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ON DETERMINATION OF STURM-LIOUVILLE OPERATOR WITH DISCONTINUITY CONDITIONS WITH RESPECT TO SPECTRAL DATA

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Abstract. In the paper, the uniqueness of the inverse problem with respect to eigenvalues and normalizing numbers was proved for Sturm-Liouville operator with a discontinuity condition on a finite segment and an algorithm for constructing a potential was given.

1. Introduction

Let us consider the equation

$$-y'' + q(x)\,y = \lambda y,\ \ 0 < x < \pi, \tag{1.1}$$

with discontinuity conditions at a point $a \in (0, \pi)$

$$y(a+0) = \alpha y(a-0)\,, \quad y'(a+0) = \alpha^{-1} y'(a-0)\,, \tag{1.2}$$

and boundary conditions

$$y(0) = y(\pi) = 0. \tag{1.3}$$

Here λ is a complex parameter, $q(x), \alpha$ are real; $q(x) \in L_2(0,\pi)\,, \alpha \neq 1, \alpha > 0$.

Let $S(x,\lambda)$ be the solution of equation (1.1) with discontinuity conditions (1.2) and initial conditions $S(0,\lambda) = 0, S^{'}(0,\lambda) = 1$.

Denote by λ_n the eigenvalues and by α_n the normalized numbers of problem (1.1) - (1.2):

$$\alpha_n = \int_0^\pi S^2(x,\lambda_n)dx. \tag{1.4}$$

The numbers $\{\ \lambda_n, \alpha_n\}$ are said to be spectral data of problem (1.1) - (1.3). In the case $q(x) \equiv 0$ denote the spectral data by $\left\{\lambda_n^0, \alpha_n^0\right\}$. We are interested in the following inverse problem: determine the function $q(x)$ in equation (1.1) with respect to spectral data $\{\lambda_n, \alpha_n\}$ of problem (1.1) - (1.3).

Another variants of inverse problems for the Sturm-Liouville operator with discontinuity conditions (the inverse problem with respect to Weyl's function, with respect to scattering data, etc.) were considered in the papers [1] - [4] (see also the references therein).

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The inverse problem with respect to spectral data, and also various variants of inverse problems for the Sturm-Liouville operator without discontinuity conditions were given in detail in monographs and review papers (see e.g. [2] - [8]).

2. Completeness of eigen functions

Let $S_\pi(x,\ \lambda)$ be the solution of equation (1.1) with discontinuity conditions (1.2), $S_\pi(\pi,\ \lambda)=0,\ \ S_\pi^{'}(\pi,\ \lambda)=1$. Then there exists a sequence β_n such that
a) $S_\pi(x,\ \lambda_n)=\ \beta_n S(x,\lambda_n),\ \beta_n\neq 0$;
b)$\beta_n\alpha_n=-\dot{\Delta}(\ \alpha_n)$, where

$$\Delta(\lambda)=S(\pi,\lambda) \tag{2.1}$$

c) $\sqrt{\lambda_n}=\sqrt{\lambda_n^0}+\frac{a_n}{\sqrt{\lambda_n^0}}+\frac{\varepsilon_n}{\sqrt{\lambda_n^0}},\{\varepsilon_n\}\in l_2,(see\,[10])$
d) $\alpha_n=\alpha_n^0+\frac{\delta_n}{\sqrt{\lambda_n^0}}\ ,\{\delta_n\}\in\ l_2$.

The Green function of problem (1.1) - (1.3) is of the form

$$G(x,t,\lambda)=-\frac{1}{\Delta(\lambda)}\begin{cases} S(x,\lambda)\,S_\pi(t,\lambda)\,, & x\leq t,\\ S(t,\lambda)\,S_\pi(x,\lambda)\,, & x\geq t.\end{cases}$$

Theorem 2.1. *1) The system of eigen functions* $\{S(x,\lambda_n)\}$ *of problem* (1.1) - (1.3) *is complete in* $L_2(0,\pi)$*;*
2) Let $f(x)\in AC[0,a]\cap AC[a,\pi]$ *and* $f(a+0)=\alpha f(a-0)\,,f(0)=f(\pi)=0$. *Then*

$$f(x)=\sum_{n=0}^{\infty}a_n S(x,\lambda_n),$$

where

$$a_n=\frac{1}{\alpha_n}\int_0^\pi f(t)\,S(t,\lambda_n)dt$$

and the series converges uniformly on $[0,\pi]$.

Proof. Consider the function

$$Y(x,\lambda)=\int_0^\pi G(x,t,\lambda)\,f(t)dt.$$

It is easy to show that the function $Y(x,\lambda)$ is the solution of the equation

$$-Y^{''}+q(x)\,Y-\lambda Y+f(x)=0, \tag{2.2}$$

satisfies discontinuity conditions (1.2) and boundary conditions (1.3). Furthermore, taking into account (1.4), we have

$$\operatorname*{Res}_{\lambda=\lambda_n}Y(x,\lambda)=\frac{1}{\alpha_n}S(x,\lambda_n)\int_0^\pi f(t)\,S(t,\lambda_n)dt. \tag{2.3}$$

Let the function $f(x)\in L_2(0,\pi)$ be such that

$$\int_0^\pi f(t)\,S(t,\lambda_n)dt=0, \qquad n=1,2,3,\ldots \tag{2.4}$$

Then from (2.3) we get $\operatorname*{Res}_{\lambda=\lambda_n} Y(x,\lambda) = 0$, and consequently for every $x \in [0,\pi]$ the function $Y(x,\lambda)$ is entire with respect to λ. On the other hand, for $\rho \in G_\delta = \{\rho : |\rho - \rho_{k,0}| \geq \delta, k = \pm 1, \pm 2, \pm 3, \dots\}$ and $|\rho| \geq \rho^*$, where $\lambda = \rho^2$, $\rho_{k,0}$ are the zeros of the function $\Delta_0(\rho) = \alpha^+ \frac{\sin\rho\pi}{\rho} + \alpha^- \frac{\sin\rho(2a-\pi)}{\rho}$, δ is a fixed positive number, ρ^* is rather large, the inequality

$$|\Delta(\lambda)| \geq \frac{C_\delta'}{|\rho|} e^{|Im\rho|\pi},$$

and consequently the inequality

$$|Y(x,\lambda)| \leq \frac{C_\delta''}{|\rho|}$$

are fulfilled.

Using the Liouville theorem, we conclude that $Y(x,\lambda) \equiv 0$.

Hence and from (2.2) it follows that $f(x) = 0$ a.e. on $[0,\pi]$. Thus, subject to equalities (2.4), we get that $f(x) = 0$ a.e. on $[0,\pi]$. Consequently, the system of eigen functions $\{S(x,\lambda_n)\}_{n=1}^\infty$ of problem (1.1) - (1.3) is complete in $L_2(0,\pi)$. Prove statement 2) of the theorem. Let $f(x) \in AC[0,a] \cap AC[a,\pi]$, $f(0) = f(\pi) = 0$ and $f(a+0) = \alpha f(a-0)$. We transform the function $Y(x,\lambda)$ to the form

$$Y(x,\lambda) = \frac{f(x)}{\lambda} - \frac{1}{\lambda}\left(z_1(x,\lambda) + z_2(x,\lambda)\right), \tag{2.5}$$

$$z_1(x,\lambda) = \frac{1}{\triangle(\lambda)}(S_\pi(s,\lambda)) \int_0^x S'(t,\lambda) f'(t)dt + S(x,\lambda) \int_x^\pi S_\pi'(t,\lambda) f'(t)dt,$$

$$z_2(x,\lambda) = \frac{1}{\dot{\Delta}(\lambda)} \left(S_\pi(x,\ \lambda)\right) \int_0^x q(t) S(t,\lambda) f(t)dt + S(x,\ \lambda) \int_x^\pi q(t) S_\pi(t,\lambda) f(t)dt$$

Using the lower bound for $\Delta(\lambda)$, by the standard method we establish that (see [10])

$$\lim_{\substack{|\rho|\to\infty \\ \rho\in G_\delta}} \max_{0\leq x\leq\pi} |z_j(x,\lambda)| = 0, \quad j = 1,2. \tag{2.6}$$

Consider the contour integral

$$I_N(x) = \frac{1}{2\pi i} \oint_{\Gamma_n} Y(x,\lambda) d\lambda,$$

where $\Gamma_n = \{\lambda : |\lambda| = |\lambda_N^0| + \frac{\beta}{2}\}$ (counter anti-clockwise), $\lambda_N^0 = \rho_{N,0}^2$, $\beta = \inf_{n\neq k} |\lambda_n^0 - \lambda_k^0| > 0$. From (2.5)-(2.6) it follows

$$I_N(x) = f(x) + \varepsilon_N(x), \ \lim_{N\to\infty} \max_{0\leq x\leq\pi} |\varepsilon_N(x)| = 0 \tag{2.7}$$

On the other hand,

$$I_N(x) = \sum_{n=1}^N \operatorname*{Res}_{\lambda=\lambda_n} Y(x,\lambda) = \sum_{n=1}^N a_n S(x,\lambda_n),$$

where

$$a_n = \frac{1}{\alpha_n}\int_0^{\pi} f(t)\, S(t,\lambda_n) dt.$$

Comparing these expression with (2.7) we arrive to statement 2) of the theorem. □

Corollary 2.1. *The system of eigen functions* $\{S(x,\lambda_n)\}$ *is complete and orthogonal in* $L_2(0,\pi)$, *therefore it forms an orthogonal basis in* $L_2(0,\pi)$ *and the Parseval equality is valid:*

$$\int_0^{\pi} |f(t)|^2\, dx = \sum_{n=1}^{\infty} \alpha_n |a_n|^2$$

Corollary 2.2. *The system of functions* $\{S_0(x,\lambda_n^0)\}_{n=1}^{\infty}$, *where* $(\lambda = \rho^2)$

$$S_0(x,\lambda) = \begin{cases} \frac{\sin\rho x}{\rho}, & 0 < x < a, \\ \alpha^+ \frac{\sin\rho x}{\rho} + \alpha^- \frac{\sin\rho(2a-x)}{\rho}, & a < x < \pi, \end{cases}$$

$$\lambda_n^0 = \rho_{n,0}^2 \quad , \alpha^{\pm} = \frac{1}{2}(\alpha \pm \frac{1}{\alpha})$$

is complete in the space $L_2(0,\pi)$.

3. Uniqueness of inverse problem and an algorithm for constructing the potential

Denote by $e(x,\lambda)$ the solution of problem (1.1) - (1.2) with initial conditions

$$e(0,\lambda) = 1,\ e'(0,\lambda) = i\lambda. \tag{3.1}$$

Obviously, $e(x,\lambda)$ satisfies the integral equation

$$e(x,\lambda) = e_0(x,\lambda) + \int_0^x g_0(x,t,\lambda)\, q(t)\, e(t,\lambda) dt, \tag{3.2}$$

where $(\lambda = \rho^2)$

$$e_0(x,\lambda) = \begin{cases} e^{i\rho x}, & 0 < x < a, \\ \alpha^+ e^{i\rho x} + \alpha^- e^{i\rho(2a-x)}, & a < x < \pi, \end{cases}$$

$$g_0(x,t,\lambda) = \begin{cases} \frac{\sin\rho(x-t)}{\rho}, & (t < x < a)\ V\ (a < t < x), \\ \alpha^+ \frac{\sin\rho(x-t)}{\rho} + \alpha^- \frac{\sin\rho(2a-x-t)}{\rho}, & t < a < x, \end{cases}$$

and vice versa, the solution of integral equation (3.2) is the solution of problem (1.1), (1.2), (3.1). The solution $e(x,\lambda)$ is representable in the form

$$e(x,\lambda) = e_0(x,\lambda) + \int_{-x}^{x} K(x,t)\, e^{i\lambda t} dt. \tag{3.3}$$

Substituting (2.2) in integral equation (3.2), for the kernel $K(x,t)$ we get the equation

$$K(x,t) = K_0(x,t) + \frac{1}{2}\int_0^x q(u)\int_{t-x+u}^{t+x-u} K(u,\xi)d\xi du$$

if $0 < x < a$, and

$$K(x,t) = K_0(x,t) + \frac{\alpha^+}{2}\int_0^a q(u)\int_{t-x+u}^{t+x-u} q(u)\,K(u,\xi)\,d\xi du +$$

$$+\frac{1}{2}\int_a^x q(u)\int_{t-x+u}^{t+x-u} K(u,\xi)\,d\xi du + \frac{\alpha^-}{2}\int_0^a q(u)\int_{t-2a+x+u}^{t+2a-x-u} K(u,\xi)\,d\xi du$$

if $a < x < \pi$.

Here

$$K_0(x,t) = \frac{\alpha^+}{2}\int_0^{\frac{x+t}{2}} q(u)du,$$

for $x < a$ and

$$K_0(x,t) = \frac{\alpha^+}{2}\int_0^{\frac{x+t}{2}} q(u)\,du +$$

$$+\frac{\alpha^-}{2}\begin{cases} 0\ , & -x < t < -(2a-x), \\ \int_0^{\frac{t+2a-x}{2}} q(u)\,du, & -(2a-x) < t < 2a-x, \\ \int_a^{\frac{x+2a-t}{2}} q(u)\,du - \int_{\frac{t+2a-x}{2}}^{a} q(u)\,du\ , & 2a-x < t < x, \end{cases} \tag{3.4}$$

for $a < x < \pi$.

Furthermore, the following conditions are fulfilled:

$$K(x,x) = \frac{1}{2}\int_0^x q(s)\,ds, \quad 0 < x < a,$$

$$K(x,x) = \frac{\alpha^+}{2}\int_0^x q(s)\,ds, \quad a < x < \pi$$

$$K(x,-x) = 0.$$

Obviously, that solutions $S(x,\lambda)$ and $e(x,\lambda)$ are connected with the equality $S(x,\lambda) = \frac{1}{2i\lambda}\left[e(x,\lambda) - \overline{e(x,\lambda)}\right]$. Therefore, taking into account (3.3), for the solution $S(x,\lambda)$ we get the following representation

$$S(x,\lambda) = S_0(x,\lambda) + \int_0^x A(x,t)\frac{\sin\rho t}{\rho}dt\ , \quad \left(\lambda = \rho^2\right) \tag{3.5}$$

where A(x, t) $=K(x,t)-K(x,-t)$ and

$$\mathrm{A}\,(\mathrm{x},\mathrm{x})=\begin{cases}\frac{1}{2}\int\limits_0^x q(s)\,ds, & 0<x<a,\\ \frac{\alpha^+}{2}\int\limits_0^x q(s)ds, & a<x<\pi\end{cases} \qquad (3.6)$$

Show that we can write representation (3.5) in the form of a transformation operator.

Indeed, from the formula for solution $S_0(x,\lambda)$ (see. Corollary 2.1), we have

$$\frac{\sin\rho x}{\rho}=\begin{cases} S_0(x,\lambda), & 0<x<a,\\ \frac{1}{\alpha^+}S_0(x,\lambda)-\frac{\alpha^-}{\alpha^+}S_0(2a-x,\lambda), & a<x<\pi.\end{cases}$$

Therefore, taking into account this formula in the integral member in representation (3.5), for $a<x<\pi$ we get:

$$S(x,\lambda)=S_0(x,\lambda)+\int\limits_0^a A(x,t)\,S_0(t,\lambda)\,dt+$$

$$+\int\limits_a^x A(x,t)\left[\frac{1}{\alpha^+}S_0(t,\lambda)-\frac{\alpha^-}{\alpha^+}S_0(2a-t,\lambda)\right]dt, \quad a<x<\pi,$$

or finally

$$S(x,\lambda)=S_0(x,\lambda)+\int\limits_0^a \tilde{A}(x,t)\,S_0(t,\lambda)\,dt, \quad 0<x<\pi, \qquad (3.7)$$

where

$$\widetilde{A}(x,t)=\begin{cases} A(x,t), & 0<t<x, \quad 0<x<a\\ A(x,t), & 0<t<2a-x, \quad a<x<\pi\\ A(x,t)-\frac{\alpha^-}{\alpha^+}A(x,2a-t), & 2a-x<t<a, \quad a<x<\pi\\ \frac{1}{\alpha^+}A(x,t), & a<t<x, \quad a<x<\pi\end{cases} \qquad (3.8)$$

Transformation operator (3.8) admits to obtain a link between the kernel $\tilde{A}(x,t)$ and spectral data of problem (1.1) - (1.3). Solving relation (3.8) with respect to $S_0(x,\lambda)$, we find

$$S_0(x,\lambda)=S(x,\lambda)+\int\limits_0^x \tilde{H}(x,\xi)S(\xi,\lambda)\,d\xi, \qquad (3.9)$$

Using (3.8) and (3.9) we get

$$\sum_{n=1}^{N}\frac{S(x,\lambda_n)\,S_0(t,\lambda_n)}{\alpha_n}=$$

$$=\sum_{n=1}^{N}\frac{S_0(x,\lambda_n)\,S_0(t,\lambda_n)}{\alpha_n}+\int\limits_0^x \tilde{A}(x,\xi)\sum_{n=1}^{N}\frac{S_0(\xi,\lambda_n)\,S_0(t,\lambda_n)}{\alpha_n}d\xi$$

$$\sum_{n=1}^{N} \frac{S(x,\lambda_n) S_0(t,\lambda_n)}{\alpha_n} =$$

$$= \sum_{n=1}^{N} \frac{S(x,\lambda_n) S(t,\lambda_n)}{\alpha_n} + \int_0^t \tilde{H}(x,\xi) \sum_{n=1}^{N} \frac{S(x,\lambda_n) S(\xi,\lambda_n)}{\alpha_n} d\xi$$

Equating the right hand sides of these equalities and assuming

$$\Phi_N(x,t) = \sum_{n=1}^{N} \left(\frac{S(x,\lambda_n) S(t,\lambda_n)}{\alpha_n} - \frac{S\left(x,\lambda_n^0\right) S\left(t,\lambda_n^0\right)}{\alpha_n^0} \right),$$

we have

$$\Phi_N(x,t) + \int_0^t \tilde{H}(t,\xi) \sum_{n=1}^{N} \frac{S(t,\lambda_n) S(\xi,\lambda_n)}{\alpha_n} d\xi =$$

$$= \sum_{n=1}^{N} \left(\frac{S_0(x,\lambda_n) S_0(t,\lambda_n)}{\alpha_n} - \frac{S_0\left(x,\lambda_n^0\right) S_0\left(t,\lambda_n^0\right)}{\alpha_n^0} \right) +$$

$$+ \int_0^x \tilde{A}(x,\xi) \sum_{n=1}^{N} \left(\frac{S_0(\xi,\lambda_n) S_0(t,\lambda_n)}{\alpha_n} - \frac{S_0\left(\xi,\lambda_n^0\right) S_0\left(t,\lambda_n^0\right)}{\alpha_n^0} \right) d\xi +$$

$$+ \int_0^x \tilde{A}(x,\xi) \sum_{n=1}^{N} \frac{S_0\left(\xi,\lambda_n^0\right) S_0\left(t,\lambda_n^0\right)}{\alpha_n^0} d\xi \tag{3.10}$$

Let $f(x) \in AC[0,a] \cap AC[a,\pi]$, $f(0) = f(\pi) = 0$ and $f(a+0) = \alpha f(a-0)$.

According to Theorem 2.1 and Corollary 2.2

$$\lim_{N\to\infty} \max_{0\le x\le\pi} \left| \int_0^\pi f(t) \Phi_N(x,t)\, dt \right| =$$

$$= \lim_{N\to\infty} \max_{0\le x\le\pi} \left| \sum_{n=1}^{N} a_n S(x,\lambda_n) - \sum_{n=1}^{N} a_n^0 S_0\left(x,\lambda_n^0\right) \right| \le$$

$$\le \lim_{N\to\infty} \max_{0\le x\le\pi} \left| f(x) - \sum_{n=1}^{N} a_n S(x,\lambda_n) \right| +$$

$$+ \lim_{N\to\infty} \max_{0\le x\le\pi} \left| f(x) - \sum_{n=1}^{N} a_n^0 S_0\left(x,\lambda_n^0\right) \right| = 0 \tag{3.11}$$

Similarly, we can show that uniformly with respect to $x \in [0,\pi]$

$$\lim_{N\to\infty} \int_0^x \tilde{A}(x,\xi) \sum_{n=1}^{N} \frac{S_0\left(\xi,\lambda_n^0\right)}{\alpha_n^0} \int_0^\pi f(t) S_0(t,\lambda_n^0)\, dt\, d\xi = \int_0^x \tilde{A}(x,\xi) f(\xi) d\xi \tag{3.12}$$

$$\lim_{N\to\infty} \int_0^\pi f(t) \sum_{n=1}^{N} \left(\frac{S_0(x,\lambda_n) S_0(t,\lambda_n)}{\alpha_n} - \frac{S_0\left(x,\lambda_n^0\right) S_0\left(t,\lambda_n^0\right)}{\alpha_n^\alpha} \right) dt =$$

$$= \int_0^\pi f(t) F(x,t) dt \tag{3.13}$$

$$\lim_{N\to\infty} \int_0^\pi f(t) \left[\int_0^x \tilde{A}(x,\xi) \sum_{n=1}^N \left(\frac{S_0(\xi,\lambda_n) S_0(t,\lambda_n)}{\alpha_n} - \frac{S_0(\xi,\lambda_n^0) S_0(t,\lambda_n^0)}{\alpha_n^0} \right) d\xi \right] dt =$$

$$= \int_0^\pi f(t) \left(\int_0^x \tilde{A}(x,\xi) F(\xi,t) d\xi \right) dt, \tag{3.14}$$

$$\lim_{N\to\infty} \int_0^\pi f(t) \left[\int_0^t \tilde{H}(t,\xi) \sum_{n=1}^N \frac{S(x,\lambda_n) S(\xi,\lambda_n)}{\alpha_n} d\xi \right] dt =$$

$$= \int_x^\pi f(t) \tilde{H}(t,x) dt, \tag{3.15}$$

where

$$F(x,t) = \sum_{n=1}^\infty \left(\frac{S_0(x,\lambda_n) S_0(t,\lambda_n)}{\alpha_n} - \frac{S_0(x,\lambda_n^0) S_0(t,\lambda_n^0)}{\alpha_n^0} \right). \tag{3.16}$$

Now multiply the both sides of relation (3.10) by $f(t)$ and integrate with respect to $t \in [0,\pi]$. Then passing to limit as $N \to \infty$ and taking into account (3.11) - (3.15) , we arrive at the equality

$$\int_x^\pi \tilde{H}(t,x) f(t) dt =$$

$$= \int_0^\pi f(t) F(x,t) dt + \int_0^\pi f(t) \left(\int_0^x \tilde{A}(x,\xi) F(\xi,t) d\xi \right) dt + \int_0^x f(t) \tilde{A}(x,t) dt.$$

Define $\tilde{A}(x,t) = \tilde{H}(x,t) = 0$ for $x < t$.

By arbitrariness of $f(x)$ we arrive at the relation

$$\tilde{H}(t,x) = F(x,t) + \int_0^x \tilde{A}(x,\xi) F(\xi,t) d\xi + \tilde{A}(x,t).$$

Hence, for $t < x$ we have

$$F(x,t) + \int_0^x \tilde{A}(x,\xi) F(\xi,t) d\xi + \tilde{A}(x,t) = 0. \tag{3.17}$$

So we proved the following theorem:

Theorem 3.1. *For every fixed $x \in (0,\pi)$ the kernel $\tilde{A}(x,t)$ from representation* (3.8) *satisfies linear integral equation* (3.17)*, and the the function $F(x,t)$ is completely determined by the spectral data $\{\lambda_n, \alpha_n\}$ according to formula* (3.16)*. Equation* (3.17) *is called the basic equation of the inverse problem.*

We now study solvability of the basic equation (3.17). At first we prove the auxiliary lemma.

Lemma 3.1. *The system of functions* $\{S_0(x,\lambda_n)\}_{n\geq 1}$ *is complete in the space* $L_2(0,\pi)$.

Proof. Let

$$\int_0^\pi f(t)\,S_0(t,\lambda_n)\,dt = 0,\ \ n = 0,1,2,\dots \tag{3.18}$$

Consider the functions

$$\Delta(\lambda) = \left(\pi\alpha^+ + (2a-\pi)\,\alpha^-\right)\prod_{n=1}^{\infty}\frac{\lambda_n-\lambda}{\lambda_n^0},\ \lambda_n^0 = \rho_{n,0}^2,$$

$$\Delta_0(\lambda) = \left(\pi\alpha^+ + (2a-\pi)\,\alpha^-\right)\prod_{n=1}^{\infty}\frac{\lambda_n^0-\lambda}{\lambda_n^0} = \alpha^+\frac{\sin\rho\pi}{\rho} + \alpha^-\frac{\sin\rho(2a-\pi)}{\rho},$$

$$F(\lambda) = \frac{1}{\Delta(\lambda)}\int_0^\pi f(t)\,S_0(t,\lambda_n)\,dt,\ \lambda\neq\lambda_n$$

From (3.18) it follows that $F(\lambda)$ is a function entire with respect to λ . Further,

$$\frac{\Delta_0(\lambda)}{\Delta(\lambda)} = \prod_{n=1}^{\infty}\left(1+\frac{\lambda_n-\lambda_n^0}{\lambda-\lambda_n}\right).$$

Hence, by virtue of the estimation

$$|\Delta_0(\lambda)| \geq \frac{Ce^{|\tau|\pi}}{\rho},\quad \left|\frac{\lambda_n-\lambda_n^0}{\lambda-\lambda_n}\right| \leq \frac{C}{\rho_{n,0}^2},\ \tau = \arg\rho,\ \lambda = \rho^2$$

in the sector $\arg\lambda \in [\delta, 2\pi-\delta]$, where δ - is some fixed positive number, we have

$$|\Delta(\lambda)| \geq \frac{C}{\rho}e^{|\tau|\pi},\ \ \lambda = \rho^2, \arg\lambda\in[\delta,2\pi-\delta].$$

Consequently, for $\arg\lambda\in[\delta,2\pi-\delta]$ we get

$$|F(\lambda)| \leq \frac{1}{|\Delta(\lambda)|}\left|\int_0^\pi f(t)\,S_0(t,\lambda)\,dt\right| \leq \frac{C}{|\rho|}.$$

Hence, by means of Fragmen – Lindelof and Liouville theorem we conclude that $F(\lambda)\equiv 0$, in particular

$$\int_0^\pi f(t)\,S_0\left(t,\lambda_n^0\right)dt = 0,\ \ n = 1,2,3,\dots.$$

By the completeness of the system of functions $\left\{S_0\left(x,\lambda_n^0\right)\right\}_{n\geq 1}$ in $L_2(0,\pi)$ (see. Corollary 2.2),hence $f(x) = 0$ a.e. The lemma is proved. □

Now we prove a theorem on solvability of the basic equation (3.17).

Theorem 3.2. *For each fixed* $x\in(0,\pi)$ *the basic equation* (3.17) *has a unique solution* $\tilde{A}(x,\cdot)\in L_2(0,x)$.

Proof. As (3.17) is the Fredholm equation of the second kind, then it suffices to prove that the homogeneous equation

$$g(t)+\int_0^x F(s,t)\,g(s)ds=0 \tag{3.19}$$

has only the zero solution $g(t)=0$. Let $g(t)$ be a solution of equation (3.19).
Then,

$$\int_0^x g^2(t)dt+\int_0^x\int_0^x F(s,t)\,g(s)g(t)dsdt=0,$$

or by (3.16)

$$0=\int_0^x g^2(t)dt+\sum_{n=1}^{\infty}\frac{1}{\alpha_n}\left(\int_0^x g(t)\,S_0(t,\lambda_n)\,dt\right)^2-\sum_{n=1}^{\infty}\frac{1}{\alpha_n^0}\left(\int_0^x g(t)\,S_0\left(t,\lambda_n^0\right)dt\right)^2$$

Using the Parseval equality

$$\int_0^x g^2(t)dt=\sum_{n=1}^{\infty}\frac{1}{\alpha_n^0}\left(\int_0^x g(t)\,S_0\left(t,\lambda_n^0\right)dt\right)^2$$

for the function $g(t)$ (for $t>x$ assume $g(t)=0$), we get

$$\sum_{n=1}^{\infty}\frac{1}{\alpha_n}\left(\int_0^x g(t)\,S_0(t,\lambda_n)\,dt\right)^2=0.$$

As $\alpha_n>0$, then

$$\int_0^x g(t)\,S_0(t,\lambda_n)\,dt=0,\ \ n=0,1,2,\ldots.$$

According to the lemma, we obtain $g(t)=0$. The theorem is proved. □

Corollary 3.1. *The coefficient* $q(x)$ *of equation* (1.1) *is uniquely determined with respect to spectral data* $\{\lambda_n,\alpha_n\}$ *of problem* (1.1) - (1.3).

The coefficient $q(x)$ is constructed according to the the following algorithm:

(1) by the given numbers $\{\lambda_n,\alpha_n\}$ we construct the function $F(x,t)$ by formula (3.16);
(2) find the function $\tilde{A}(x,t)$ from the equation (3.17);
(3) calculate $q(x)$ by formula (see (3.6) and (3.8))

$$\tilde{A}(x,x)=\frac{1}{2}\int_0^x q(s)\,ds.$$

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