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SWITCHING LUMPED CONTROLS

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Abstract. In this article, we consider the 1-d heat equation endowed with arbitrary number (finite) of lumped controls and under suitable conditions, we show that our approach allows building switching controls. For achieving this goal, we first introduce a new functional based on the adjoint system whose minimizers yield the switching controls. We show that, due to the time analyticity of the solutions, under suitable conditions on the location of the controllers, lumped switching controls exist in the 1-d heat equation.

1. Introduction

As an introduction part, first of all we defined the problem of controllability in PDEs/ODEs. Roughly speaking, it consists in observing whether the solution of the PDEs/ODEs can be driven to a given final target by means of a suitable control. More precisely, the controllability problem may be characterized as follows. Consider an evolution system with given a time interval $t \in (0, T)$, initial and final states. We try to find a suitable control such that the solution matches both the initial state at time $t = 0$ and the final one at time $t = T$. This is a type of exact controllability problem. There are other type of controllability problems beside this one. For instance, when the final target is achieved to zero, then the system is null controllable or when the set of reachable states (set of final targets) is dense in the space where the evolution system is satisfied, then the system is approximate controllable.
Control systems are often endowed with several actuators. It is then desirable to design switching control strategies guaranteeing that, at each instant of time, only one control is activated. The goal is to control the system by switching from an actuator to another in a systematic way so that, at each instant of time, only one actuator is active. In [3], the author developed a first analysis of this problem of switching controls addressing some model cases.
This paper deals with some of general results in null controllability of 1-d heat equation with switching lumped controls.

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2. Lumped Controls

Let $f_0 = f_0(x)$, $f_1 = f_1(x)$ and $f_2 = f_2(x)$ be three control profiles in $L^2(0,1)$. Consider the heat equation:

$$\begin{cases} y_t - y_{xx} = u_0(t)f_0 + u_1(t)f_1 + u_2(t)f_2, & 0 < x < 1, 0 < t < T, \\ y(0,t) = y(1,t) = 0, & 0 < t < T, \\ y(x,0) = y^0(x), & 0 < x < 1. \end{cases} \tag{2.1}$$

For a given initial datum $y^0 \in L^2(0,1)$ we look for controls $u_0(t), u_1(t), u_2(t) \in L^2(0,T)$ such that $y(x,T) = 0$ and the switching condition satisfies:

$$u_0(t)u_1(t) = 0, \quad u_0(t)u_2(t) = 0, \quad u_1(t)u_2(t) = 0, \quad \text{a.e.} \quad t \in (0,T). \tag{2.2}$$

For φ^0 in $L^2(0,1)$, we consider the solution $\varphi : [0,1] \times [0,T] \to C([0,T], L^2(0,1))$, of the following backward Cauchy linear problem:

$$\begin{cases} \varphi_t + \varphi_{xx} = 0, & 0 < x < 1, 0 < t < T, \\ \varphi(0,t) = \varphi(1,t) = 0, & 0 < t < T, \\ \varphi(x,T) = \varphi^0(x), & 0 < x < 1. \end{cases} \tag{2.3}$$

This linear system is called the adjoint system corresponding to the 1-d heat equation with Dirichlet's boundary condition. (see, e.g. [3], [4])
We may compute the null control of 1-d heat equation by minimizing the following quadratic functional (see, e.g., [3])

$$\begin{aligned} \hat{J}(\varphi^0) &= \frac{1}{2}\int_0^T \max\Big[\Big|\int_0^1 f_0(x)\varphi dx\Big|^2, \Big|\int_0^1 f_1(x)\varphi dx\Big|^2, \Big|\int_0^1 f_2(x)\varphi dx\Big|^2\Big] dt \\ &- \int_0^T y^0(x)\varphi(x,0)dx \end{aligned}$$

over the class $\hat{\mathcal{H}}$ of initial data given by

$$\hat{\mathcal{H}} = \{\varphi^0 : \int_0^T \Big[\Big|\int_0^1 f_0(x)\varphi dx\Big|^2 + \Big|\int_0^1 f_1(x)\varphi dx\Big|^2 + \Big|\int_0^1 f_2(x)\varphi dx\Big|^2\Big] dt < \infty\}$$

where $\varphi(x,t)$ is the solution of the adjoint system (2.3) associated to the final target φ^0. We will consider $\hat{\mathcal{H}}$ space endowed with the canonical norm

$$||\varphi^0||^2_{\hat{\mathcal{H}}} = \int_0^T \Big[\Big|\int_0^1 f_0(x)\varphi dx\Big|^2 + \Big|\int_0^1 f_1(x)\varphi dx\Big|^2 + \Big|\int_0^1 f_2(x)\varphi dx\Big|^2\Big] dt$$

constitutes a Hilbert space (see, e.g., [3]). Let us analyse the positivity of the norm $||\cdot||_{\hat{\mathcal{H}}}$ in $\hat{\mathcal{H}}$. Before proving the positivity of the norm, we will give very important lemma on families of real exponentials. This lemma is known as estimates on families of real exponentials (see, e.g. [3], [5]).

Lemma 2.1 *In our case, it is guaranteed that*

$$\int_0^T \Big|\sum_{k\geq 1} \beta_k e^{\pi^2 k^2 (t-T)}\Big|^2 dt \geq c_1 \sum_{k\geq 1} e^{-2\pi^2 k^2 T}\beta_k^2$$

for a suitable positive constants $c_1 > 0$ which is independent from $\{\beta_k\}_{k\geq 1}$.

Assume that the controls f_0, f_1 and f_2 have Fourier series expansions of the form

$$f_0(x) = \sum_{k\geq 1} f_{0,k}\omega_k(x), \quad f_1(x) = \sum_{k\geq 1} f_{1,k}\omega_k(x), \quad f_2(x) = \sum_{k\geq 1} f_{2,k}\omega_k(x). \tag{2.4}$$

Hence after some calculation, we have

$$||\varphi^0||^2_{\hat{\mathcal{H}}} = \int_0^T \Big[\Big|\sum_{k\geq 1}\beta_k e^{\pi^2k^2(t-T)} f_{0,k}\Big|^2 + \Big|\sum_{k\geq 1}\beta_k e^{\pi^2k^2(t-T)} f_{1,k}\Big|^2 + \Big|\sum_{k\geq 1}\beta_k e^{\pi^2k^2(t-T)} f_{2,k}\Big|^2\Big] dt$$

Now using Lemma 2.1, we then get weighted observability inequality

$$||\varphi^0||^2_{\hat{\mathcal{H}}} \geq c_1 \sum_{k\geq 1} e^{-2\pi^2k^2T}\Big[|f_{0,k}|^2 + |f_{1,k}|^2 + |f_{2,k}|^2\Big]\beta_k^2 \tag{2.5}$$

where positive constant c_1 is independent from $\{\beta_k\}_{k\geq 1}$.

In addition, since the adjoint system (2.3) is well posed, the fuctional $\hat{J}(\varphi^0)$ is obviously continuous in $\hat{\mathcal{H}}$, and the convexity (strictly) of $\hat{J}(\varphi^0)$ comes from Lemma 2.1.

Firstly, we get approximate controllability of (2.1), i.e., for $\epsilon > 0$ we could find approximate controls $u_0^\epsilon, u_1^\epsilon$, u_2^ϵ such that the solution y_ϵ of heat equation satisfies the following condition

$$||y_\epsilon(x,T) - y(x,T)||_{L^2(0,1)} \leq \epsilon. \tag{2.6}$$

For this, we consider new functional very similar with $\hat{J}$: for any $\epsilon > 0$ and any $y^1 \in L^2(0,1)$

$$\begin{aligned}\hat{J}_\epsilon(\varphi^0) \;&=\; \frac{1}{2}\int_0^T \max\Big[\Big|\int_0^1 f_0(x)\varphi dx\Big|^2, \Big|\int_0^1 f_1(x)\varphi dx\Big|^2, \Big|\int_0^1 f_2(x)\varphi dx\Big|^2\Big] dt \\ &\quad + \epsilon||(I-\pi_E)\varphi^0||_{L^2(0,1)} + \int_0^1 \varphi^0 y^1 dx - \int_0^1 y^0(x)\varphi(x,0)dx\end{aligned}$$

where E is finite dimensional subspace of $L^2(0,1)$ and π_E denotes the ortogonal projection from $L^2(0,1)$ over E.

Lemma 2.2 *Assume that the following unique continuation property*

$$\mu\Big\{t \in (0,T) : \Big|\int_0^1 f_0\varphi dx\Big| = \Big|\int_0^1 f_1\varphi dx\Big| = \Big|\int_0^1 f_2\varphi dx\Big|\Big\} > 0 \Rightarrow \varphi \equiv 0 \tag{2.7}$$

holds. Then the heat system (2.1) is approximate controllable.

Proof. For obtaining approximate controllability of (2.1), we should minimize $\hat{J}_\epsilon$ over $\hat{\mathcal{H}}$. We have already known that $\hat{J}_\epsilon$ is strictly convex and continuous in $\hat{\mathcal{H}}$. Also, in view of the unique continuation property above, one can prove the following coercivity property of $\hat{J}_\epsilon$

$$\lim_{||\varphi^0||_{L^2(0,1)}\to\infty} \frac{\hat{J}_\epsilon(\varphi^0)}{||\varphi^0||_{L^2(0,1)}} \geq \epsilon.$$

Hence, we have proved that $\hat{J}_\epsilon$ is convex, continuous and coercive in $\hat{\mathcal{H}}$. Therefore $\hat{J}_\epsilon$ admits an unique minimizer $\hat{\varphi}^0 \in \hat{\mathcal{H}}$, i.e., for any $\psi^0 \in L^2(0,1)$ and $h \in \mathbb{R}$ sufficiently small, we have $\hat{J}_\epsilon(\hat{\varphi^0}) \le \hat{J}_\epsilon(\widehat{\varphi^0 + h\psi^0})$. More precisely,

$$\begin{aligned}
\Delta\hat{J}_\epsilon &= \int_{I_0} h\int_0^1 f_0(x)\hat{\varphi}(x,t)dx\int_0^1 f_0(x)\psi(x,t)dxdt \\
&+ \int_{I_1} h\int_0^1 f_1(x)\hat{\varphi}(x,t)dx\int_0^1 f_1(x)\psi(x,t)dxdt \\
&+ \int_{I_2} h\int_0^1 f_2(x)\hat{\varphi}(x,t)dx\int_0^1 f_2(x)\psi(x,t)dxdt \\
&+ \int_{I_0} h^2\Big|\int_0^1 f_0(x)\psi(x,t)dx\Big|^2 dt + \int_{I_1} h^2\Big|\int_0^1 f_1(x)\psi(x,t)dx\Big|^2 dt \\
&+ \int_{I_2} h^2\Big|\int_0^1 f_2(x)\psi(x,t)dx\Big|^2 dt - \int_0^1 hy^0(x)\psi(x,0)dx + \int_0^1 h\psi^0y^1dx \\
&+ \epsilon\Big[||(I-\pi_E)(\hat{\varphi}^0 + h\psi^0)||_{L^2(0,1)} - ||(I-\pi_E)\hat{\varphi}^0||_{L^2(0,1)}\Big] \ge 0
\end{aligned}$$

where

$$I_0 \stackrel{\text{def}}{=} \Big\{t\in(0,T) : \Big|\int_0^1 f_0(x)\varphi dx\Big| > \max\Big(\Big|\int_0^1 f_1(x)\varphi dx\Big|, \Big|\int_0^1 f_2(x)\varphi dx\Big|\Big)\Big\}$$

$$I_1 \stackrel{\text{def}}{=} \Big\{t\in(0,T) : \Big|\int_0^1 f_1(x)\varphi dx\Big| > \max\Big(\Big|\int_0^1 f_0(x)\varphi dx\Big|, \Big|\int_0^1 f_2(x)\varphi dx\Big|\Big)\Big\}$$

$$I_2 \stackrel{\text{def}}{=} \Big\{t\in(0,T) : \Big|\int_0^1 f_2(x)\varphi dx\Big| > \max\Big(\Big|\int_0^1 f_0(x)\varphi dx\Big|, \Big|\int_0^1 f_1(x)\varphi dx\Big|\Big)\Big\}$$

Let us define

$$\begin{aligned}
\hat{\mathcal{A}} &\stackrel{\text{def}}{=} \int_{I_0}\int_0^1 f_0(x)\hat{\varphi}dx\int_0^1 f_0(x)\psi dxdt + \int_{I_1}\int_0^1 f_1(x)\hat{\varphi}dx\int_0^1 f_1(x)\psi dxdt \\
&+ \int_{I_2}\int_0^1 f_2(x)\hat{\varphi}dx\int_0^1 f_2(x)\psi dxdt
\end{aligned}$$

After considering cases: $h > 0$, $h < 0$ and taking $h \to 0$, at the end, we have

$$\Big|\hat{\mathcal{A}} + \int_0^1 \psi^0y^1dx - \int_0^1 \psi(x,0)y^0(x)dx\Big| \le \epsilon\Big[||((I-\pi_E)\psi^0||_{L^2(0,1)}\Big]. \tag{2.8}$$

Now, if we take

$$u_0^\epsilon(t) = -\chi_{I_0}\int_0^1 f_0(x)\hat{\varphi}_\epsilon(x,t)dx, \tag{2.9}$$

$$u_1^\epsilon(t) = -\chi_{I_1}\int_0^1 f_1(x)\hat{\varphi}_\epsilon(x,t)dx, \tag{2.10}$$

$$u_2^\epsilon(t) = -\chi_{I_2}\int_0^1 f_2(x)\hat{\varphi}_\epsilon(x,t)dx, \tag{2.11}$$

where χ_{I_i} is the characteristic function defined on the set I_i which gets 1 in I_i and 0 otherwise for i = 0,1,2. Now, multiplying the heat equation (2.1) with initial data $y^0(x) \in L^2(0,1)$ by ψ which is the solution of adjoint system (2.3) with initial data ψ^0 and integrating by parts we finally get

$$\hat{\mathcal{A}} = \int_0^1 \psi(x,0)y^0(x)dx - \int_0^1 \psi^0 y(x,T)dx$$

and putting this identity into (2.8), and letting $E = 0$, we finally get

$$\Big|\int_0^1 \psi^0(y(x,T) - y^1)dx\Big| \leq \epsilon||\psi^0||_{L^2(0,1)}$$

for every $\psi^0 \in L^2(0,1)$ which is equivalent to (2.6) i.e., (2.1) is approximate controllable (see, e.g., [4]) □

□

From Lemma 2.2, we understand that, for approximate controllability of (2.1) it suffices to obtain (2.7). Observe that

$$I_0^1 \stackrel{\text{def}}{=} \Big\{t \in (0,T) : \Big|\int_0^1 f_0(x)\varphi(x,t)dx\Big| = \Big|\int_0^1 f_1(x)\varphi(x,t)dx\Big|\Big\},$$
$$I_1^2 \stackrel{\text{def}}{=} \Big\{t \in (0,T) : \Big|\int_0^1 f_1(x)\varphi(x,t)dx\Big| = \Big|\int_0^1 f_2(x)\varphi(x,t)dx\Big|\Big\},$$
$$I_2^0 \stackrel{\text{def}}{=} \Big\{t \in (0,T) : \Big|\int_0^1 f_2(x)\varphi(x,t)dx\Big| = \Big|\int_0^1 f_0(x)\varphi(x,t)dx\Big|\Big\},$$

are of positive measure. Now using (2.4), we have

$$\int_0^1 f_0(x)\varphi(x,t)dx \pm \int_0^1 f_1(x)\varphi(x,t)dx = \sum_{k\geq 1} \beta_k e^{k^2(t-T)}(f_{1,k} \pm f_{0,k})$$

The function $\int_0^1 \varphi(x,t)(f_0(x) \pm f_1(x))dx$ are time analytic for $t \leq T$ (see, e.g., [3]). Consequently, if they vanish for a set of time instants of positive measure, then they vanish for all $t \leq T$. It is then easy to see, by multiplying above identity by $e^{-\eta^2(t-T)}$ successively, starting from $\eta = 1$ and taking limits as $t \to -\infty$, that

$$\beta_k(f_{1,k} \pm f_{0,k}) = 0, \qquad \forall k \geq 1.$$

To conclude that $\beta_k = 0$ for all $k \geq 1$, it is sufficient to assume that

$$f_{1,k} \pm f_{0,k} \neq 0 \qquad \forall k \geq 1. \tag{2.12}$$

Similarly, we would have:

$$f_{2,k} \pm f_{1,k} \neq 0 \qquad \forall k \geq 1. \tag{2.13}$$
$$f_{0,k} \pm f_{2,k} \neq 0 \qquad \forall k \geq 1. \tag{2.14}$$

As a result, under the assumption of (2.12), (2.13) and (2.14), we prove that (2.7) satisfies. Now, we would like to say that for each $\epsilon > 0$, we must have the fact that $u_0^\epsilon(t), u_1^\epsilon(t), u_2^\epsilon(t)$ are uniformly bounded in $L^2(0,T)$.

Lemma 2.3 *Assume that the Fourier coefficient of the initial datum y^0 satisfying*

$$\sum_{k\geq 1} \frac{e^{2\pi^2 k^2 T}}{|f_{0,k}|^2 + |f_{1,k}|^2 + |f_{2,k}|^2} |y_k^0|^2 < \infty. \tag{2.15}$$

Then, $y^0 \in \hat{\mathcal{H}}'$ which is the dual space of $\hat{\mathcal{H}}$ and our approximate controls $u_0^\epsilon(t), u_1^\epsilon(t), u_2^\epsilon(t)$ would be uniformly bounded in $L^2(0,T)$.

Proof. We skip the proof of the first part, i.e., $y^0 \in \hat{\mathcal{H}}'$ which comes from direct application of Cauchy-Schwartz (CS) inequality. Now, let us prove that our approximate controls $u_0^\epsilon(t), u_1^\epsilon(t), u_2^\epsilon(t)$ are uniformly bounded in $L^2(0,T)$. Observe that at the minimizer $\hat{\varphi}_\epsilon^0$ we have
$J_\epsilon(\hat{\varphi}_\epsilon^0) \leq J_\epsilon(0) = 0$. This implies that

$$\frac{1}{6}\int_0^T \Big[\Big|\int_0^1 f_0(x)\hat{\varphi}_\epsilon dx\Big|^2 + \Big|\int_0^1 f_1(x)\hat{\varphi}_\epsilon dx\Big|^2 + \Big|\int_0^1 f_2(x)\hat{\varphi}_\epsilon dx\Big|^2\Big]dt$$
$$\leq \frac{1}{2}\int_0^T \max\Big[\Big|\int_0^1 f_0(x)\hat{\varphi}_\epsilon dx\Big|^2, \Big|\int_0^1 f_1(x)\hat{\varphi}_\epsilon dx\Big|^2, \Big|\int_0^1 f_2(x)\hat{\varphi}_\epsilon dx\Big|^2\Big]dt$$
$$\leq \Big|\int_0^1 y^0(x)\hat{\varphi}_\epsilon(x,0)dx\Big|.$$

From (2.5), we have

$$\int_0^T \Big[\Big|\int_0^1 f_0(x)\hat{\varphi}_\epsilon dx\Big|^2 + \Big|\int_0^1 f_1(x)\hat{\varphi}_\epsilon dx\Big|^2 + \Big|\int_0^1 f_2(x)\hat{\varphi}_\epsilon dx\Big|^2\Big]dt$$
$$\leq \frac{\hat{C}\Big|\int_0^1 y^0(x)\hat{\varphi}_\epsilon(x,0)dx\Big|^2}{\Big|\sum_{k\geq 1}\beta_k^2 e^{-2\pi^2k^2T}\{|f_{0,k}|^2+|f_{1,k}|^2+|f_{2,k}|^2\}\Big|}$$

for suitable $\hat{C} > 0$ which is independent from $\{\beta_k\}_{k\geq 1}$.

Since $\{\omega_k(x)\}_{k\geq 1}$ form orthonormal basis in $L^2(0,1)$ after some simplification, we have

$$\int_0^T \Big[\Big|\int_0^1 f_0(x)\hat{\varphi}_\epsilon(x,t)dx\Big|^2 + \Big|\int_0^1 f_1(x)\hat{\varphi}_\epsilon(x,t)dx\Big|^2 + \Big|\int_0^1 f_2(x)\hat{\varphi}_\epsilon(x,t)dx\Big|^2\Big]dt$$
$$\leq \frac{\hat{C}\Big|\sum_{k\geq 1} y_k^0\beta_k\Big|^2}{\Big|\sum_{k\geq 1}\beta_k^2 e^{-2\pi^2k^2T}\{|f_{0,k}|^2+|f_{1,k}|^2+|f_{2,k}|^2\}\Big|}$$

But applying Cauchy-Schwarz inequality, we have

$$\frac{\hat{C}\Big|\sum_{k\geq 1} y_k^0\beta_k\Big|^2}{\Big|\sum_{k\geq 1}\beta_k^2 e^{-2\pi^2k^2T}\{|f_{0,k}|^2+|f_{1,k}|^2+|f_{2,k}|^2\}\Big|} = \frac{\hat{C}\Big|\sum_{k\geq 1} y_k^0\hat{\upsilon}_k\beta_k\hat{\upsilon}_k^{-1}\Big|^2}{\Big|\sum_{k\geq 1}(\beta_k\hat{\upsilon}_k^{-1})^2\Big|}$$
$$\overset{\text{CS}}{\leq} \frac{\hat{C}\Big|\sum_{k\geq 1}(y_k^0\hat{\upsilon}_k)^2\Big|\Big|\sum_{k\geq 1}(\beta_k\hat{\upsilon}_k^{-1})^2\Big|}{\Big|\sum_{k\geq 1}(\beta_k\hat{\upsilon}_k^{-1})^2\Big|}$$
$$= \hat{C}\sum_{k\geq 1}(y_k^0\hat{\upsilon}_k)^2 < \infty.$$

where

$$\hat{\upsilon}_k = \Big|\frac{e^{2\pi^2k^2T}}{|f_{0,k}|^2+|f_{1,k}|^2+|f_{2,k}|^2}\Big|^{\frac{1}{2}}$$

and $\hat{C} > 0$ which is independent from $\{\beta_k\}_{k\geq 1}$.
We conclude that $\forall \epsilon > 0$, $u_0^\epsilon(t), u_1^\epsilon(t)$ and $u_2^\epsilon(t)$ are uniformly bounded in $L^2(0,T)$□
□

As a result, if we would obtain the condition that $\{u_0^\epsilon\}_{\epsilon>0}$, $\{u_1^\epsilon\}_{\epsilon>0}$ and $\{u_2^\epsilon\}_{\epsilon>0}$ are uniformly bounded in $L^2(0,T)$, by extracting subsequences, we would have $u_0^\epsilon \rightharpoonup u_0$, $u_1^\epsilon \rightharpoonup u_1$ and $u_2^\epsilon \rightharpoonup u_2$ weakly in $L^2(0,T)$. Hence using the continuous dependence of the solution of the heat equation, we can show that $y_\epsilon(x,T)$ converges to $y(x,T)$ weakly in $L^2(0,T)$ which implies that $y(x,T) = 0$ i.e., the limit controls u_0, u_1 and u_2 fulfill the null controllability requirement.
Consequently, we obtain the following result

Theorem 2.1 *Assume that $f_0(x)$, $f_1(x)$ and $f_2(x)$ are three control profiles in $L^2(0,1)$ in which their Fourier coefficients satisfy (2.12), (2.13), and (2.14). Let the initial datum y^0 be in the dual space of $\hat{\mathcal{H}}$. More precisely, let Fourier coefficients of y^0 satisfy (2.15). Then, for all $T > 0$, there exist switching controls*

$$u_0(t) = -\chi_{I_0} \int_0^1 f_0(x)\hat{\varphi}(x,t)dx,$$

$$u_1(t) = -\chi_{I_1} \int_0^1 f_1(x)\hat{\varphi}(x,t)dx,$$

$$u_2(t) = -\chi_{I_2} \int_0^1 f_2(x)\hat{\varphi}(x,t)dx,$$

where χ is the characteristic function. These controls satisfying the switching property (2.2) and solution of heat equation (2.1) satisfies

$$y(x,T) = 0.$$

These switching controls can be obtained by minimizing the functional $\hat{J}$ over $\hat{\mathcal{H}}$.

In general, we could examine the case in which $n \in \mathbb{N}$ control profiles given in $L^2(0,1)$. Consider the heat equation

$$\begin{cases} y_t - y_{xx} = \sum_{i=1}^n u_i(t) f_i(x), & 0 < x < 1, 0 < t < T, \\ y(0,t) = y(1,t) = 0, & 0 < t < T, \\ y(x,0) = y^0(x), & 0 < x < 1. \end{cases} \tag{2.16}$$

Here now, given an initial datum $y^0 \in L^2(0,1)$ we are looking for controls $\{u_i\}_{i=1}^{i=n} \in L^2(0,T)$ such that null controllability of heat equation holds, i.e, $y(x,T) = 0$ and switching condition satisfies:

$$u_i(t)u_j(t) = 0, \quad \forall i \neq j, \qquad \text{a.e. } t \in (0,T) \tag{2.17}$$

At first, we consider the approximate controllability problem. To obtain approximate controls, one should minimize an appropriate quadratic functional over suitable Hilbert space, and under some conditions on the Fourier coefficients of y^0, we will get our desired null switching controls satisfying switching property.

In conclusion, we obtain following general result for switching lumped controls.

Theorem 2.2 *Assume that $\{f_i(x)\}_{i=1}^{i=n}$ are n control profiles in $L^2(0,1)$ and their Fourier expansions are*

$$f_i(x) = \sum_{k\geq 1} f_{i,k}\omega_k(x), \qquad i \in \{1,2,..,n\},$$

and satisfying

$$(f_{i,k} \pm f_{j,k}) \neq 0, \quad i \neq j, \quad i,j \in \{1,2,..,n\}, \ \forall k \geq 1.$$

Now, let the initial datum y^0 *be in* H_n' *which is the dual space of the class of initial data given by*

$$H_n = \{\varphi^0 : \int_0^T \Big[\sum_{i=1}^n \Big| \int_0^1 f_i(x)\varphi(x,t)dx\Big|^2\Big] dt < \infty\}.$$

More precisely, let the Fourier coefficients of y^0 *satisfy*

$$\sum_{k\geq 1} \frac{e^{2\pi^2 k^2 T}}{\sum_{i=1}^{i=n} |f_{i,k}|^2} |y_k^0|^2 < \infty$$

Then, for all $T > 0$*, there exist switching controls* $\{u_i(t)\}_{i=1}^{i=n} \in L^2(0,T)$ *satisfying (2.17) and solution of heat equation with* $\{f_i(x)\}_{i=1}^{i=n}$ *control profiles satisfies null controllability condition, i.e.,* $y(x,T) = 0$*. These controls are*

$$u_i(t) = -\int_0^1 f_i(x)\hat{\varphi}(x,t)dx, \quad u_j(t) = 0, \ \forall j \neq i, \quad in \quad S_i, \quad \forall i \in \{1,2,..,n\}.$$

where $\{S_i\}_{i=1}^{i=n}$ *defined by*

$$S_i = \Big\{ t \in (0,T) : \Big| \int_0^1 f_i(x)\hat{\varphi}(x,t)\Big| > \max_{\substack{1\leq j\leq n \\ j\neq i}} \Big(\Big| \int_0^1 f_j(x)\hat{\varphi}(x,t)\Big| \Big) \Big\}$$

These switching controls can be obtained by minimizing the functional

$$\hat{J_s^n}(\varphi^0) = \frac{1}{2}\int_0^T \max_{1\leq i\leq n} \Big\{ \Big| \int_0^1 f_i(x)\varphi(x,t)dx\Big|^2 \Big\} dt - \int_0^1 y^0(x)\varphi(x,0)dx$$

over the Hilbert space H_n*.*

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