Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 42, Number 1, 2016, Pages 116–125

GLOBAL BIFURCATION OF A SECOND ORDER NONLINEAR ELLIPTIC PROBLEM WITH AN INDEFINITE WEIGHT FUNCTION

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Abstract. In this paper we consider bifurcation of solutions of nonlinear eigenvalue problems for second order elliptic operator with indefinite weight function and Dirichlet boundary condition. We show the existence of an unbounded continua of positive or negative solutions bifurcating from trivial solutions corresponding to the principal eigenvalues.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, and let L be the differential operator in Ω defined by

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + c(x)\,u.$$

We assume that the $a_{ij}(x) \in C^1(\bar{\Omega})$, $a_{ij}(x) = a_{ji}(x)$ for $x \in \overline{\Omega}$, $c(x) \in C(\bar{\Omega})$, $c(x) \geq 0$ for $x \in \overline{\Omega}$, and L is uniformly elliptic in Ω, i.e., there exists positive constant β such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \beta|\xi|^2$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Moreover, let $a(x)$ be a continuous function on $\overline{\Omega}$ such that $|\Omega_a^\sigma| > 0$ for $\sigma \in \{+\,,\,-\}$, where $\Omega_a^\sigma = \{x \in \Omega : \sigma a(x) > 0\}$ and $|\Omega_a^\sigma| = \text{meas}\{\Omega_a^\sigma\}$.

We consider the following nonlinear eigenvalue problem

$$\begin{aligned} &Lu = \lambda(\,a(x)\,u + h(x,u,Du,\,\lambda)) \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here $Du = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n})$, λ is a real parameter, and the nonlinear term h is a continuous function on $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ such that

$$h(x,u,v,\lambda) = o(|u| + |s|) \text{ as } |u| + |v| \to 0, \tag{1.2}$$

uniformly in $x \in \Omega$ and $\lambda \in \Lambda$, for every bounded interval $\Lambda \subset \mathbb{R}$.

2010 *Mathematics Subject Classification.* 35J15, 35J25, 35J65, 35P05, 35P30, 47J10, 47J15.
Key words and phrases. nonlinear elliptic problem, bifurcation point, principal eigenvalue, global bifurcation, indefinite weight.

The global bifurcation for nonlinear eigenvalue problem were studied by Rabinowitz in well known paper [19] for the case $a(x) > 0$, $x \in \overline{\Omega}$, and all the coefficients and the nonlinear terms are of class $C^1(\overline{\Omega})$. Note that in the case $a(x) > 0$, $x \in \overline{\Omega}$, the linear problem obtained from (1.1) by setting $h \equiv 0$ possesses unique simple principal eigenvalue μ_1, where by a principal eigenvalue we mean a value of $\lambda \in \mathbb{R}$ for which problem (1.1) for $h \equiv 0$ admits a positive solution u. In [19] Rabinowitz showed that if $\mathcal{L} : E \to E$ (E be a real Banach space) linear compact operator and μ be a characteristic value of $\mathcal{L}$ of odd multiplicity, then the closure of the set of nontrivial solutions of (1.1) possesses a continuum $\mathfrak{L}_\mu$ such that $(\mu, 0) \in \mathfrak{L}_\mu$ and $\mathfrak{L}_\mu$ either (i) meets infinity in $\mathbb{R} \times E$, or (ii) meets $(\tilde{\mu}, 0)$, where $\tilde{\mu} \neq \mu$ is a characteristic value of $\mathcal{L}$. Using a positivity argument in the partial differential equation Rabinowitz [19] prove that the continuum $\mathfrak{L}_{\mu_1}$ of positive or negative solutions of problem (1.1) for $a > 0$ bifurcating from $(\mu_1, 0)$ is unbounded. In the our case the corresponding linear problem has two simple positive and negative principal eigenvalues λ_1 and λ_{-1} respectively. At first glance, it seems that a continuum $\mathfrak{L}_{\lambda_1}$ bifurcating from the point $(\lambda_1, 0)$ of may also contain a bifurcation point $(\lambda_1, 0)$, and is therefore bounded. But we using maximum principle, Berestycki [2] type approximation and regularization, and Dancer [6] theorem shown that each of continuum $\mathfrak{L}_{\lambda_{-1}}$ and $\mathfrak{L}_{\lambda_1}$ decompose into two subcontinua which contain the points $(\lambda_{-1}, 0)$ and $(\lambda_1, 0)$ respectively, and are both unbounded.

In the papers [2, 3, 14, 15, 16, 18, 21] were studied global bifurcation for some nonlinearizable second and fourth order elliptic ordinary and partial differential equations with definite weight functions.

2. On the negative and positive principal eigenvalues of corresponding linear problem

In this section we are interested in the basic properties of principal eigenvalues of linear problem obtained from (1.1) by setting $h \equiv 0$, i.e. of eigenvalue problem

$$\begin{aligned} &Lu = \lambda\, a(x)\, u \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

The existence of a principal eigenvalue of (2.1) was first shown by Manes and Micheletti [17], Hess and Kato [8] and Lopez-Gomez [13] extended the theorem of Manes and Micheletti to cover the case when L is not necessarily selfadjoint. Independently, Brown and Lin [4] obtained the theorem when $L = -\Delta$. Basically, the following is known: if a does not change sign, then (2.1) admits one principal eigenvalue; if a changes sign, then problem (2.1) admits two principal eigenvalues; one positive and the other negative. The proofs of Manes and Micheletti, Brown and Lin and Lopez-Gomez are based on the variational characterization of the principal eigenvalue; the proof of Hess and Kato (see also [7]) uses well known Krein-Rutman's theorem (see [12]).

Here we show the existence and present the basic properties of the principal eigenvalues of the problem (2.1) using the method Brown and Lin [4].

For any integer $k \in \mathbb{N}$, let $C^k(\overline{\Omega})$ denote the usual Banach space of real-valued, continuously differentiable (to order k) functions on $\overline{\Omega}$ and, for $\alpha \in (0, 1)$, let

$C^{k,\alpha}(\overline{\Omega})$ denote the set of functions in $C^k((\overline{\Omega})$ whose k-th order derivatives are Hölder continuous with exponent α. We let $|\cdot|_k$ and $|\cdot|_{k,\alpha}$ denote the standard sup-norms on these spaces. For $p > 1$, let $W^{k,p}(\overline{\Omega})$ denote the standard Sobolev space of functions whose distributional derivatives, up to order k, belong to $L^p(\Omega)$. We let $||\cdot||_p$ and $||\cdot||_{k,p}$ denote the norm on $L_p(\Omega)$ and $W^{k,p}(\Omega)$, respectively.

It is well known that the differential operator L (which determined in § 1) for all $u \in D(L) = \{u \in W^{2,2}(\Omega) : u(x) = 0 \text{ for } x \in \partial\Omega\}$ is a densely defined self-adjoint operator on $L_2(\Omega)$ whose spectrum contains only the eigenvalues [10]

$$0 < \mu_1 < \mu_2 \leq ... \leq \mu_k \mapsto +\infty.$$

For all $u \in D(L)$ let

$$V_\lambda(v) = (Lv, v) - \lambda \int\limits_\Omega av^2 dx$$

Lemma 2.1. *If there exists a nonnegative eigenfunction corresponding to an eigenvalue λ of problem (2.1), then $V_\lambda(v) \geq 0$ for all $v \in D(L)$.*

Proof. Let u is nonnegative eigenfunction associated to the eigenvalue λ. Then u is an eigenfunction corresponding to the eigenvalue of the spectral problem

$$\begin{aligned} &Lu - \lambda\, a(x)\, u = \mu u \quad \text{in } \Omega, \\ &u = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{2.2}$$

Consider the differential operator $A : D(L) \to L^2(\Omega)$ defined by

$$Au = Lu - \lambda au.$$

It is well known [5, 10] that A is a self-adjoint operator whose spectrum contains only eigenvalues $\eta_1 < \eta_2 < ... < \eta_k \mapsto +\infty$ and that the smallest eigenvalue η_1 is simple and the corresponding eigenfunction v_1 does not change sign in Ω. Note that u is not orthogonal to v_1. Since eigenfunctions corresponding to distinct eigenvalues of self-adjoint operators are orthogonal, it follows that u must be an eigenfunction corresponding η_1, i.e. $\eta_1 = 0$. It follows by the spectral theorem (see [10]) that $(Av, v) > \eta_1(v, v) = 0$ for all $v \in D(A)$, which is equivalent to the inequality $V_\lambda(v) \geq 0$ for all $v \in D(L)$. The proof of lemma is complete.

Now consider the Rayleigh quotient

$$R(v) = \frac{\int\limits_\Omega a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \int\limits_\Omega cv^2 dx}{\int\limits_\Omega av^2 dx}. \tag{2.3}$$

Lemma 2.2. *Suppose that*

$$\lambda_1 = \inf\{\, R(v) \;:\; v \in D(L), \int\limits_\Omega au^2 dx > 0\}. \tag{2.4}$$

Then $\lambda_1 > 0$.

Proof. By (2.4) it follows from (2.3) that

$$V_{\lambda_1}(v) = \int\limits_\Omega a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \int\limits_\Omega cv^2 dx - \lambda_1 \int\limits_\Omega av^2 dx \geq 0 \text{ for all } v \in D(L).$$

It is known by the spectral theorem that

$$(Lv, v) \geq \tau_1(v, v) \text{ for all } v \in D(L),$$

where τ_1 is a smallest eigenvalue of operator L. Because, if $v \in D(L)$ and $\int\limits_{\Omega} av^2dx > 0$, then we have

$$R(v) = \frac{(Lv,v)}{\int\limits_{\Omega} au^2dx} \geq \tau_1 \frac{(v,v)}{\int\limits_{\Omega} au^2dx} \geq \frac{\tau_1}{\max\limits_{x\in\overline{\Omega}} |a(x)|}$$

from which it follows that

$$\lambda_1 \geq \frac{\tau_1}{\max\limits_{x\in\overline{\Omega}} |a(x)|} > 0.$$

The proof of Lemma 2.2 is complete.

Lemma 2.3. *Let $\lambda \in (0, \lambda_1)$. Then there exists positive number $\varkappa_0$ (a depends on λ) such that $V_\lambda(v) > \varkappa_0 ||v||_2^2$ for all $v \in D(L)$.*

Proof. Suppose that $\lambda = (1-s)\lambda_1$ where $s \in (0,1)$. Then we have

$$V_\lambda(v) = (Lv,v) - \lambda \int\limits_{\Omega} au^2dx = \tfrac{\lambda}{\lambda_1} V_{\lambda_1}(v) +$$
$$\left(1 - \tfrac{\lambda}{\lambda_1}\right)(Lv,v) \geq s\,(Lv,v) \geq s\mu_1(v,v) = s\mu_1||v||_2^2.$$

Here we have taken into account the fact that by the spectral theorem the inequality $(Lv,v) \geq \mu(v,v)$ holds for all $v \in D(L)$. Then, assuming $\varkappa_0 = s\mu_1$ we get

$$V_\lambda(v) > \varkappa_0(v,v) \text{ for all } v \in D(L),$$

which completes proof of Lemma 2.3.

Lemma 2.4. *If $\lambda > 0$ and $\lambda \neq \lambda_1$, then λ is not an eigenvalue of problem* (2.1) *having a non-negative eigenfunction.*

Proof. If $\lambda > \lambda_1$ then it follows from the definition of λ_1 (see (2.4)) that there exists $v \in D(L)$ such that

$$\int\limits_{\Omega} av^2dx > 0 \text{ and } \lambda > R(v) = \frac{(Lv,v)}{\int\limits_{\Omega} av^2dx}.$$

Then by (2.3) we obtain

$$V_\lambda(v) = (Lv,v) - \lambda \int\limits_{\Omega} av^2dx < 0$$

which contrary to Lemma 2.1.

In the case $0 < \lambda < \lambda_1$ by Lemma 2.3 we have

$$V_\lambda(v) = (Lv,v) - \lambda \int\limits_{\Omega} av^2dx \geq \varkappa_0||v||_2^2 > 0 \text{ for all } v \in D(L). \qquad (2.5)$$

But on the other hand if λ is an eigenvalue of problem (2.1) and u_λ is a corresponding eigenfunction then by multiplying (2.1) by u_λ and then integrating over Ω we obtain

$$\int\limits_{\Omega} a_{ij} \frac{\partial u_\lambda}{\partial x_i} \frac{\partial u_\lambda}{\partial x_j} dx + \int\limits_{\Omega} cu_\lambda^2 dx = \lambda \int\limits_{\Omega} au_\lambda^2 dx$$

from which it follows that

$$V_\lambda(u_\lambda) = (Lu_\lambda, u_\lambda) - \lambda \int_\Omega a {u_\lambda}^2 dx =$$

$$= \int_\Omega a_{ij} \frac{\partial u_\lambda}{\partial x_i} \frac{\partial u_\lambda}{\partial x_j} dx + \int_\Omega c u_\lambda^2 dx - \lambda \int_\Omega a u_\lambda^2 dx = 0$$

which contradicts inequality (2.5). The proof lemma is complete.

Theorem 2.1. *λ_1 is an eigenvalue of spectral problem* (2.1). *Moreover λ_1 is simple and the corresponding eigenfunction u_1^+ can be chosen so that $u_1^+(x) > 0$ for all $x \in \Omega$ and $\frac{\partial u_1^+(x)}{\partial \omega} < 0$ for all $x \in \partial\Omega$, where $\frac{\partial u_1^+(x)}{\partial \omega}$ is the outward normal derivative to $\partial\Omega$.*

Proof. We consider the following eigenvalue problem

$$\begin{aligned} &Lu - \lambda\, a(x)\, u = \mu u \quad \text{in } \Omega, \\ &u = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{2.6}$$

Let operator $A_1 : D(L) \to L_2(\Omega)$ id determined by

$$(A_1 u)(x) = (Lu)(x) - \lambda_1 a(x) u(x).$$

It is obvious that λ_1 is an eigenvalue of problem (2.1) with corresponding eigenfunction u_1^+ if and only if 0 is an eigenvalue of operator A_1 and by virtue of (2.6) with corresponding eigenfunction u_1^+. The smallest eigenvalue τ_1 of operator A_1 is given by

$$\begin{aligned} \tau_1 = \inf &\left\{ \int_\Omega a_{ij} \tfrac{\partial v}{\partial x_i} \tfrac{\partial v}{\partial x_j} dx + \int_\Omega c v^2 dx - \lambda_1 \int_\Omega a u^2 dx : v \in D(L) \right\} = \\ &= \inf \left\{ V_{\lambda_1}(v) \,:\, v \in D(L) \right\}. \end{aligned} \tag{2.7}$$

From the definition of λ_1 it follows that $V_{\lambda_1}(v) \geq 0$ for all $v \in D(L)$. Hence we have $\tau_1 \geq 0$. Moreover, there exists a sequence $\{v_m\}_{m=1}^{\infty} \subseteq D(L)$ such that

$$\int_\Omega a v_m^2 dx = 1,$$

and

$$\lim_{m\to\infty} R(v_m) = \lim_{m\to\infty} \left\{ \int_\Omega a_{ij} \frac{\partial v_m}{\partial x_i} \frac{\partial v_m}{\partial x_j} dx + \int_\Omega c v_m^2 dx \right\} = \lambda_1.$$

Thus $\lim\limits_{m\to\infty} V(v_m) = 0$ and by (2.7) $\tau_1 \leq 0$ which implies that $\tau_1 = 0$ is the first eigenvalue of problem (2.7) and consequently τ_1 is simple and the corresponding eigenfunction u_1^+ can be chosen to be positive on Ω.

Now we choose a function $a_1 \in C(\overline{\Omega})$ so to satisfy the relation

$$a(x) + a_1(x) > 0 \ \text{ for all } \ x \in \overline{\Omega}.$$

Note that λ_1 is an eigenvalue of linear problem

$$\begin{aligned} &Lu + a_1(x) u = \lambda\, \tilde{a}(x)\, u \quad \text{in } \Omega, \\ &u = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{2.8}$$

with corresponding positive eigenfunction u_1^+, where $\tilde{a}(x) = a(x) + a_1(x)$ and by above $\tilde{a}(x) > 0$ for all $x \in \overline{\Omega}$. Hence λ_1 is the smallest eigenvalue of the spectral problem (2.8). Then by a theorem of Krein-Rutman [5] it follows that $\frac{\partial u_1^+(x)}{\partial \omega} < 0$ for all $x \in \partial\Omega$. This completes the proof.

Theorem 2.2. *The problem* (2.1) *has first negative eigenvalue* λ_{-1} *which is simple and the corresponding eigenfunction* u_{-1}^+ *can be chosen so that* $u_{-1}^+(x) > 0$ *for all* $x \in \Omega$ *and* $\frac{\partial u_{-1}^+(x)}{\partial \omega} < 0$ *for all* $x \in \partial\Omega$.

Proof. It is clear that $\Omega_a^- = -\Omega_{-a}^+$. Then we have $|\Omega_{-a}^+| = |\Omega_a^-| > 0$. The problem (2.1) can be rewritten in the following equivalent form

$$\begin{aligned} &Lu = \hat{\lambda}\, \hat{a}(x)\, u \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.9}$$

where $\hat{\lambda} = -\lambda$ and $\hat{a}(x) = -a(x)$, $x \in \overline{\Omega}$. By Theorem 2.1 the problem (2.9) possesses a smallest positive eigenvalue $\hat{\lambda}_1$, which is simple, and corresponding eigenfunction can be chosen so that $\hat{u}_1^+(x) > 0$ for all $x \in \Omega$ and $\frac{\partial \hat{u}_1^+(x)}{\partial \omega} < 0$ for all $x \in \partial\Omega$. If we put $\lambda_{-1} = -\hat{\lambda}_1$ and $u_{-1}^+ = \hat{u}_1^+(x)$ then we get the need result. The proof of theorem is complete.

Corollary. *The negative eigenvalue* λ_{-1} *is defined from the following relation*

$$\lambda_{-1} = \sup\{\, R(v) \,:\, v \in D(L), \int\limits_{\Omega} a u^2 dx < 0\}\,.$$

Remark 2.1. λ_1 (λ_{-1}) is a unique positive (negative) principal eigenvalue of problem (2.1).

Remark 2.2. In what follows we shall assume that $|u_k^+|_{1,\alpha} = 1$ for $k \in \{-,+\}$. Hence it will make u_k^+, $k \in \{-,+\}$, unique.

3. Global bifurcation of solutions of problem (1.1) from principal eigenvalues

It is well known that, when $p > N$, there exists a constant γ such that

$$|u|_{1,1-n/p} \le \gamma\, ||u||_{2,p} \ \text{ for all } \ u \in W^{2,p}.$$

In the following, $\alpha \in (0,1)$ is given and p will denote a real number such that $p > n$ and $\alpha < 1 - n/p$. Thus $W^{2,p}$ is compactly embedded in $C^{1,\alpha}$ (see [1, 9]).

Let $E = \{u \in C^{1,\alpha}(\overline{\Omega}) \,:\, u = 0 \text{ on } \partial\Omega\}$ be the Banach space with the usual norm $|\cdot|_{1,\alpha}$. (λ, u) is called a solution of problem (1.1) if $u \in W^{2,p}(\Omega)$ and (λ, u) satisfies (1.1) (see Remark 3.1 below). Let $P^+ = \{u \in E \,:\, u > 0$ in Ω and $\frac{\partial u}{\partial \omega} < 0$ on $\partial\Omega\}$ and $P^- = -P^+$, $P = P^- \cup P^+$. The sets P^-, P^+ and P are open subsets of E. Moreover, if $(\lambda, u) \in \partial P$ then the function u has either an interior zero in Ω or $\frac{\partial u}{\partial \omega} = 0$ at some point on $\partial\Omega$.

The closure of the set of nontrivial solutions of (1.1) will be denoted by $\mathfrak{L}$ and let $\mathcal{E} = \mathbb{R} \times E$, $\mathcal{P}^\nu = \mathbb{R} \times P^\nu$, $\nu \in \{+,-\}$ and $\mathcal{P} = \mathbb{R} \times P$.

The main result of this paper is the following theorem.

Theorem 3.1. *For each $k \in \{-1, 1\}$ and each $\nu \in \{-, +\}$ there exists a continuum $\mathfrak{L}_k^\nu$ of solutions of problem* (1.1) *in* $\mathcal{P}^\nu \cup \{(\lambda_k, 0)\}$ *which contain* $(\lambda_k, 0)$ *and is unbounded in* $\mathcal{E}$.

Proof. Step 1. We assume that $a_{ij} \in C^2(\overline{\Omega})$, c, $a \in C^1(\overline{\Omega})$ and $h \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R})$.

We define nonlinear solution operator of $\mathcal{E} \to E$ as follows: for each $(\lambda, u) \in \mathcal{E}$, let $v = G(\lambda, u)$ be the solution of the following non-homogenous linear problem

$$\begin{aligned} &Lv = \lambda(a(x)\, u + h(x, u, Du,\, \lambda)) \quad \text{in } \Omega, \\ &v = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.1}$$

It follows by linear existence theory of the linear uniformly elliptic partial differential equation that there is a unique solution of (3.1) (see [9, 11]). The Schauder theory (see, eg., [9]) implies that the operator $\mathcal{H}$ is completely continuous. Then each solution (λ, u) of problem (1.1) is a solution of equation

$$u = G(\lambda, u), \tag{3.2}$$

and conversely.

For $(\lambda, u) \in \mathcal{E}$, let $w \equiv K(\lambda, u)$ denote the solution operator of

$$Lw = \lambda au \text{ in } \Omega, \quad w = 0 \text{ in } \partial\Omega. \tag{3.3}$$

It obvious that $K(\lambda, u) = \lambda\mathcal{L}u$, where, as above $\mathcal{L}$ is linear compact operator of $E \to E$ (see [13]).

Remark 3.1. The operator $\mathcal{L}$ is injective, since L is closable in E. In fact, L admits a closed extension in $L_p(\Omega)$, $1 < p < \infty$, having domain $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$.

Moreover, by the condition (1.2) it follows that

$$H(\lambda, u) \equiv G(\lambda, u) - \lambda\mathcal{L}u = o(|u|_{1,\alpha}) \text{ as } |u|_{1,\alpha} \to 0 \tag{3.4}$$

uniformly for $\lambda \in \Lambda$. Thus the problem (1.1) can be rewritten in the equivalent form

$$u = \lambda\mathcal{L}u + \mathcal{H}(\lambda, u). \tag{3.5}$$

By [11; Ch. 4, § 2, Lemma 2.1] and condition (3.4) problem (3.5) is linearizable in the neighborhood of zero, and the linearization of (3.5) at $u = 0$ is the spectral problem

$$u = \lambda\mathcal{L}u. \tag{3.6}$$

Obviously, the problem (3.6) is equivalent to the spectral problem (2.1). Hence the principal eigenvalues λ_{-1} and λ_1 of problem (2.1) are the characteristic values of $\mathcal{L}$ and are simple. Then by [11; Ch. 4, § 2, Theoerm 2.1] $(\lambda_{-1}, 0)$ and $(\lambda_1, 0)$ are bifurcation points of (3.5), and these points correspond to continuous branches of non-trivial solutions. Moreover, by [19; Theorem 1.3] for each $k \in \{-1, 1\}$ there exists a continuum $\mathfrak{L}_k$ of $\mathfrak{L}$ such that $(\lambda_k, 0) \in \mathfrak{L}_k$ and $\mathfrak{L}_k$ either (i) is unbounded in $\mathcal{E}$, or (ii) contain $(\lambda_s, 0)$, where $\lambda_k \neq \lambda_s \in X(\mathcal{L})$, $X(\mathcal{L})$ denote the set of characteristic values of $\mathcal{L}$. By [19; Lemma 1.24] it follows that if $(\lambda, u) \in \mathfrak{L}_k$ and near $(\lambda_k, 0)$, then $u = \beta u_k^+ + w$ with $\beta = o(|\beta|)$ as $\beta \to 0$. Since P^ν open subset in E and $u_k^+ \in P$ by Theorems 2.1 and 2.2 and Remarks 2.1 and 2.2, then we have

$$(\lambda, u) \in \mathcal{P} \text{ and } ((\mathfrak{L}_k \backslash \{(\lambda_k, 0)\}) \cap B_{y,k}) \subset \mathcal{P}.$$

for all small $y > 0$, where $B_{y,\,k}$ is an open ball in $\mathcal{E}$ of radius y centered at $(\lambda_k, 0)$. Note that if $(\lambda, u) \in \mathfrak{L}_k$, then $\lambda \neq 0$. Indeed, in this case $\lambda = 0$ is an eigenvalue of linear eigenvalue problem (2.1) which is impossible by maximum principle (see [5, 9, 13]). Moreover, if $(\lambda, u) \in \mathcal{E}$ is a solution of problem (2.1), then (λ, u) is also a solution of the following nonlinear problem

$$Lu + (-a^-(x))u = \lambda(a^+(x)\, u + h(x, u, Du,\, \lambda)) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on} \quad \partial\Omega,$$

where $a^+(x) = \max\{a(x), 0\}$ and $a^-(x) = \min\{a(x), 0\}$ respectively. Hence using [20; Lemma VIII.10] and an argument similar to [19, Corollary 2.13] we can show that $(\mathfrak{L}_k \backslash \{(\lambda_k, 0)\}) \cap \partial\mathcal{P} = \emptyset$. Consequently, using the fact that the eigenfunctions of the problem (2.1) corresponding to the eigenvalues which are different from $\lambda_{\pm 1}$ have interior zeros in Ω, we have that $\mathfrak{L}_k$ lies in $\mathcal{P} \cup \{(\lambda_k, 0)\}$ and for each $k \in \{\, -1\,,\, 1\,\}$ alternative (ii) of Theorem 1.3 from [19] is not possible.

It remains to decompose $\mathfrak{L}_k$ into two subcontinua which contain $(\lambda_k, 0)$, lies in $\mathcal{P}^\nu \cup \{(\lambda_k, 0)\}$, $\nu \in \{\, -\,,\, +\,\}$, and unbounded in $\mathcal{E}$. For $k \in \{\, -1\,,\, 1\,\}$ let $\ell_k \in E^*$ be such that $\ell_k = \lambda_k \mathcal{L}^* \ell_k$ and $\langle\, \ell_k\,,\ u_k^+\,\rangle = 1$, where E^* is the dual of E, $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$ and $\langle\, \cdot\,,\ \cdot\,\rangle$ is the duality between of E and E^*. If $0 < y < 1$, define

$$M_{k,\,y} = \{(\lambda, u) \in \mathcal{E} : |\langle\, \ell_k\,,\, u\,\rangle| > y\, |u|_{1,\,\alpha}\},$$
$$M^+_{k,\,y} = \{(\lambda, u) \in M_{k,\,y} : \langle\, \ell_k\,,\, u\,\rangle > 0\},$$

and $M^-_{k,\,y} = M_{k,\,y} \backslash M^+_{k,\,y}$. This defines two pair "wedges" in $\mathcal{E}$ with vertices lying along the line $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$. Each of these pair wedges are independent of λ but it is obvious that different pair wedges are associated with each characteristic value. In [19], corresponding wedges are defined for a range of λ centered at the characteristic value λ_k.

By virtue of [19; Lemma 1.24] it follows that there exists an $S > 0$ such that $(\mathfrak{L} \backslash (\lambda_k, 0)) \cap \overline{B}_S(\lambda_k) \subseteq M_{k,\,y}$, where $\overline{B}_{S,\,k}$ denotes the closure of open ball $B_{S,\,k}$. The subcontinua $\mathfrak{L}_k^\nu$, $\nu \in \{\, -\,,\, +\}$, can now be defined (see [6]). For $0 < \varepsilon \leq S$, $k \in \{\, -1, 1\,\}$ and $\nu \in \{\, -\,,\, +\}$ define $D^\nu_{k,\,\varepsilon}$ to be component of $(\mathfrak{L} \cap \overline{B}_\varepsilon(\lambda_k) \cap M^\nu_{k,\,y}) \cup \{(\lambda_k, 0)\}$ containing $(\lambda_k, 0)$, $\mathfrak{L}^\nu_{k,\,\varepsilon}$ to be component of $\overline{\mathfrak{L}_k \backslash D^{-\nu}_{k,\,\varepsilon}}$ containing $(\lambda_k, 0)$, where $-\nu$ is interpreted in the natural way, and $\mathfrak{L}_k^\nu$ to be closure of $\bigcup\limits_{0\,<\varepsilon \leq S} \mathfrak{L}^\nu_{k,\varepsilon}$. Then $\mathfrak{L}_k^\nu$ is connected, and by [19; Theorem 1.25], $\mathfrak{L}_k = \mathfrak{L}_k^+ \cup \mathfrak{L}_k^-$. Note that this definition is independent of the choice of y but the choice of S is dependent on y; however, by [19; Lemma 1.24] given any $0 < y < 1$, an any $S > 0$ can always be chosen such that the above holds.

Again writing $u = \beta u_k^+ + w$ for $\lambda, u) \in (\mathfrak{L}_k \backslash \{(\lambda_k, 0)\}$ near $(\lambda_k, 0)$ we have $\beta u_k^+ \in \mathcal{P}^\nu$ if $\beta \in \mathbb{R}^\nu \backslash \{0\}$ where $\mathbb{R}^\nu = \{z \in \mathbb{R} : 0 \leq \nu z \leq +\infty\}$ and, therefore,

$$((\mathfrak{L}_k^+ \backslash \{(\lambda_k, 0)\}) \cap B_{y,\,k}) \subset \mathcal{P}^+,$$

$$((\mathfrak{L}_k^- \backslash \{(\lambda_k, 0)\}) \cap B_{y,\,k}) \subset \mathcal{P}^-$$

for all $y > 0$ small. Since $\mathfrak{L}_k^\nu \backslash \{(\lambda_k, 0)\}$ cannot leave P^ν outside of a neighborhood of $(\lambda_k, 0)$ and $\mathcal{P}^+ \cap \mathcal{P}^- = \emptyset$, then by [6; Theorem 2] $\mathfrak{L}_k^+$ and $\mathfrak{L}_k^-$ are both unbounded in $\mathcal{E}$.

Step 2. To complete prove of this theorem, we approximate (1.1) by a family of linearizable equations, as in [3; Section 4]. However, with a view to applying the result of Step 1, we need to approximate (1.1) by equations where all the coefficients and the nonlinear terms are smooth and we pass to the limit using by a positivity argument in the uniformly elliptic partial differential equation and compactly embedding from above. This completes the proof.

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Received: March 14, 2016; Accepted: May 16, 2016