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BASIS PROPERTIES IN THE SPACE $L_p(0,1)$ OF ROOT FUNCTIONS OF THE BOUNDARY VALUE PROBLEM WITH SPECTRAL PARAMETER IN THE BOUNDARY CONDITIONS

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Abstract. In this paper we consider the Sturm-Liouville problem with spectral parameter in the boundary conditions. We study the basis properties in the space $L_p(0,1)$, $1 < p < \infty$, of systems of root functions of this problem.

1. Introduction

In this paper, we continue the study [9] of the boundary value problem

$$-y''(x) = \lambda y(x), \;\; 0 < x < 1, \tag{1.1}$$

$$y'(0) = -a_0 \lambda y(0), \tag{1.2}$$

$$y'(1) = (a_1 \lambda + b_1) y(1), \tag{1.3}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, a_0, a_1, b_1 are real constants, and $a_0 \neq 0$, $a_1 \neq 0$.

The structure of root subspaces and location of eigenvalues on the real axis of problem (1.1)-(1.3) were studied by Kapustin [9] for the case where $a_0 > 0$, $a_1 > 0$, $b_1 = 0$ and by Aliev [1, 3] for the cases where $a_0 > 0$, $a_1 < 0$, $b_1 = 0$ and $a_0 < 0$, $a_1 < 0$, $b_1 = 0$. In these papers, studied also basis properties of the system of root functions of the boundary value problem (1.1)-(1.3) in the space $L_p(0,1)$, $1 < p < \infty$, where obtained necessary and sufficient conditions for the basicity of subsystems of root functions. In [4, 5, 12-14] studied the eigenvalue problem for a second order differential equation with spectral parameter in the boundary conditions in the more general case, where investigated oscillation properties of eigenfunctions and obtained the sufficient condition for basicity of subsystem of eigenfunctions in the space $L_p(0,1)$, $1 < p < \infty$.

In [9] were studied general characteristic of the location of eigenvalues on the real axis (the complex plane) and the structure of root subspaces of problem (1.1)-(1.3) in the case $a_0 < 0$, $a_1 < 0$, $b_1 \neq 0$. The subject of the present paper is the study of the basis property in the space $L_p(0,1), 1 < p < \infty$, of the system of root functions of the boundary value problem (1.1)-(1.3) in the case $a_0 < 0$, $a_1 < 0$, $b_1 \neq 0$.

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2. Preliminaries

The considered problem (1.1)-(1.3) can be reduced to the eigenvalue problem for the linear operator L in the Hilbert space $H = L_2(0,1) \oplus \mathbb{C}^2$ with inner product

$$(\hat{u},\hat{v})_H = (\{u(x),\, m,\, n\},\, \{v(x), s, t\})_H = (u,v)_{L_2} + |a_0|^{-1}\, m\bar{s} + |a_1|^{-1} n\bar{t} \quad (2.1)$$

where $(\cdot\,,\cdot)_{L_2}$ is an inner product in $L_2(0,1)$ and

$$L\hat{y} = L\{\, y(x), m, n\} = \{-y''(x),\, y'(0),\, y'(1) - b_1 y(1)\}$$

is an operator with the domain

$$D(L) = \{\, \hat{y} \in H \mid y(x),\, y'(x) \in AC\,[0,1],\, m = -a_0 y(0),\, n = a_1 y(1)\}$$

dense everywhere in H (see [14]). Obviously, the operator L is well defined in H. Problem (1.1)-(1.3) takes the form

$$L\hat{y} = \lambda\hat{y},\ \ \hat{y} \in D(L),$$

i.e., the eigenvalues λ_k, $k \in \mathbb{N}$, of the operator L and problem (1.1)-(1.3) coincide together with their multiplicities, and between the root functions, there is a one-to-one correspondence

$$y_k(x) \leftrightarrow \hat{y}_k = \{y_k(x),\, m_k,\, n_k\},\ \ m_k = -a_0 y_k(0),\ \ n_k = a_1 y_k(1).$$

Throughout the following, we assume that

$$a_0 < 0 \ \text{ and } \ a_1 < 0.$$

We introduce an operator $J : H \to H$ as follows:

$$J\{\, y, m, n\} = \{\, y, -m, -n\}.$$

The operator J is unitary and symmetric in H with spectrum consisting of two eigenvalues, -1 with multiplicity 2 and 1 with infinite multiplicity [9, Theorem 2.1]. Consequently, this operator generates the Pontryagin space $\Pi_2 = L_2(0,1) \oplus \mathbb{C}^2$ with inner product ($J-$ metric) [8, Ch.1]

$$(\hat{u},\hat{v})_{\Pi_2} = (\{u(x),\, m,\, n\},\, \{v(x), s, t\})_{\Pi_2} = (u,v)_{L_2} + a_0^{-1}\, m\bar{s} + a_1^{-1} n\bar{t}. \quad (2.2)$$

Theorem 2.1 [9, Theorem 2.2, Corollary 2.1]. *The operator JL is self-adjoint in H, i.e. L is $J-$self-adjoint in Π_2.*

Lemma 2.1 [6]. *Let L^* be the adjoint of the operator L. Then $L^* = JLJ$.*

The solution

$$y(x,\lambda) = a_0\sqrt{\lambda}\sin\sqrt{\lambda}\, x - \cos\sqrt{\lambda}\, x \quad (2.3)$$

of Eq. (1.1) satisfies the initial conditions

$$y(0,\lambda) = -1,\ \text{ and }\ y'(0,\lambda) = a_0\lambda. \quad (2.4)$$

The eigenvalues of the boundary value problem

$$-y''(x) = \lambda y(x),\ \ 0 < x < 1,$$

$$a_0\lambda y(0) + y'(0) = 0,\ \ y(1) = 0,$$

are real and simple and form an infinitely increasing sequence

$$\mu_1 < 0 < \mu_2 < ... < \mu_k < ... .$$

Set $B_k = (\mu_{k-1}, \mu_k)$, $k = 1,\, 2,\, ...$, where $\mu_0 = -\infty$.

Theorem 2.2 [9, Theorem 3.1]. *If $b_1 < 0$, then all eigenvalues of problem* (1.1)-(1.3) *are real and simple; in this case B_2 contains two eigenvalues, and B_k, $k = 1, 3, 4, \ldots$, contain one eigenvalue. If $b_1 > 0$, then one of the following assertions holds: (i) all eigenvalues of problem* (1.1)-(1.3) *are real; in this case, B_2 contains algebraically two eigenvalues (either two simple eigenvalues or one double eigenvalue), and B_k, $k = 1, 3, 4, \ldots$, contains one simple eigenvalue; (ii) all eigenvalues of problem* (1.1)-(1.3) *are real; in this case, B_2 contains no eigenvalues, while there exists a positive integer M ($M > 2$) such that B_M contains algebraically three eigenvalues (either three simple eigenvalues, or one double eigenvalue and one simple eigenvalue, or one triple eigenvalue), and B_k, $k = 1, 3, 4, \ldots$, $k \neq M$, contains one simple eigenvalue; (iii) problem* (1.1)-(1.3) *has one pair of nonreal complex conjugate eigenvalues; in this case, B_2 contains no eigenvalues, and B_k, $k = 1, 3, 4, \ldots$, contains one simple eigenvalue.*

We denote by $\rho(\lambda)$ the algebraic multiplicity of an eigenvalue λ of the boundary value problem (1.1)-(1.3).

By Theorem 3.1, we have $\rho(\lambda_k) = 2$ (i.e., $\lambda_k = \lambda_{k+1}$) if $k = M - 1$ or $k = M$ and $\rho(\lambda_k) = 3$ (i.e., $\lambda_k = \lambda_{k+1} = \lambda_{k+2}$) if $k = M - 1$. (If assertion (i) of second part in Theorem 3.1 holds, then we set $M = 2$.)

Let $\{y_k(x)\}_{k=1}^{\infty}$ be the system of eigenfunctions and associated functions corresponding to the system of eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ of problem (1.1)-(1.3):

$y_k(x) = y(x, \lambda_k)$ if $\rho(\lambda_k) = 1$;

$y_k(x) = y(x, \lambda_k)$, $y_{k+1}(x) = y^*_{k+1}(x) + c_k y_k(x)$, where $y^*_{k+1}(x) = \partial y(x, \lambda_k)/\partial\lambda$, and c_k is an arbitrary constant if $\rho(\lambda_k) = 2$;

$y_k(x) = y(x, \lambda_k)$, $y_{k+1}(x) = y^*_{k+1}(x) + d_k y_k(x)$, $y_{k+2}(x) = y^*_{k+2}(x) + d_k y^*_{k+1} + h_k y_k(x)$, where $y^*_{k+2}(x) = 2^{-1}\partial^2 y(x, \lambda_k)/\partial\lambda^2$, d_k and h_k are arbitrary constants if $\rho(\lambda_k) = 3$.

Note that, the functions $y_k(x)$ and $y_{k+1}(x)$ for $\rho(\lambda_k) = 2$ and the functions $y_k(x)$, $y_{k+1}(x)$ and $y_{k+2}(x)$ for $\rho(\lambda_k) = 3$ form a chain of an eigenfunction and associated functions.

3. Basis properties of systems of root functions of problem (1.1)-(1.3)

Each element $\hat{y}_k = \{y_k(x), m_k, n_k\}$, $m_k = -a_0 y_k(0)$ and $n_k = a_1 y_k(1)$, of the system $\{\hat{y}_k\}_{k=1}^{\infty}$ of root vectors of the operator L satisfies the relation

$$L\hat{y}_k = \lambda_k \hat{y}_k + \theta_k \hat{y}_{k-1}, \tag{3.1}$$

where θ_k is equal either to zero (in this case, $\hat{y}_k$ is an eigenvector) or to unit (in this case, $\lambda_k = \lambda_{k-1}$ and $\hat{y}_k$ is an associated vector) (e.g., see [10]).

Let $\{\hat{v}^*_k\}_{k=1}^{\infty}$, where $\hat{v}^*_k = \{v^*_k(x), s_k, t_k\}$, is the system of root vectors of the operator L^*, i.e.,

$$L^*\hat{v}^*_k = \bar{\lambda}_k \hat{v}^*_k + \theta_{k+1}\hat{v}^*_{k+1}. \tag{3.2}$$

By Lemma 2.1 and relations (3.1), (3.2), the following assertion holds.

Lemma 3.1. *The following relations are true : $\hat{v}^*_k = J\,\overline{\hat{y}_k}$ if $\rho(\lambda_k) = 1$; $\hat{v}^*_k = J\,\hat{y}^*_{k+1} + \tilde{c}_k J\hat{y}_k$ and $\hat{v}^*_{k+1} = J\hat{y}_k$ if $\rho(\lambda_k) = 2$; $\hat{v}^*_k = J\hat{y}^*_{k+2} + \tilde{d}_k J\hat{y}^*_{k+1} + \tilde{h}_k J\hat{y}_k$, $\hat{v}^*_{k+1} = J\hat{y}^*_{k+1} + \tilde{d}_k J\hat{y}_k$ and $\hat{v}^*_{k+2} = J\hat{y}_k$ if $\rho(\lambda_k) = 3$, where $\hat{y}^*_{k+1} = \{y^*_{k+1}(x), m^*_{k+1}, n^*_{k+1}\}$, $m^*_{k+1} = m'(\lambda_k)$, $n^*_{k+1} = n'(\lambda_k)$, $\hat{y}^*_{k+2} = \{y^*_{k+2}(x), m^*_{k+2}, n^*_{k+2}\}$,*

$m^*_{k+2} = (1/2)\,m''(\lambda_k)$, $m^*_{k+2} = (1/2)\,n''(\lambda_k)$; $\tilde{c}_k$, $\tilde{d}_k$ *and* $\tilde{h}_k$ *are arbitrary constants.*

Lemma 3.2. *Let*

$$\delta_k = \begin{cases} ||y_k||^2_{L_2} + a_0 y_k^2(0) + a_1 y_k^2(1), & if \quad \rho(\lambda_k) = 1, \\ (\hat{y}_k, \hat{y}^*_{k+1})_{\Pi_2}, & if \quad \rho(\lambda_k) = 2, \\ ||\hat{y}^*_{k+1}||^2_{\Pi_2}, & if \quad \rho(\lambda_k) = 3, \end{cases}$$

where $||\cdot||_{\Pi_2}$ *is the norm in the space* Π_2. *Then* $\delta_k \neq 0,\ k = 1, 2, \ldots$.

The proof of this lemma is similar to that of [2, Lemma 6.2].

Lemma 3.3. *An element* $\hat{v}_k = \{v_k(x),\, s_k,\, t_k\}$ *of the system* $\{\hat{v}_k\}_{k=1}^\infty$ *adjoint to the system* $\{\hat{y}_k\}_{k=1}^\infty$ *is given by the relation*

$$\hat{v}_k = \bar{\delta}_k^{-1} \hat{v}^*_k, \tag{3.3}$$

where the $\hat{v}^*_k$, $k = 1,\, 2,\, \ldots$, *are defined in Lemma 3.1, and the* $\delta_k, k = 1,\, 2,\, \ldots$, *are defined in Lemma 3.2; in addition* $\tilde{c}_k = -c_k - \delta_k^{-1} ||\hat{y}^*_{k+1}||^2_{\Pi_2}$ *if* $\rho(\lambda_k) = 2$; $\tilde{d}_k = -d_k - \delta_k^{-1} (\hat{y}^*_{k+1}, \hat{y}^*_{k+2})_{\Pi_2}$ *and* $\tilde{h}_k = -h_k - \delta_k^{-1} ||\hat{y}^*_{k+2}||^2_{\Pi_2} + \delta_k^{-2} (\hat{y}^*_{k+1}, \hat{y}^*_{k+2})^2_{\Pi_2} + d_k (d_k + \delta_k^{-1} (\hat{y}^*_{k+1}, \hat{y}^*_{k+2})_{\Pi_2})$ *if* $\rho(\lambda_k) = 3$.

The proof of Lemma 3.3 is similar to that of [2, Lemma 6.3].

If $\rho(\lambda_k) = 1$, then it follows by (2.3) that

$$m_k = -a_0 y_k(0) = -a_0 y(0, \lambda_k) = a_0. \tag{3.4}$$

The following result is a straightforward consequence of Lemma 3.3 and formula (2.3).

Corollary 3.1. *The following assertions hold:*

(i) if $\rho(\lambda_k) = 1$, *then* $s_k = -\bar{\delta}_k^{-1}\ \overline{m}_k = -a_0 \bar{\delta}_k^{-1}$ *and* $t_k = -\bar{\delta}_k^{-1}\ \overline{n}_k = -a_1 \bar{\delta}_k^{-1} y_k(1)$;

(ii) if $\rho(\lambda_k) = 2$, *then* $s_k = -\delta_k^{-1} \tilde{c}_k m_k = -a_0 \delta_k^{-1} \tilde{c}_k$, $s_{k+1} = -\delta_k^{-1} m_k = -a_0 \delta_k^{-1}$, $t_k = -\delta_k^{-1} \{n^*_{k+1} + \tilde{c}_k n_k\}$ *and* $t_{k+1} = -\delta_k^{-1} n_k = -a_1 \delta_k^{-1} y_k(1)$;

(iii) if $\rho(\lambda_k) = 3$, *then* $s_k = -\delta_k^{-1} \tilde{h}_k m_k = -a_0 \delta_k^{-1} \tilde{h}_k$, $s_{k+1} = -\delta_k^{-1} \tilde{d}_k m_k = -a_0 \delta_k^{-1} \tilde{d}_k$, $s_{k+2} = -\delta_k^{-1} m_k = -a_0 \delta_k^{-1}$, $t_k = -\delta_k^{-1} \{n^*_{k+2} + \tilde{d}_k n^*_{k+1} + \tilde{h}_k n_k\}$, $t_{k+1} = -\delta_k^{-1} \{n^*_{k+1} + \tilde{d}_k n_k\}$ *and* $t_{k+2} = -\delta_k^{-1} n_k = -a_1 \delta_k^{-1} y_k(1)$.

Let $r, l, r \neq l$ be arbitrary fixed positive integers and

$$\Delta_{r,\,l} = \begin{vmatrix} s_r & s_l \\ t_r & t_l \end{vmatrix}.$$

Theorem 3.1. *If* $\Delta_{r,\,l} \neq 0$, *then the system of root functions* $\{y_k(x)\}_{k=1,\ k \neq\ r,\,l}^\infty$ *of problem* (1.1)-(1.3) *forms a basis in the space* $L_p(0,1)$, $1 < p < \infty$ *(even a Riesz basis for* $p = 2$*); if* $\Delta_{r,\,l} = 0$, *then this system is neither complete nor minimal in the space* $L_p(0,1)$, $1 < p < \infty$.

The proof of this theorem is similar to that of [5, Theorem 4.1].

Theorem 3.2. *There exists positive integer* $\tilde{k}$ *such that the system of eigenfunctions* $\{y_k(x)\}_{k=1,\ k \neq\ r,\,l}^\infty$ *of problem* (1.1)- (1.3) *for* $r,\, l \geq \tilde{k}$ *forms a basis in the space* $L_p(0,1)$, $1 < p < \infty$ *(even a Riesz basis for* $p = 2$*).*

Proof. If $\rho(\lambda_k) = 1$, then by [13, formula (10)] we obtain

$$y_k(1) = (-1)^k \left(\frac{a_0^2 \lambda_k^2 + \lambda_k}{(a_1 \lambda_k + b_1)^2 + \lambda_k} \right)^{1/2}. \tag{3.5}$$

Let $\rho(\lambda_r) = \rho(\lambda_l) = 1$. Then by Corollary 3.1 it follows that

$$\Delta_{r,l} = \begin{vmatrix} -a_0\bar{\delta}_r^{-1} & -\, a_0\bar{\delta}_l^{-1} \\ -a_1\bar{\delta}_r^{-1}y_r(1) & -\, a_1\bar{\delta}_r^{-1}y_r(1) \end{vmatrix} = a_0a_1\bar{\delta}_r^{-1}\bar{\delta}_l^{-1} \begin{vmatrix} 1 & 1 \\ y_r(1) & y_l(1) \end{vmatrix}. \quad (3.6)$$

Hence in view of (3.5), by (3.6) we obtain

$$\Delta_{r,l} = (-1)^{r+l}a_0a_1\bar{\delta}_r^{-1}\bar{\delta}_l^{-1} \begin{vmatrix} 1 & 1 \\ \left(\frac{a_0^2\lambda_r^2+\lambda_r}{(a_1\lambda_r+b_1)^2+\lambda_r}\right)^{1/2} & \left(\frac{a_0^2\lambda_l^2+\lambda_l}{(a_1\lambda_l+b_1)^2+\lambda_l}\right)^{1/2} \end{vmatrix}. \quad (3.7)$$

We introduce the function

$$F(\lambda) = \left(\frac{a_0^2\lambda_k^2+\lambda_k}{(a_1\lambda_k+b_1)^2+\lambda_k}\right)^{1/2}. \quad (3.8)$$

By virtue of [13, formula (18)] that

$$F'(\lambda) = \frac{1}{2}\,(F(\lambda))^{-1/2}\,\frac{\left(2a_0^2a_1b_1 + a_0^2 - a_1^2\right)\lambda^2 + 2a_0^2b_1^2\lambda + b_1^2}{\left\{(a_1\lambda+b_1)^2+\lambda\right\}^2}. \quad (3.9)$$

Since $a_0b_1 \neq 0$, it follows from (3.9) that there exists $\tilde{\lambda} > 0$ such that $F'(\lambda) \neq 0$ for $\lambda > \tilde{\lambda}$. Hence the function $F(\lambda)$ is strictly monotonic in the interval $(\tilde{\lambda}, +\infty)$. Consequently, there exists $\tilde{k} \in \mathbb{N}$ such that $\Delta_{r,l} \neq 0$ for $r, l \geq \tilde{k}$, by (3.7) and (3.8). Then on the base of Theorem 3.1 the eigenfunctions system $\{y_k(x)\}_{k=1,\, k\neq\, r,\, l}^{\infty}$ of problem (1.1)- (1.3) for $r,\ l \geq \tilde{k}$ forms a basis in the space $L_p(0,1)$, $1 < p < \infty$ (even a Riesz basis for $p = 2$). The proof of Theorem 3.2 is complete.

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