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NECESSARY CONDITIONS OF RIESZ PROPERTY OF ROOT VECTOR-FUNCTIONS OF DIRAC DISCONTINUOUS OPERATOR WITH SUMMABLE COEFFICIENT

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Abstract. In the paper we consider Dirac discontinuous operator on a finite interval $G = (a, b)$. It is assumed that its coefficient is a complex-valued matrix-function summable on G. Necessary conditions of the Riesz property of the systems of vector-functions of the given discontinuous operator are established.

1. Introduction and formulation of results

The problem of validity of the Riesz inequality for the system of functions not possessing the completeness and orthonormalization properties was stated in the paper [2] and this inequality was proved for root-functions of the Laplace operator. In [3] necessary and sufficient conditions of the Bessel property and unconditional basicity in L_2 of the system of root functions of second order ordinary differential equations was established. Later on, these and other issues for ordinary operators of second and higher orders were studied in the papers [1,4,6-9,13-15]. For the Dirac operator these issues were studied in [10-12].

In the present paper we study the validity of the Riesz inequality for the systems of root vector-functions of Dirac's discontinuous operator and establish necessary conditions for it to be fulfilled.

Let the points $\{\xi_i\}_{i=0}^{m}$, $a = \xi_0 < \xi_1 < ... < \xi_m = b$ realize the partition of the interval $G = (a, b)$. Denote $G_l = (\xi_{l-1}, \xi_l)$, $l = \overline{1, m}$. Denote by A_l a class of absolutely continuous two-component vector-functions on $\overline{G_l}$. Define the class $A(a, b)$ in the following way: if $f(x) \in A(a, b)$, then for every $l = \overline{1, m}$ there exists a vector function $f_l(x) \in A_l$ such that $f(x) = f_l(x)$ for $\xi_{l-1} < x < \xi_l$.

Consider the Dirac operator

$$Dy = B\frac{dy}{dx} + \Omega(x)\, y, \quad x \in \bigcup_{l=1}^{m} G_l,$$

where $B = (b_{ij})_{ij=1}^{2}$, $b_{i,3-i} = (-1)^{i-1}$, $b_{ii} = 0$, $y(x) = (y_1(x), y_2(x))^T$, $\Omega(x) = diag\,(p(x), q(x))$, moreover $p(x)$ and $q(x)$ are complex-valued functions summable on G.

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Following [3] we understand that the root vector-functions of the operator D regardless to the form of boundary conditions and "sewing conditions", more exactly, under the eigen vector-function of the operator D, responding to the eigen value λ we understand any identically nonzero complex-valued vector-function $\overset{0}{y}(x) \in A(a,b)$ satisfying almost everywhere in G the equation $D\overset{0}{y} = \lambda\overset{0}{y}$. Similarly, under the associated vector-function of order $r, r \geq 1$, responding to the same λ and the eigen vector-function $\overset{0}{y}(x)$ we understand any complex-valued vector-function $\overset{r}{y}(x) \in A(a,b)$ satisfying almost everywhere in G the equation $D\overset{r}{y} = \lambda\overset{r}{y} + \overset{r-1}{y}$.

Let $\{u_k(x)\}_{k=1}^{\infty}$ be an arbitrary system composed of the root (eigen and associated) vector-functions of the operator D, $\{\lambda_k\}_{k=\infty}^{\infty}$ be the appropriate system of eigen-values. Furthermore, every vector-function $u_k(x)$ enters into the system $\{u_k(x)\}_{k=1}^{\infty}$ together with its appropriate associated functions of less order. This means that each element $u_k(x)$ of the system $\{u_k(x)\}_{k=1}^{\infty}$ almost everywhere in G satisfies either the equation

$$Du_k = \lambda_k u_k, \tag{1.1}$$

in this case $u_k(x)$ is an eigen vector-function, or the equation

$$Du_k = \lambda_k u_k + u_{\nu(k)}, \tag{1.2}$$

where the number $\nu(k)$ is uniquely determined by the number k and $\nu(k_1) \neq \nu(k_2)$ for $k_1 \neq k_2$ (in this case $\lambda_k = \lambda_{\nu(k)}$, $u_k(x)$ is an associated vector-function of order $r \geq 1$, $u_{\nu(k)}(x)$ is an associated vector-function of order $r-1$) (see [5]). In the case when the lengths of the chains of associated functions is uniformly bounded, in equality (1.2) we should take $\nu(k) = k-1$, $u_{\nu(k)} = \theta_k u_{k-1}$. Therewith θ_k equals either 0 (in this case $u_k(x)$ is an eigen vector-function) or 1 (in this case $u_k(x)$ is an associated function, $\lambda_{k-1} = \lambda_k$).

Let $L_p^2(G)$, $p \geq 1$ be a space of two-component vector-functions $f(x) = (f_1(x), f_2(x))^{\varsigma}$ with the norm

$$\|f\|_{p,2} \equiv \|f\|_{p,2,G} = \left(\int_G |f(x)|^p\, dx\right)^{1/p}$$

$$\text{(in the case } p=\infty \quad \|f\|_{\infty,2,G} \equiv \|f\|_{\infty,2} = \sup_{x\in\overline{G}} vrai\,|f(x)|\,).$$

For $f(x) \in L_p^2(G)$, $g(x) \in L_p^q(G)$, $p^{-1} + q^{-1} = 1$, $p \geq 1$ the "scalar product" $(f,g) = \int\limits_a^b \sum\limits_{j=1}^{2} f_j(x)\overline{g_j(x)}dx$ was determined.

We say that for the given system $\{\varphi_k(x)\}_{k=1}^{\infty}$, $\varphi_k(x) \in L_q^2(a,b)$ the Riesz inequality is fulfilled if there exists a constant $M(p)$ such that for an arbitrary $f(x) \in L_p^2(G)$, $1 < p \leq 2$ the following inequality is fulfilled:

$$\sum_{k=1}^{\infty} |(f,\varphi_k)|^q \leq M\,\|f\|_{p,2}^2\,, \tag{1.3}$$

where $p^{-1} + q^{-1} = 1$.

In the paper the following theorems are proved:

Theorem 1.1. *Let the functions* $p(x)$ *and* $q(x)$ *belong to the class* $L_p(G)$, $1<p\leq 2$ *, the lengths of the chains of the root vector-functions be uniformly bounded. Then for the system* $\{\varphi_k(x)\}_{k=1}^{\infty}$*, where* $\varphi_k(x)=u_k(x)\|u_k\|_{q,2}^{-1}$*, to satisfy the Riesz inequality it is necessary*

$$\sum_{|\mathrm{Re}\lambda_k-\nu|\leq 1}\frac{|u_k(x)|^q}{\|u_k\|_{q,G}^q}\leq K_1\left(1+\sup_{|\mathrm{Re}\lambda_k-\nu|\leq 1}|\mathrm{Im}\lambda_k|\right),\quad x\in\overline{G}, \tag{1.4}$$

where ν *is an arbitrary real number;* K_1 *is a constant independent of* ν*;* $u_k(a)=u_k(a+0)$, $u_k(b)=u_k(b-0)$*;* $u_k(\xi_i)$ *equals any of the values* $u_k(\xi_i-0)$, $u_k(\xi+0)$, $i=\overline{1,m-1}$*; the summation is taken only over eigen vector-functions.*

Corollary 1.1. *Subject to conditions of Theorem 1.1 for the Riesz property of the system* $\{\varphi_k(x)\}_{k=1}^{\infty}$*, where* $\varphi_k(x)=u_k(x)\|u_k\|_{q,2}^{-1}$*, it is necessary*

$$\sum_{|\mathrm{Re}\lambda_k-\nu|\leq 1}\leq K_2\left(1+\sup_{|\mathrm{Re}\lambda_k-\nu|\leq 1}|\mathrm{Im}\lambda_k|\right), \tag{1.5}$$

where K_2 *is a constant independent of* ν *and summation is taken with regard to multiplicity of the number* λ_k*.*

Theorem 1.2. *Let the functions* $p(x)$ *and* $q(x)$ *belong to the class* $L_p(G)$, $1<p\leq 2$ *and let the anti-a priori estimation*

$$\left\|u_{\nu(k)}\right\|_{q,2,G_l}\leq C_0\left(1+|\lambda_k|\right)^{1/p}\|u_k\|_{q,2,G_l} \tag{1.6}$$

be fulfilled, where C_0 *is independent on order of the associated functions,* $l=\overline{1,m}$, $p^{-1}+q^{-1}=1$*. Then for the Riesz property of the system* $\{\varphi_k(x)\}_{k=1}^{\infty}$*, where* $\varphi_k(x)=u_k(x)\|u_k\|_{q,2}^{-1}$*, inequality (1.4) should be fulfilled, where summation is taken over all root vector-functions.*

Corollary 1.2. *Let a priori estimation (1.6) be fulfilled. Then for the Riesz property of the system* $\{\varphi_k(x)\}_{k=1}^{\infty}$ *where* $\varphi_k(x)=u_k(x)\|u_k\|_{q,2}^{-1}$*, the inequality (1.5) should be fulfilled, where summation is taken with regard to multiplicity of the number* λ_k*.*

Remark 1.1. Theorems 1.1 and 1.2 remain valid also in the case when $\Omega(x)$ is arbitrary, but not necessarily a diagonal matrix-function.

2. Auxiliary statements

Cite some necessary statements that will be used in the proofs of Theorems 1.1 and 1.2.

Statement 2.1 (see [10). If $p(x)$ and $q(x)$ belong to the class $L_1^{loc}(G_l)$ and the points $x-t$, x, $x+t$ are in the domain G_l, then the following formulas are valid

$$\begin{aligned}u_k(x\pm t)&=(\cos\lambda_k t\cdot I\mp\sin\lambda_k t\cdot B)\,u_k(x)+\\&+\int_x^{x\pm t}(\sin\lambda_k(t-|\xi-x|)\,I\pm\cos\lambda_k(t-|\xi-x|)\,B)\times\\&\times\left[\Omega(\xi)\,u_k(\xi)-u_{\nu(k)}(\xi)\right]d\xi;\end{aligned} \tag{2.1}$$

$$u_k(x-t)+u_k(x+t)=2u_k(x)\cos\lambda_k t+$$
$$+\int_{x-t}^{x+t}\left(\sin\lambda_k(t-|\xi-x|)I+\sin(\xi-x)\cos\lambda_k(t-|x-\xi|)\right)\times$$
$$\times\left[\Omega(\xi)u_k(\xi)-u_{v(k)}(\xi)\right]d\xi, \tag{2.2}$$

where I is a unit operator in E^2.

Statement 2.2 (see [10]). Let the functions $p(x)$ and $q(x)$ belong to the class $L_1(G_1)$. Then there exists the constants $C_i(n_k, G_l)$, $i=1,2$ independent of λ_k, such that the following estimations are valid

$$\left\|u_{\nu(k)}\right\|_{\infty,2,G_l}\leq C_1(n_k,G_l)\left(1+|\mathrm{Im}\lambda_\mathrm{k}|\right)\|u_k\|_{\infty,2,G_l} \tag{2.3}$$

$$\|u_k\|_{\infty,2,G_l}\leq C_2(n_k,G_l)\left(1+|\mathrm{Im}\lambda_\mathrm{k}|\right)^{1/r}\|u_k\|_{r,2,G_l} \tag{2.4}$$

where n_k is the order of the associated vector-function $u_k(x)$, $r\geq 1$, $l=\overline{1,m}$.

Statement 2.3. Let the functions $p(x)$ and $q(x)$ belong to the class $L_1(G)$. Then for the root functions $u_k(x)$ the following estimation is valid:

$$\|u_k\|_{\infty,2,G_l}\leq C_3\left(1+|\mathrm{Im}\lambda_\mathrm{k}|\right)^{1/r}\times$$
$$\times\left\{\|u_k\|_{r,2,G_l}+\left(1+|\mathrm{Im}\lambda_\mathrm{k}|\right)^{-1}\left\|u_{\nu(k)}\right\|_{r,2,G_l}\right\},\quad r\geq 1, \tag{2.5}$$

where the constant C_3 is independent of λ_k, order n_k of the root vector-function $u_k(x)$ and on $l, l=\overline{1,m}$.

Proof. Denote by R the number $\left(K\left(1+|\mathrm{Im}\lambda_\mathrm{k}|\right)\right)^{-1}$, where the number $K\geq\max\left\{1,2\left(\min\limits_l|G_l|\right)^{-1}\right\}$ is chosen so that $R\leq\min\limits_l{}^{|G_l|}/4$, and for any set $E\subset\overline{G}, mesE\leq 2R$, it is fulfilled $\omega(R)=\sup\limits_{E\subset\overline{G}}\left\{\|\Omega\|_{1,E}\right\}\leq\frac{1}{8}$, where $\|\Omega\|_{1,E}=\int\limits_E\left(|p(x)|+|q(x)|\right)dx$.

Let $x\in\left[\xi_{l-1},{}^{(\xi_{l-1}+\xi_l)}/_2\right]$. Having written formula (2.2) for the points x, $x+t$, $x+2t$, where $t\in[0,R]$ and used the inequality (for $|\mathrm{Imz}|\leq 1$)

$$|\sin z|, |\cos z|\leq 2, |\sin z|\leq 2|z| \tag{2.6}$$

and the Holder inequality, we get

$$|u_k(x)|\leq 4|u_k(x+t)|+|u_k(x+2t)|+4\omega(R)\|u_k\|_{\infty,2,G_l}+$$
$$+4(2t)^{1/r'}\left\|u_{\nu(k)}\right\|_{r,2,G_l},\quad r^{-1}+r'^{-1}=1.$$

Having applied the operation $R^{-1}\int\limits_0^R dt$, to each side of the last inequality, we have

$$|u_k(x)|\leq 5R^{-1/r}\|u_k\|_{r,2,G_l}+4\omega(R)\|u_k\|_{\infty,2,G_l}+4R^{1/r'}\left\|u_{\nu(k)}\right\|_{r,2,G_l}.$$

This inequality is fulfilled in the case $x\in\left[\frac{(\xi_{l-1}+\xi_l)}{2},\xi_l\right]$. Consequently, with regard to the inequality $\omega(R)\leq 1/8$, we get

$$\|u_k\|_{\infty,2,G_l}\leq 10R^{-1/r}\|u_k\|_{r,2,G_l}+8R^{1/r'}\left\|u_{\nu(k)}\right\|_{r,2,G_l}.$$

Hence, by virtue of definition of the number R it follows

$$\|u_k\|_{\infty,2,G_l} \le 10K^{1/r}\left(1+|\mathrm{Im}\lambda_k|\right)^{1/r}\|u_k\|_{r,2,G_l}+$$
$$+\frac{8}{K^{1/r'}\left(1+|\mathrm{Im}\lambda_k|\right)^{1/r'}}\left\|u_{\nu(k)}\right\|_{r,2,G_l}.$$

Taking into account $r\ge 1,\ K\ge 1$, hence we get estimation (2.5) with the constant $C_3=10K$. Statement 2.3 is proved. □

3. Proof of results

Proof of theorem 1.1.

Introduce the indices set $I_{\mu,\nu} = \{k : |\mathrm{Re}\lambda_k-\nu|\le 1, |\mathrm{Im}\lambda_k|\le\mu\}$, where $\mu = \sup\limits_{|\mathrm{Re}\lambda_k - v|\le 1}|\mathrm{Im}\lambda_k|$, where v is an arbitrary real number. In the case $\mu=\infty$ the statements of the theorem are obvious. Therefore, it suffices to consider the case $\mu<\infty$.

Choose the number $K\ge\max\left\{1,2\left(\min\limits_l|G_l|\right)^{-1}\right\}$ so that $R=(K(1+\mu))^{-1}\le$ $\le\min\limits_l\frac{|G_l|}{4}$ and for any $E\subset\overline{G}$, $|E|\le 2R$, $\omega_p(R)\le L^{-1}$ be fulfilled, where L is a positive number, whose choice of the value will be determined later, $\omega_p(R)=\sup\limits_{E\subset\overline{G}}\left\{\|\Omega\|_{P,E}\right\}$ $\|\Omega\|_{P,E}=\left(\int\limits_E\left(|p(x)|^p+|q(x)|^p\right)dx\right)^{1/p}$.

Let $x\in\left[\xi_{l-1},\frac{(\xi_{l-1}+\xi_l)}{2}\right]$. Write the mean value formula (2.2) for the points $x, x+t, x+2t$, where $t\in[0,R]$.

$$u_k(x)=2u_k(x+t)\cos\lambda_k t-u_k(x+2t)+$$
$$+\int\limits_x^{x+2t}\left\{\sin\lambda_k(t-|x+t-\xi|)I+sgn(\xi-x-t)\cos\lambda_k(t-|x+t-\xi|)B\right\}\times$$
$$\times\left[\Omega(\xi)u_k(\xi)-\theta_k u_{k-1}(\xi)\right]d\xi.$$

Adding and subtracting $2u_k(x+t)\cos\nu t$, $|\mathrm{Re}\lambda_k-\nu|\le 1$, in the right hand side of this equality, we represent this formula in the form

$$u_k(x)=2u_k(x+t)\cos\nu t-u_k(x+2t)+4u_k(x+t)\sin\frac{\lambda_k+\nu}{2}t\sin\frac{\nu-\lambda_k}{2}t+$$
$$+\int\limits_x^{x+2t}\left\{\sin\lambda_k(t-|x+t-\xi|)I+sgn(\xi-x-t)\cos\lambda_k(t-|x+t-\xi|)B\right\}\times$$
$$\times\left[\Omega(\xi)u_k(\xi)-\theta_k u_{k-1}(\xi)\right]d\xi.$$

At first in the third addend we apply formula (2.1) for $u_k(x+t)$, and the apply the operation $R^{-1}\int\limits_0^R dt$:

$$u_k(x)=R^{-1}\int\limits_G u_k(t)V(t)dt+$$

$$+R^{-1}\int_0^R (\cos\lambda_k tI - \sin\lambda_k tB)\sin\frac{\nu+\lambda_k}{2}t\sin\frac{\nu-\lambda_k}{2}tdtu_k(x)+$$

$$+R^{-1}\int_0^R\int_x^{x+t}(\sin\lambda_k(t-|\xi-x|)I+\cos\lambda_k(t-|\xi-x|)B)\times$$

$$\times[\Omega(\xi)u_k(\xi)-\theta_k u_{k-1}(\xi)]d\xi\sin\frac{\nu+\lambda_k}{2}t\sin\frac{\nu-\lambda_k}{2}tdt+$$

$$+R^{-1}\int_0^R\int_x^{x+2t}(\sin\lambda_k(t-|x+t-\xi|)I+sgn(\xi-x-t)\cos\lambda_k(t-|x+t-\xi|)B)\times$$

$$\times[\Omega(\xi)u_k(\xi)-\theta_k u_{k-1}(\xi)]d\xi dt = R^{-1}\int_{G_l}u_k(t)V(t)dt+J_1+J_2+J_3, \quad (3.1)$$

where $V(t) = 2\cos\nu(x-t)-\frac{1}{2}$ for $x\le t\le x+R$, $V(t)=-\frac{1}{2}$ for $x+R<t\le x+2R$ and $V(t)=0$ for $t\notin[x,x+2R]$.

Let $k\in I_{\mu,\nu}$. Using inequality (2.6), we find

$$|J_1|\le 16R|\nu-\lambda_k||u_k(x)|\le 16R(1+|\mathrm{Im}\lambda_\mathrm{k}|)|u_k(x)|\le\frac{16}{K}|u_k(x)|. \quad (3.2)$$

Having applied inequality (2.6) and the Holder inequality, we find

$$|J_2|\le 64R^{-1}\left(\omega_p(R)R\|u_k\|_{q,2,G_l}+\|\theta_k u_{k-1}\|_{\infty,2,G_l}\frac{R^2}{2}\right)\le$$

$$\le 64\left(\omega_p(R)\|u_k\|_{q,2,G_l}+\frac{R}{2}\|\theta_k u_{k-1}\|_{\infty,2,G_l}\right) \quad (3.3)$$

$$|J_3|\le 4\left(\omega_p(R)\|u_k\|_{q,2,G_l}+R\|\theta_k u_{k-1}\|_{\infty,2,G_l}\right). \quad (3.4)$$

Allowing for estimations (3.2)-(3.4), from (3.1) we get

$$|u_k(x)|\le R^{-1}\left|\int_{G_l}u_k(t)V(t)dt\right|+$$

$$+\frac{16}{K}|u_k(x)|+68\omega_p(R)\|u_k\|_{q,2,G_l}+36R\|\theta_k u_{k-1}\|_{\infty,2,G_l}.$$

This inequality is proved for $x\in\left[\frac{(\xi_{l-1}+\xi_l)}{2},\xi_l\right]$ in the same way. In this case $V(t)=-\frac{1}{2}$ for $x-2R\le t<x-R$,

$$V(t)=2\cos\nu(x-t)-\frac{1}{2} \text{ for } x-R\le t\le x, V(t)=0 \text{ for } t\notin[x-2R,x].$$

Chosing $K\ge 32$, we hence get that for any $x\in\overline{G_l}$ the following inequalities are valid:

$$|u_k(x)|\le\frac{2}{R}\left|\int_{G_l}u_k(t)V(t)dt\right|+136\omega_p(R)\|u_k\|_{q,2,G_l}+72R\|\theta_k u_{k-1}\|_{\infty,2,G_l}.$$

By virtue of the inequality $\left|\sum\limits_{i=1}^{m}a_i\right|^q \leq m^{q-1}\sum\limits_{i=1}^{m}|a_i|^q$ we get that for any $x\in\overline{G_l}$ it holds the inequality

$$\frac{|u_k(x)|}{\|u_k\|_{q,2}^q} \leq \frac{2^q}{R^q}4^{q-1}\left\{\left|\int\limits_{G_l}u_k^1(t)\,V(t)\,dt\right|^q\|u_k\|_{q,2}^{-q}+\left|\int\limits_{G_l}u_k^2(t)\,V(t)\,dt\right|^q\|u_k\|_{q,2}^{-q}\right\}+$$

$$+4^{q-1}\left\{(136\omega_p(R))^q+(72R)^q\|\theta_k u_{k-1}\|_{\infty,2,G_l}^q\|u_k\|_{q,2}^{-q}\right\}, \qquad (3.5)$$

where $u_k(t)=\left(u_k^1(t),u_k^2(t)\right)^T$.

Consider only eigen vector-functions. In this case $\theta_k=0$. Then for any $x\in\overline{G_l}$ by the Riesz inequality from (3.5) we have

$$\sum_{k\in J}|u_k(x)|^q\|u_k\|_{q,2}^{-q} \leq \frac{2^{q+1}}{R}4^{q-1}\|V\|_p^q+4^{q-1}(136\omega_p(R))^q\sum_{k\in J}1,$$

where $J\subset I_{\mu,\nu}$ is an arbitrary finite set of indices. From the expression of the function $V(t)$ it is seen that $\|V\|_p\leq 3R^{1/p}$. Consequently, it holds the inequality

$$\sum_{k\in J}|u_k(x)|^q\|u_k\|_{q,2}^{-q}\leq 3^q\cdot 2^{3q-1}R^{\frac{q}{p}-q}+4^{q-1}(136\omega_p(R))^q\sum_{k\in J}1,\, x\in\overline{G_l}. \qquad (3.6)$$

As the number R is realized independent of specific G_l, then inequality (3.6) will be valid for arbitrary $x\in\overline{G}$. Integrating this inequality with respect to $x\in\overline{G}$ and chosing R so small (the number K so large) that $4^{q-1}(136\omega_p(R))^q\leq 4^{q-1}\left(136L^{-1}\right)^q<\frac{1}{2mesG}$. As a result we arrive at the inequality

$$\sum_{k\in J}1\leq 3^q2^{3q}(mesG)\,R^{-1}. \qquad (3.7)$$

By virtue of arbitrariness of the finite set $J\subset I_{\mu,\nu}$ and definition of the number R, we get

$$\sum_{k\in I_{\mu,\nu}}1\leq const\left(1+\sup_{k\in I_{\mu,\nu}}|\mathrm{Im}\lambda_{\mathrm{k}}|\right), \qquad (3.8)$$

where the summation is taken only over eigen vector-functions.

As the lengths of the chains of the root vector-functions are uniformly bounded, then estimation (1.5) follows from (3.8). Allowing for (3.7), from inequality (3.6) we find that for $x\in\overline{G}$

$$\sum_{k\in J}|u_k(x)|^q\|u_k\|_{q,2}^{-q}\leq\left(3^q2^{3q-1}+3^q2^{3q-1}\right)R^{-1}=3^q2^{3q}R^{-1},$$

where the summation is taken only over eigen vector-functions. Hence by virtue of arbitrariness of the finite set $J\subset I_{\mu,\nu}$ and definition of the number R, estimation (1.4) follows for $x\in\overline{G}$. As this estimation is valid at every $\overline{G_l}$, $l=\overline{1,m}$, then it is fulfilled on $\overline{G}$, if we consider that $u_k^{(a)}=u_k(a+0)$, $u_k(b)=u_k(b-0)$, $u_k(\xi_i)$, $i=\overline{1,m-1}$, equals any of the values $u_k(\xi_i-0)$, $u_k(\xi_i+0)$. Theorem 1.1 is proved.

Proof of theorem 1.2. At first we notice that in formula (3.1) the expression $\theta_k u_{k-1}(\xi)$ should be replaced by $u_{\nu(k)}(\xi)$. Then for J_2 and J_3 the following estimations are valid

$$|J_2| \leq 64\left(\omega_p(R)\|u_k\|_{q,2,G_l} + R^{1/p}\left\|u_{\nu(k)}\right\|_{q,2,G_l}\right),$$

$$|J_3| \leq 4\left(\omega_p(R)\|u_k\|_{q,2,G_l} + R^{\frac{1}{p}}\left\|u_{\nu(k)}\right\|_{q,2,G_l}\right).$$

And instead of (3.5) the following inequality will hold

$$\frac{|u_k(x)|^q}{\|u_k\|_{q,2}^q} \leq \frac{2^{3q-2}}{R^q}\left\{\left|\int_{G_l} u_k^1(t)V(t)\,dt\right|^q \|u_k\|_{q,2}^{-q} + \left|\int_{G_l} u_k^2(t)V(t)\,dt\right|^q \|u_k\|_{q,2}^{-q}\right\} +$$
$$+2^{2q-2}\left\{(136\omega_p(R))^q + \left(136R^{\frac{1}{p}}\right)^q \left\|u_{\nu(k)}\right\|_{q,2,G_l}^q \|u_k\|_{q,2}^{-q}\right\} \qquad (3.5')$$

Here we use anti a-priori estimation (1.6). As a result we have

$$\frac{|u_k(x)|^q}{\|u_k\|_{q,2}^q} \leq \frac{2^{3q-2}}{R^q}\left\{\left|\int_{G_l} u_k^1(t)V(t)\,dt\right|^q \|u_k\|_{q,2}^{-q} + \left|\int_{G_l} u_k^2(t)V(t)\,dt\right|^q \|u_k\|_{q,2}^{-q}\right\} +$$
$$+2^{2q-2}\left\{(136\omega_p(R))^q + \left(136R^{\frac{1}{p}}\right)^q C_0^q\left(1+|\mathrm{Im}\lambda_k|\right)^{\frac{q}{p}}\right\} \leq$$
$$\leq \frac{2^{3q-2}}{R^q}\left\{\left|\int_{G_l} u_k^1(t)V(t)\,dt\right|^q \|u_k\|_{q,2}^{-q} + \left|\int_{G_l} u_k^2(t)V(t)\,dt\right|^q \|u_k\|_{q,2}^{-q}\right\} +$$
$$+2^{2q-2}\left\{(136\omega_p(R))^q + (136C_0)^q\left(R(1+\mu)\right)^{\frac{q}{p}}\right\} \leq$$
$$\leq \frac{2^{3q-2}}{R^q}\left\{\left|\int_{G_l} u_k^1(t)V(t)\,dt\right|^q \|u_k\|_{q,2}^{-q} + \left|\int_{G_l} U_k^2(t)V(t)\,dt\right|^q \|u_k\|_{q,2}^{-q}\right\} +$$
$$+2^{2q-2}\left\{(136\omega_p(R))^q + (136C_0)^q K^{\frac{q}{p}}\right\},$$

where $x \in \overline{G_l}$, $k \in I_{\mu,\nu}$.

By virtue of the Riesz inequality, from this inequality if follows

$$\sum_{k\in J}|u_k(x)|^q\|u_k\|_{q,2}^{-q} \leq \frac{2^{3q-1}}{R^q}\|V\|_p^q +$$
$$+2^{2q-2}\left\{(136\omega_p(R))^q + (136C_0)^q K^{-\frac{q}{p}}\right\}\sum_{k\in J}1$$

where $J \subset I_{\mu,\nu}$ is an arbitrary finite set of indices.

Here taking $\|V\|_p \leq 3R^{\frac{1}{p}}$ into account, we find

$$\sum_{k\in J}|u_k(x)|^q\|u_k\|_{q,2}^{-q} \leq$$
$$\leq 3^q 2^{3q-1}R^{-1} + 2^{2q-2}\left\{(136\omega_p(R))^q + (136C_0)^q K^{-\frac{q}{p}}\right\}\sum_{k\in J}1,\ x \in \overline{G_l} \qquad (3.6')$$

This inequality will be valid on $\overline{G}$ as the choice of the number K is independent on specific G_l.

Integrating this inequality with respect to $x \in \overline{G_l}$ and chosing K so large (R rather small) so that

$$2^{2q-2}\left\{(136\omega_p(R))^q + (136C_0)^q K^{-\frac{q}{p}}\right\} < \frac{1}{2mesG},$$

we arrive at inequality (3.7). Hence in its turn estimation (3.8) follows. Therewith, summation is taken not only over eigen vector-functions, but also over all root vector-functions. The proof of Theorem 1.2 is completed in the same way as the proof of theorem 1.1, with regard to the last obtained inequalities (3.8) and (3.6′). Theorem 1.2 is proved.

References

[1] V.D. Budaev, Criteria for the Bessel property and the Riesz basis property of system of root functions of differential operators. *I, Diff. Uravnenia.*, **27** (12) (1991), 2033-2044.

[2] V.A. Il'in, Inequalities of Bessel and Hausdorff-Young-Riesz type in the system of eigenfunctions of the Laplace operator for functions in the class of radial functions, *Dokl. Akad. Nauk, SSSR,* **291** (2) (1986), 284-288.

[3] V.A. Il'in, Unconditional basis property on a closed interval of systems of eigen and associated functions of a second-order differential operator, *Dokl. Akad. Nauk, SSSR,* **273** (5) (1983), 1048-1053.

[4] N.B. Kerimov, Unconditional basis property of a system of eigen and associated functions of a fourth-order differential operator. *Dokl. Akad. Nauk, SSSR,* **286** (4) (1986), 803-808.

[5] N.B. Kerimov, On the Basis properlry and uniform minimality of system of root functions of differential operators. *I, Diff. Uravnenia,* **32** (3) (1996), 317-322.

[6] V.M. Kurbanov, E.D. Ibadov, On the properties of systems of root fuirctions of a second-order discontinuous operator, *ISSN 1064-5624, Doklady Mathematics*, **80** (1) (2009), 516-529.

[7] V.M. Kurbanov, On the distribution of eigenvalues and a criterion for the Bessel property of root functions of a differential operator. *I, Diff. Uravnenia,* **41** (4) (2005), 464-478.

[8] V.M. Kurbanov, On the distribution of eigenvalues and a criterion for the Bessel property of root functions of a differential operator. *II, Diff. Uravnenia,* **41** (5) (2005), 623-659.

[9] V.M. Kurbanov, On the Hausdorff-Young inequality for systems of root vector functions for an nth order differential operator, *Diff. Uravnenia,* **33** (3) (1997), 358-367.

[10] V.M. Kurbanov, On the Bessel property and the unconditional basis property of systems of root vector functions of the *Dirac operator, Diff. Uravnenia,* **32** (12) (1996), 1608-1617.

[11] V.M. Kurbanov, A.I. Ismailova, Properties of root vector functions for the one-dimensional Dirac operator, *Dokl. Akad. Nauk,* **433** (6) (2010), 736-740.

[12] V.M. Kurbanov, A.I. Ismailova, Componentwise uniform equiconvergence of expansions in root vector functions of the Dirac operator with the trigonometric expansion, *ISSN 0012-2661, Differential Equations,* **48** (5) (2012), 655-669.

[13] L.V. Kritskov, A uniform estimate for the order of associated functions, and the distribution of eigenvalues of a one-dimensional Schrddinger operator, *Diff. Uravnenia,* **25** (7) (1989), 1121-1129.

[14] I.S. Lomov, The Bessel inequality, the Riesz theorem, and the unconditional asis property for root vectors of ordinary differential operatorc, Vestnik Moskov. *Univ. Ser. I Mat. Mekh.*, (5) (1992), 42-52.

[15] A.M. Sarsenbi, Criteria for the Riesz basis property of systems of eigen and associated functions for higher-order differential operators on an interval, *Dokl. Aknd. Nauk,* **419** (5) (2008), 601-603.

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