Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 41, Number 2, 2015, Pages 124–129

VARIATIONAL PRINCIPLE FOR TWO-PARAMETER SPECTRAL PROBLEM UNDER LEFT DEFINITENESS CONDITION

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Abstract. In the paper we consider two-parameter spectral problem

$$\begin{cases} \lambda_1 K_{r1}\varphi_r + \lambda_2 K_{r2}\varphi_r = \varphi_r, \quad \varphi_r \in H_r, \\ r = 1, 2 \end{cases}$$

with compact self-adjoint operators in Hilbert space under left definiteness condition. The analog of the variational principle was obtained for the two-parameter spectral problem.

1. Introduction

In some problems given by means of differential operators, at separation of variables we get multi-parameter spectral problems mainly. All these problems are reduced to the from of weakly connected system of integral equations with spectral parameters. The number of parameters equals the number of the variables of the given initial problem.Therefore, it is interesting to consider the spectral problem of the form

$$\sum_{s=1}^{n} \lambda_s K_{rs}\varphi_r = \varphi_r; \quad \varphi_r \in H_r;\ r = 1, ..., n \tag{1.1}$$

where K_{rs} are compact self-adjoint operators in Hilbert space H_r, $r = 1, ..., n$.

It is known that for any compact self-adjoint operator A, the number

$$\lambda_0 = \sup_{\varphi \in H} \frac{(A\varphi, \varphi)}{(\varphi, \varphi)}$$

is an eigenvalue, i.e. we can solve the spectral problem $A\varphi = \lambda\varphi$ by the variational method (see[5],[7]). There arises a question if we can generalize this principle for problem (1.1).

At different conditions of definiteness, the variational principle for a multi-parameter problem of the form

2010 *Mathematics Subject Classification.* 47J22, 47A10, 47A75.

Key words and phrases. multiparameter problem, spectrum, eigen elements, eigenvalues, compact operators, variational principle.

$$T_m x_m = \sum_{k=1}^{n} \lambda_k V_{mk} x_m, \quad 0 \neq x_m \in H_m \quad m = 1; 2; ...; n,$$

was studied in the papers [1-4], where V_{mk}, $m, k = 1, ..., n$ bounded operators, T_m $m = 1, ..., n$ are densely determined linear operators in Hilbert space H_m $m = 1, ..., n$ In these papers, both in finite-dimensional and in infinite-dimensional cases the principle is given in the form of R^n -valued function in $H_1 \otimes, ..., \otimes H_m$ or $H_1 \times, ..., \times H_n$ as the extremum of the function in definite sense. In the papers [2] and [4] it is proved that to each multi-index $i = (i_1, i_2, ..., i_n)$ ($i_r \geq 0$ are integers) there corresponds such an eigenvalue λ^i and eigenvector $x_i = x_{1i_1} \otimes ... \otimes x_{ni_n}$ that

$$\rho_r^{i_r}\left(\lambda^i\right) = 0, \quad W_r\left(\lambda^i\right) x_{ri_r} = 0,$$

where

$$\rho_r^{ir}\left(\lambda^i\right) = \max_{\substack{y_m \in H_r \\ 1 \leq m \leq i_r}} \; \min_{\substack{u_r \in S_r \cap D(T_r) \\ (u_r, y_j) = 0}} \left(W_r(\lambda) u_r, u_r\right),$$

S_r is a unit ball in $H_r, W_r(\lambda) = T_r - \sum\limits_{k=1}^{n} \lambda_k V_{rk}$. The following geometrical property is also proved. If $j \geq i$ (i.e.$j_r \geq i_r$, $r = 1; ...; n$), then $\lambda^j \in \lambda^i + C$. In particular, the spectrum $\sigma \subset \lambda^0 + C$, where

$$C = \{\ a \in R^n : V(u)a \geq 0 \ for\, some\, u_r \in S_r, r = 1, ..., n\}$$

It was proved that under some conditions (singularity conditions) , the cones C is non-singular, i.e. the cones C is convex, closed and does not contain a straight line. In the paper, a more constructive variational method for two-parameter problem (1.1) is found and this enables to find the eigen value by means of the extremum of a simple functional.

In the paper we study such a problem for a two-parameter problem.

2. Two-parameter variational problem

Let us consider the two-parameter problem

$$\begin{cases} \lambda_1 K_{r1} \varphi_r + \lambda_2 K_{r2} \varphi_r = \varphi_r, \quad \varphi_r \in H_r, \\ r = 1, 2 \end{cases} \tag{2.1}$$

where H_r, $r = 1; 2$ are Hilbert spaces, K_{r1}, K_{r2} are compact self-adjoint operators in the space H_r, $r = 1; 2$.

By virtue of compactuness of the operators K_{r1}, K_{r2}; $r = 1; 2$, the operators

$$\Delta_0 = K_{11} \otimes K_{22} - K_{12} \otimes K_{21}; \;\; \Delta_1 = J_1 \otimes K_{22} - K_{12} \otimes J_2$$

$$\Delta_2 = K_{11} \otimes J_2 - J_1 \otimes K_{21}$$

are bounded operators in the space $H = H_1 \otimes H_2$. Here the symbol $\otimes$ means a tensor product of two spaces or two operators, respectively. Let in problem (2.1) the left definiteness condition be fulfilled, i.e.

$$\Delta_1 > 0; \qquad \Delta_2 > 0; \tag{2.2}$$

Let us consider the functional

$$f(\varphi) = \frac{(\Delta_0\varphi, \varphi)^2}{(\Delta_1\varphi, \varphi)\ (\Delta_2\varphi, \varphi)} \tag{2.3}$$

Theorem 2.1. *Under conditions* (2.2)

(1) *the functional $f(\varphi)$ is bounded,*
(2) *there exists a sequence $\{\varphi^n\} \subset H$ and $\exists \varphi^0 \in H$ such that*

$$\varphi^n \to \varphi^0 \in H,$$
$$f(\varphi^n) \to f\left(\varphi^0\right) = \sup_{\varphi\in H} f(\varphi)$$

(3) *$\varphi^0 = \varphi_1^0 \otimes \varphi_2^0 \in H$ is an eigenfunction of problem (2.1), corresponding to the eigenvalue*

$$\left(\lambda_1^0,\ \lambda_2^0\right) = \left(\frac{\left(\Delta_1\varphi^0, \varphi^0\right)}{\left(\Delta_0\varphi^0, \varphi^0\right)},\ \frac{\left(\Delta_2\varphi^0, \varphi^0\right)}{\left(\Delta_0\varphi^0, \varphi^0\right)}\right)$$

Proof. It is easy to prove that

$$\left|\frac{(\Delta_0\varphi, \varphi)}{(\Delta_1\varphi, \varphi)}\right| \le C; \qquad \left|\frac{(\Delta_0\varphi, \varphi)}{(\Delta_2\varphi, \varphi)}\right| \le C$$

Indeed, under conditions (2.2) the following inequalities are fulfilled

$$K_{r,r}^t \ge 0; \quad K_{r,3-r}^t \le 0; \quad r = 1; 2,$$

where $K_{1,r}^t = K_{1,r} \otimes J_2; K_{2,r}^t = J_1 \otimes K_{21}$.

Using the permutability of the operators $K_{1,r}^t;\ K_{2,r}^t$, we get

$$\begin{gathered}
|(\Delta_0\varphi, \varphi)| = \left|\left((K_{1,1}^t K_{2,2}^t - K_{1,2}^t K_{2,1}^t)\varphi, \varphi\right)\right| \le \\
\le \left|\left(K_{1,1}^t K_{2,2}^t \varphi, \varphi\right)\right| + \left|\left(K_{1,2}^t K_{2,1}^t \varphi, \varphi\right)\right| = \\
= \left|\left(K_{1,1}^t (K_{2,2}^t)^{\frac{1}{2}} \varphi, (K_{2,2}^t)^{\frac{1}{2}} \varphi\right)\right| + \left|\left(-K_{2,1}^t (-K_{1,2}^t)^{\frac{1}{2}}\, \varphi, (-K_{1,2}^t)^{\frac{1}{2}}\, \varphi\right)\right| \le \\
\le \left\|K_{1,1}^t\right\| (K_{2,2}^t \varphi, \varphi) + \left\|K_{2,1}^t\right\| \ (-K_{1,2}^t \varphi, \varphi) \le \\
\le C_1 \ \left[(K_{2,2}^t \varphi, \varphi) - (K_{1,2}^t \varphi, \varphi)\right] = C_1 (\Delta_1\varphi, \varphi) \\
|(\Delta_0\varphi, \varphi)| \le\ C_1 \,(\Delta_1\varphi, \varphi),\ \ \forall \varphi \in H = H_1 \otimes H_2
\end{gathered}$$

In the same way we can prove $|(\Delta_0\varphi, \varphi)\,| \le\ C_2\,(\Delta_2\varphi, \varphi), \quad \forall\varphi \in\ H = H_1 \otimes H_2$

Consequently the functional $f(\varphi)$ is bounded. □

Let $\sup\limits_{\varphi\in H} f(\varphi) = \alpha$. Then there exist a sequence $\{\varphi^n\} \in H$ such that

$$\lim_{n\to\infty} f(\varphi^n) = \alpha \tag{2.4}$$

Without loss of generality, we can assume $|\varphi^n|_H = 1$. Therefore, from the sequence $\{\varphi^n\}$ we can extract a weakly convergent subsequence. Without loss of generality, we assume that the sequence $\{\varphi^n\}$ weakly converges to the element $\varphi^0 \in H$. Prove that $f\left(\varphi^0\right) = \alpha$. Let, vice versa, the inequality

$$f\left(\varphi^0\right) < \alpha \tag{2.5}$$

be fulfilled.

The operator Δ_0 is a compact operator, therefore for the weakly convergent sequence $\{\varphi^n\}$ the relation

$$\lim_{n\to\infty}(\Delta_0\varphi^n,\varphi^n)=\left(\Delta_0\varphi^0,\varphi^0\right)$$

is fulfilled.

The operators Δ_1 and Δ_2 are not compact operators. But inspite of this fact, we prove that under conditions (2.4), (2.5) the relations

$$\begin{array}{ll}(\Delta_j\varphi^n,\varphi^n)\to\left(\Delta_j\varphi^0,\varphi^0\right); & (\Delta_j\varphi^n,\varphi^n)\to\left(\Delta_j\varphi^0,\varphi^0\right)\\ j=1,2 & j=1,2\end{array}$$

are fulfilled.

Δ_1 and Δ_2 are positive and bounded operators. Therefore, from the sequences $(\Delta_j\varphi^n,\varphi^n)$, $\quad j=1,2$ one can choose a convergent subsequence. Consequently, without loss of generality, assume that these sequences converge to some numbers d_j, i.e.

$$(\Delta_j\varphi^n,\varphi^n)\to d_j,\quad j=1,2$$

Compare the numbers d_j and $\left(\Delta_j\varphi^0,\varphi^0\right)$, $\quad j=1,2$.

Note that the inequalities

$$d_1>\left(\Delta_1\varphi^0,\varphi^0\right),\ d_2>\left(\Delta_2\varphi^0,\varphi^0\right)$$

may not be fulfilled simultaneously. As therewith

$$\frac{\left(\Delta_0\varphi^0,\varphi^0\right)^2}{(\Delta_1\varphi^0,\varphi^0)(\Delta_2\varphi^0,\varphi^0)}>\frac{\left(\Delta_0\varphi^0,\varphi^0\right)^2}{d_1,d_2}=\frac{\lim\limits_{n\to\infty}(\Delta_0\varphi^n,\varphi^n)^2}{\lim\limits_{n,\to\infty}(\Delta_1\varphi^n,\varphi^n)\lim\limits_{n,\to\infty},(\Delta_2\varphi^n,\varphi^n)}=$$
$$=\lim_{n\to\infty}\frac{(\Delta_0\varphi^n,\varphi^n)^2}{(\Delta_1\varphi^n,\varphi^n)(\Delta_2\varphi^n,\varphi^n)}=\alpha$$

This contradicts equality (2.5), i.e. definition of the number α. In the similar way, it is proved that the relations

$$d_1>\left(\Delta_1\varphi^0,\varphi^0\right),\ d_2=\left(\Delta_2\varphi^0,\varphi^0\right)$$

or

$$d_1=\left(\Delta_1\varphi^0,\varphi^0\right),\ d_2>\left(\Delta_2\varphi^0,\varphi^0\right)$$

may not be fulfilled simultaneously.

None of ineqalities

$$d_1<\left(\Delta_1\varphi^0,\varphi^0\right)\,or\ d_2<\left(\Delta_2\varphi^0,\varphi^0\right)$$

not be true. As if the inequality

$$d_2<\left(\Delta_2\varphi^0,\varphi^0\right)\tag{2.6}$$

is fulfilled, then there exists a natural number N such that for all $n>N$ the inequality

$$(\Delta_2\varphi^n,\varphi^n)<\left(\Delta_2\varphi^0,\varphi^0\right)$$

is fulfilled.

Using this inequality, we write

$$\left(\Delta_2\left(\varphi^n-\varphi^0\right),\varphi^n-\varphi^0\right)=(\Delta_2\varphi^n,\varphi^n)-2\left(\Delta_2\varphi^n,\varphi^0\right)+\left(\Delta_2\varphi^0,\varphi^0\right)<$$
$$<2\left(\Delta_2\varphi^0,\varphi^0\right)-2\left(\Delta_2\varphi^n,\varphi^0\right)\to2\left(\Delta_2\varphi^0,\varphi^0\right)-2\left(\Delta_2\varphi^0,\varphi^0\right)=0$$

i.e.

$$\left(\Delta_2\left(\varphi^n-\varphi^0\right),\left(\varphi^n-\varphi^0\right)\right)\to0$$

The operator Δ_2 is a positive self-adjoint operator and consequently there exists the operator $\Delta_2^{1/2}$ and this operator determines a new norm in the space H.

$$\|\Psi\|_{\Delta_2} = (\Delta_2\Psi, \Psi)^{1/2}; \quad \forall\Psi \in H$$

For this norm the triangle inequality is fulfilled:

$$\left| \|\varphi^n\|_{\Delta_2} - \|\varphi^0\|_{\Delta_2} \right| \leq \|\varphi^n - \varphi^0\|_{\Delta_2}$$

Therefore

$$\left| (\Delta_2\varphi^n, \varphi^n)^{1/2} - \left(\Delta_2\varphi^0, \varphi^0\right)^{1/2} \right| \leq \left(\Delta_2\left(\varphi^n - \varphi^0\right), \left(\varphi^n - \varphi^0\right)\right) \to 0$$

or

$$\left(\Delta_2\varphi^n, \varphi^0\right)^{1/2} \to \left(\Delta_2\varphi^0, \varphi^0\right)^{1/2}$$

$$\lim_{n\to\infty} (\Delta_2\varphi^n, \varphi^n) = \left(\Delta_2\varphi^0, \varphi^0\right) \Rightarrow \left(\Delta_2\varphi^0, \varphi^0\right) = d_2 \tag{2.7}$$

This equality contradicts condition $d_2 > \left(\Delta_2\varphi^0, \ \varphi^0\right)$.

In the same way we can prove that $d_1 > \left(\Delta_1\varphi^0, \ \varphi^0\right)$ is impossible

So, we get

$$d_1 = \lim_{n\to\infty} (\Delta_1\varphi^n, \varphi^n) = \left(\Delta_1\varphi^0, \varphi^0\right)$$

$$d_2 = \lim_{n\to\infty} (\Delta_2\varphi^n, \varphi^n) = \left(\Delta_2\varphi^0, \varphi^0\right)$$

From the relation

$$\lim_{n\to\infty} \frac{(\Delta_0\varphi^n, \varphi^n)^2}{(\Delta_1\varphi^n, \varphi^n)(\Delta_2\varphi^n, \varphi^n)} = \alpha$$

we write $f\left(\varphi^0\right) = \alpha$. This contradicts the proposition $f\left(\varphi^0\right) < \alpha$. So, the equality

$$\lim_{n\to\infty} \frac{(\Delta_0\varphi^n, \varphi^n)^2}{(\Delta_1\varphi^n, \varphi^n)(\Delta_2\varphi^n, \varphi^n)} = \frac{\left(\Delta_0\varphi^0, \varphi^0\right)}{\left(\Delta_1\varphi^0, \varphi^0\right)\left(\Delta_2\varphi^0, \varphi^0\right)}$$

is valid.

Now prove that the elements φ^0 is an eigenfunction of problem corresponding to the eigenvalue

$$(\lambda_1; \lambda_2) = \left(\frac{\left(\Delta_1\varphi^0, \varphi^0\right)}{\left(\Delta_0\varphi^0, \varphi^0\right)}; \frac{\left(\Delta_2\varphi^0, \varphi^0\right)}{\left(\Delta_0\varphi^0, \varphi^0\right)} \right) \tag{2.8}$$

Indeed, pair (2.8) may not belong to the set of regular points as for the above considered sequence $\{\varphi^n\} \subset H$ the relation

$$\frac{\left(\left(\Delta_1\varphi^0, \varphi^0\right)\right)}{\left(\Delta_0\varphi^0, \varphi^0\right)} (\Delta_0\varphi^n, \varphi^n) - \left(\Delta_1\varphi^0, \varphi^0\right) \to 0 \qquad i = 1, 2$$

is fulfilled, i.e. operators $\lambda_1\Delta_0 - \Delta_1$ may not have bounded inverse.

It is known that the spectral set of problem (1.1) and of the problem

$$\begin{cases} \lambda_1 K_{r1}^t\varphi + \lambda_2 K_{r2}^t\varphi = \varphi, & \varphi \in H, \\ r = 1, 2 \end{cases}$$

coincide, and they consist only of eigen elements (see[2]). Therefore, pair (2.8) being the element of the spectral set is an eigenvalues of this problem, consequently, φ^0 is an eigenfunction of problem (2.1). The theorem is proved.

It is known that the points

$$\left\{ (\lambda_1, \lambda_2) / \lambda_1 = \frac{(\Delta_1 \varphi, \varphi)}{(\Delta_0 \varphi, \varphi)};\ \lambda_2 = \frac{(\Delta_2 \varphi, \varphi)}{(\Delta_0 \varphi, \varphi)};\ \forall \varphi \in H = H_1 \otimes H_2 \right\}$$

set is called a numerical set. The value of the functional (2.3) for each $\varphi \in H$ is the product of coordinates of the point of numerical domain corresponding to this φ. For each indicated c, the points of the numerical domain satisfying the condition

$$f(\varphi) = \frac{(\Delta_1 \varphi, \varphi)}{(\Delta_0 \varphi, \varphi)} \cdot \frac{(\Delta_2 \varphi, \varphi)}{(\Delta_0 \varphi, \varphi)} = c = const$$

will be on the hyperbola $\lambda_1 \cdot \lambda_2 = c$. Therefore functional (2.3) is said to be a hyperbolic functional.

We can consider the above theorem as confirmation of the known (see [1],[3]) theorem on the largest eigenvalue and eigenfunction for two-parameter problem (2.1).

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Received: October 8, 2015; Accepted: November 30, 2015