let $\varepsilon > 0$ be an arbitrary number. Then it follows from property $iii)$, $\exists n_\varepsilon \in N$:

$$\left\|\{0;\, \ldots;\, 0;\, \nu_{n_\varepsilon};\, \nu_{n_\varepsilon+1};\, \ldots\}\right\|_{\mathscr{K}^*} < \varepsilon.$$

Taking into account the expression for $\varphi_n(\cdot)$ and the property $iv)$, from (3.3) we have

$$\left\|\sum_{n=0}^{\infty} c_n \left(\varphi_n(x) - \cos\lambda_n x\right)\right\|_{p(\cdot)} \leq M \left\|\{0;\, \ldots;\, 0;\, \nu_{n_\varepsilon};\, \nu_{n_\varepsilon+1};\, \ldots\}\right\|_{\mathscr{K}^*} \times$$

$$\times \left\|\sum_n c_n \cos\lambda_n x\right\|_{p(\cdot)} \leq M\varepsilon \left\|\sum_n c_n \cos\lambda_n x\right\|_{p(\cdot)}. \tag{3.4}$$

Take ε , such that $M\varepsilon < 1$. Then by Paley-Wiener theorem, from the inequality (3.4) it follows that the system $\{\varphi_n(\cdot)\}_{n\in Z_+}$ forms a basis for $L_{p(\cdot)}(0,\, \pi)$, equivalent to the basis $\{\cos\lambda_n x\}_{n\in Z_+}$.

In the subsequent, we replace the elements $\{\varphi_0;\, \ldots;\, \varphi_{n_\varepsilon-1}\}$ of the system $\{\varphi_n\}_{n\in Z_+}$ on $\{\cos\mu_0 x;\, \ldots;\, \cos\mu_{n_\varepsilon-1}\}$. Corollary 3.1 implies that the system $\{\cos\mu_n x\}_{n\in Z_+}$ is complete in $L_{p(\cdot)}(0,\, \pi)$. Then, as follows from Corollary 2.1, it forms a basis for $L_{p(\cdot)}(0,\, \pi)$, equivalent to the basis $\{\cos\lambda_n x\}_{n\in Z_+}$. Thus, the following theorem is true.

Theorem 3.3. *Let $1 < p^- \leq p^+ < +\infty$; and $\{\lambda_n;\, \mu_n\}_{n\in Z_+} \subset R$ be some sequence of different numbers. Let $\mathscr{K}$ be some K-space satisfying the conditions $i) - iv)$, and the system of cosines $\{\cos\lambda_n x\}_{n\in Z_+}$ forms a $\mathscr{K}$-Hilbertian basis for $L_{p(\cdot)}(0,\, \pi)$, isomorphic to the basis $\{\cos nx\}_{n\in Z_+}$. Then if $\{|\lambda_n - \mu_n|\}_{n\in Z_+} \in \mathscr{K}^*$, then the system $\{\cos\mu_n x\}_{n\in Z_+}$ also forms a basis for $L_{p(\cdot)}(0,\, \pi)$, equivalent to the basis $\{\cos\lambda_n x\}_{n\in Z_+}$.*

Let $p(\cdot) \in WL_0$. Then, it is known that (see e.g. [2, 14]) the system of cosines $\{\cos nx\}_{n\in Z_+}$ forms a basis for $L_{p(\cdot)}(0,\, \pi)$. By $\mathscr{K}_c$ we denote the space of its coefficients. Assume that the following continuous embedding $\mathscr{K}_c \subset \mathscr{K}$ is valid, i.e. the following inequality is true.

$$\left\|\{\lambda_n\}_{n\in N}\right\|_{\mathscr{K}} \leq c \left\|\{\lambda_n\}_{n\in N}\right\|_{\mathscr{K}_c},\, \forall\, \{\lambda_n\}_{n\in N} \in \mathscr{K}_c,$$

where $c > 0$ is some constant. Then the system $\{\cos nx\}_{n\in N}$ forms a $\mathscr{K}$*-Hilbert* basis for $L_{p(\cdot)}(0,\, \pi)$. Consequently, for arbitrary $\{\lambda_n\} \in l_0$ the inequality

$$\left\|\{\lambda_n\}\right\|_{\mathscr{K}} \leq c \left\|\sum_n \lambda_n \cos nx\right\|_{p(\cdot)},$$

is true. If the system $\{\cos\lambda_n x\}_{n\in N}$ forms a basis for $L_{p(\cdot)}(0,\, \pi)$ isomorphic to the basis $\{\cos nx\}_{n\in N}$, then it is clear that it is also a $\mathscr{K}$-Hilbert. So, the following corollary is true.

Corollary 3.2. *Let $p(\cdot) \in WL_0$, $p^- > 1$, and $\mathscr{K}$ be some K-space, satisfying the conditions $i) - iv)$; $\mathscr{K}_c$ is a space coefficients of basis $\{\cos nx\}_{n\in Z_+}$ in $L_{p(\cdot)}(0,\, \pi)$, the continuous embedding $\mathscr{K}_c \subset \mathscr{K}$ is true. If $\{\cos\lambda_n x\}_{n\in Z_+}$ forms a $\mathscr{K}$-Hilbert basis for $L_{p(\cdot)}(0,\, \pi)$, isomorphic to the basis $\{\cos nx\}_{n\in Z_+}$ and $\{|\lambda_n - \mu_n|\}_{n\in Z_+} \in \mathscr{K}^*$, then the system $\{\cos\mu_n x\}_{n\in Z_+}$ also forms a basis for $L_{p(\cdot)}(0,\, \pi)$, isomorphic to the basis $\{\cos nx\}_{n\in Z_+}$.*

Consider the following special case. Denote $p = \min\{2; p^-\}$ As $\mathscr{K}$-space K we take $\mathscr{K} = l_q$, where $\frac{1}{p} + \frac{1}{q} = 1$. The following continuous embeddings are valid

$$L_{p(.)}(0,\pi) \subset L_{p^-}(0,\pi) \subset L_p(0,\pi). \tag{3.5}$$

Let the system $\{\cos\lambda_n x\}_{n\in Z_+}$ form a basis for $L_{p(\cdot)}(0,\ \pi)$ equivalent to the basis $\{\cos nx\}_{n\in Z_+}$. Then $\exists M > 0$:

$$M^{-1}\left\|\sum_n c_n\cos\lambda_n x\right\|_{p(\cdot)} \leq \left\|\sum_n c_n\cos nx\right\|_{p(.)} \leq M\left\|\sum_n c_n\cos\lambda_n x\right\|_{p(.)} \tag{3.6}$$

From the classical Hausdorff- Young theorem it follows

$$\|\{c_n\}\|_{l_q} \leq M_\alpha\left\|\sum_n c_n\cos nx\right\|_{L_{p(0,\pi)}}.$$

Taking into account the relation (3.6) we obtain

$$\|\{c_n\}\|_{l_q} \leq MM_\alpha\left\|\sum_n c_n\cos\lambda_n x\right\|_{p(\cdot)},$$

i.e. the system $\{\cos\lambda_n x\}_{n\in N}$ is l_q-Hilbertian. Theorem 3.3 directly has the following

Corollary 3.3. *Let* $1 < p^- \leq p^+ < +\infty$, *and* $\{\lambda_n;\mu_n\}_{n\in Z_+} \subset R$ *be a sequence of different numbers, such that*

$$\sum_{n=0}^{\infty}|\lambda_n - \mu_n|^p < +\infty.$$

If the system $\{\cos\lambda_n x\}_{n\in Z_+}$ *forms a basis for* $L_{p(\cdot)}(0,\pi)$, *equivalent to the basis* $\{\cos nx\}_{n\in Z_+}$, *then the system* $\{\cos\mu_n x\}_{n\in Z_+}$ *also forms a basis for* $L_{p(\cdot)}(0,\pi)$, *equivalent to the basis* $\{\cos nx\}_{n\in Z_+}$.

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