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THE "ALGEBRAIC ZERO" CONDITION FOR ORTHOGONAL POLYNOMIALS OVER A CONTOUR IN THE WEIGHTED LEBESGUE SPACES

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Abstract. In this work, we continue to investigate the order of the height of the modulus of orthogonal polynomials over a contour and also arbitrary algebraic polynomials with respect to the weighted Lebesgue space, where the contour and the weight functions have some singularities. In this work we investigate the case of "algebraic zero" conditions with respect to weight and contour.

1. Introduction

Let $\mathbb{C}$ be a complex plane, $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$; $L \subset \mathbb{C}$ be a closed rectifiable Jordan curve, $G := intL$, with $0 \in G$, $\Omega := extL$. Let $h(z)$ be nonnegative, summable on L and nonzero except possible on a set of measure zero function. The systems of polynomials $\{K_n(z)\}$, $K_n(z) = a_n z^n + ...$, $\deg K_n = n$, $n = 0, 1, 2, ...$, satisfying the condition

$$\int\limits_L h(z) K_n(z) \overline{K_m(z)} \, |dz| = \begin{cases} 1, & n = m, \\ 0, & n \neq m, \end{cases}$$

are called orthonormal polynomials for the pair (L, h). These polynomials are determined uniquely if the coefficient $a_n > 0$.

These polynomials were first studied in [32], [33]. In [31], [19] and [16], these polynomials were investigated under the various conditions on the weight function $h(z)$ and contour L. In [36], many properties of the polynomials $K_n(z)$ were investigated for smooth contour and weight function $h(z)$ which is zero or infinite at finite number points on contour L. In [20] and [15], some properties of the polynomials $K_n(z)$ were considered for piecewise analytic contour L with finite number corners. In [37], some estimates for the rate of growth of the polynomials $K_n(z)$ were obtained on the contour L, depending of the singularity of the weight function $h(z)$ on L and of the contour L.

By $w = \Phi(z)$ denote the univalent conformal mapping of Ω onto $\Delta := \{w : |w| > 1\}$ with normalization $\Phi(\infty) = \infty$, $\lim_{z\to\infty} \frac{\Phi(z)}{z} > 0$ and

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$\Psi := \Phi^{-1}$. For $t \geq 1$, we set

$$L_t := \{z : \ |\Phi(z)| = t\}, \ L_1 \equiv L, \ G_t := intL_t, \ \Omega_t := extL_t.$$

Let $\{z_j\}_{j=1}^m$ be the fixed system of distinct points on curve L. For some fixed R_0, $1 < R_0 < \infty$, and $z \in \overline{G}_{R_0} \backslash G$, consider generalized Jacobi weight function $h(z)$, which is defined as follows:

$$h(z) := h_0(z) \prod_{j=1}^m |z - z_j|^{\gamma_j}, \tag{1.1}$$

where $\gamma_j > -1$, for all $j = 1, 2, ..., m$, and h_0 is uniformly separated from zero in L, i.e. there exists a constant $c_0(L) > 0$ such that for all $z \in G_{R_0}$

$$h_0(z) \geq c_0(L) > 0.$$

Let a rectifiable Jordan curve L, has a natural parametrization $z = z(s)$, $0 \leq s \leq l := mesL$. It is said to be $L \in C(1, \alpha)$, $0 < \alpha < 1$, if $z(s)$ is continuously differentiable and $z'(s) \in Lip\alpha$. Let L belong to $C(1, \alpha)$ everywhere except for a single point $z_1 \in L$, i.e., the derivative $z'(s)$ satisfies the Lipschitz condition on the $[0, l]$ and $z(0) = z(l) = z_1$, but $z'(0) \neq z'(l)$. Assume that L has a corner at z_1 with exterior angle $\omega_1\pi$, $0 < \omega_1 \leq 2$, and denote the set of such curves by $C(1, \alpha, \omega_1)$.

P.K. Suetin [37] investigated this problem for $K_n(z)$ with the weight function $h(z)$ defined as in (1.1) and for the curve $L \in C(1, \alpha, \omega_1)$. He showed that the condition of "pay off" singularity curve and weight function at the points z_1 can be given as follows:

$$(1 + \gamma_1)\,\omega_1 = 1, \tag{1.2}$$

and, under this condition, for $K_n(z)$ provided the following estimation:

$$|K_n(z)| \leq c\sqrt{n+1}, \ z \in L, \tag{1.3}$$

where $c = c(L) > 0$ is a constant independent on n.

In [37], the case, where $(1 + \gamma_1)\,\omega_1 \neq 1$, were also investigated. In particular, it is shown, if the singularity of a curve and weight function at the points z_1 satisfy the condition:

$$(1 + \gamma_1)\,\omega_1 > 1, \tag{1.4}$$

then for $|K_n(z)|$, the following estimation is true

$$|z - z_1|^{\mu_1} |K_n(z)| \leq c_1\sqrt{n+1}, \ z \in L, \tag{1.5}$$

$$|K_n(z_1)| \leq c_2\,(n+1)^{s_1}, \tag{1.6}$$

where

$$s_1 = \frac{1}{2}\,(1 + \gamma_1)\,\omega_1, \ \mu_1 = \frac{1}{2}\left(1 + \gamma_1 - \frac{1}{\omega_1}\right),$$

and $c_1 = c_1(L) > 0$, $c_2 = c_2(L) > 0$ are the constants independent of n.

In this work we study the estimations of the (1.5) and (1.6)-type, under the condition (1.4), for more general contours of the complex plane and we obtain the analogue of the estimations (1.5) and (1.6) for more general case. In addition, we study the growth of arbitrary algebraic polynomials with respect to their seminorm in the weighted Lebesgue space, under the condition of (1.4)-type.

2. Definitions and Main Results

Throughout this paper, $c, c_0, c_1, c_2, ...$ are positive and $\varepsilon_0, \varepsilon_1, \varepsilon_2, ...$ are sufficiently small positive constants (generally, different in different relations), which depends on G in general and, on parameters inessential for the argument; otherwise, such dependence will be explicitly stated.

Let $\wp_n$ denote the class of arbitrary algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N} := \{1, 2, ...\} \cup \{0\}$.

Without loss of generality, the number R_0 in the definition of the weight functions, we can take $R_0 = 2$. Otherwise the natural number n can be chosen $n \geq \left[\frac{\varepsilon_0}{R_0 - 1}\right]$, where ε_0, $0 < \varepsilon_0 < 1$, some fixed small constant.

Let $0 < p \leq \infty$. For a rectifiable Jordan curve L, we denote

$$\begin{aligned} \|P_n\|_{\mathcal{L}_p} &:= \|P_n\|_{\mathcal{L}_p(h,L)} := \left(\int_L h(z) \left|P_n(z)\right|^p |dz| \right)^{1/p}, \ 0 < p < \infty, \\ \|P_n\|_{\mathcal{L}_\infty} &:= \|P_n\|_{\mathcal{L}_\infty(1,L)} := \max_{z \in L} |P_n(z)|, \ p = \infty. \end{aligned}$$

For any $k \geq 0$ and $m > k$, notation $i = \overline{k, m}$ means $i = k, k+1, ..., m$.

Let $z = \psi(w)$ be the univalent conformal mapping of $B := \{w : |w| < 1\}$ onto the $G = intL$ normalized by $\psi(0) = 0$, $\psi'(0) > 0$. In [27, pp.286-294], a bounded Jordan region G is called κ *-quasidisk,* $0 \leq \kappa < 1$, if any conformal mapping ψ can be extended to a K -quasiconformal, $K = \frac{1+\kappa}{1-\kappa}$, the homeomorphism of the plane $\overline{\mathbb{C}}$ on plane $\overline{\mathbb{C}}$. In that case, the curve $L := \partial G$ is called a κ *-quasicircle.* The region G (curve L) is called a *quasidisk (quasicircle)*, if it is κ*-quasidisk (*κ *-quasicircle)* for some $0 \leq \kappa < 1$.

We denote the class of κ *-quasicircle* by $Q(\kappa)$, $0 \leq \kappa < 1$, and write $L \in Q$, if $L \in Q(\kappa)$, for some $0 \leq \kappa < 1$. It is well-known that the quasicircle may not even be locally rectifiable (see [21, p.104]).

We say that $L \in \widetilde{Q}(\kappa)$, $0 \leq \kappa < 1$, if $L \in Q(\kappa)$ and L is rectifiable. Analogously, $L \in \widetilde{Q}$, if $L \in \widetilde{Q}(\kappa)$, for some $0 \leq \kappa < 1$.

Definition 2.1. We say that $L \in Q_\alpha$, $0 < \alpha \leq 1$, if $L \in Q$ and $\Phi \in Lip\alpha$, $z \in \overline{\Omega}$.

We note that the class Q_α is sufficiently wide. A detailed account on it and the related topics are contained in [28], [22], [38] and the references cited therein. We consider only some cases:

Remark 2.1. a) If $L = \partial G$ is a Dini-smooth curve [28, p.48], then $L \in Q_1$.

b)If $L = \partial G$ is a piecewise Dini-smooth curve and largest exterior angle at L has opening $\alpha\pi, 0 < \alpha \leq 1$, [28, p.52], then $L \in Q_\alpha$.

c)If $L = \partial G$ is a smooth curve having continuous tangent line, then $L \in Q_\alpha$ for all $0 < \alpha < 1$.

d) If L is quasismooth (in the sense of Lavrentiev), that is, for every pair $z_1, z_2 \in L$, if $s(z_1, z_2)$ represents the smallest of the lengths of the arcs joining z_1 to z_2 on L, there exists a constant $c > 1$ such that $s(z_1, z_2) \leq c\,|z_1 - z_2|$, then $\Phi \in Lip\ \alpha$ for $\alpha = \frac{1}{2}(1 - \frac{1}{\pi}\arcsin\frac{1}{c})^{-1}$ [38].

e) If L is "c-quasiconformal" (see, for example, [22]), then $\Phi \in Lip\ \alpha$ for $\alpha = \frac{\pi}{2(\pi - \arcsin\frac{1}{c})}$. Also, if L is an asymptotic conformal curve, then $\Phi \in Lip\ \alpha$ for all $0 < \alpha < 1$ [22].

Definition 2.2. It is said that $L \in \widetilde{Q}_\alpha$, $0 < \alpha \leq 1$, if $L \in Q_\alpha$ and L is rectifiable.

In this case, we have the following:

Theorem A. [26] *Let $p > 0$. Suppose that $L \in \widetilde{Q}_\alpha$, for some $0 < \alpha \leq 1$and $h(z)$ defined as in (1.1) with $\gamma_j = 0$, for all $j = \overline{1, m}$. Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$,there exists $c_3 = c_3(L, p, \alpha) > 0$ such that*

$$\|P_n\|_{\mathcal{L}_\infty} \leq c_3 \|P_n\|_{\mathcal{L}_p(h_0,L)} \begin{cases} (n+1)^{\frac{1}{\alpha p}}, & \frac{1}{2} \leq \alpha \leq 1, \\ (n+1)^{\frac{\delta}{p}}, & 0 < \alpha < \frac{1}{2}, \end{cases} \tag{2.1}$$

and consequently,

$$\|K_n\|_{\mathcal{L}_\infty} \leq c_3 \begin{cases} (n+1)^{\frac{1}{\alpha p}}, & \frac{1}{2} \leq \alpha \leq 1, \\ (n+1)^{\frac{\delta}{p}}, & 0 < \alpha < \frac{1}{2}, \end{cases}$$

where $\delta = \delta(L)$, $\delta \in [1, 2]$, is a certain number.

Thus, Theorem A provides an opportunity to observe the growth of $|P_n(z)|$ on the curve L. Note that, Theorem A for $L := \{z : |z| = 1\}$ (i.e. $\kappa = 0$) provided in [18]. The other classical results are similar to (2.1) we can find in [34]. The evaluations of (2.1)-type for $0 < p < \infty$, $h(z) \equiv 1$ (or $h(z) \neq 1$) was also investigated in [35], [23], [24], [25, pp.122-133], [29], [14, Theorem 6], [2]-[9] and others (see also the references cited therein), for different Jordan curves having special properties.

According to 2.1, we can calculate α in the right parts of estimation (2.1) for each case, respectively. In addition, for $L \in \widetilde{Q}(\kappa)$, $0 \leq \kappa < 1$ the estimation (2.1) satisfies with $\alpha = \frac{1}{1+\kappa}$ [9].

Now, let's introduce "special" singular points on the curve L.

Definition 2.3. We say that $L \in \widetilde{Q}[\nu]$, $0 < \nu < 2$, if

a) $L \in \widetilde{Q}$,
b) For $\forall z \in L$, there exists a $r := r(L, z) > 0$ and $\nu := \nu(L, z)$, $0 < \nu < 2$, such that for some $0 \leq \theta_0 < 2$ a closed maximal circular sector $S(z; r, \nu) := \left\{\zeta : \zeta = z + re^{i\theta\pi},\ \theta_0 < \theta < \theta_0 + \nu\right\}$ of radius r and opening $\nu\pi$ lies in $\overline{G} = \overline{intL}$ with vetrex at z.

It is well known that each quasicircle satisfies the condition b). Nevertheless, this condition imposed on L gives a new geometric characterization of the curve. For example, if the contour L^* defined by

$$L^* := [0, i] \cup \left\{z : z = e^{i\theta\pi},\ \frac{1}{2} < \theta < 2\right\} \cup [1, 0],$$

then the coefficient of quasiconformality k of the L^* does not obtain so easily, whereas $L^* \in Q\left[\frac{3}{2}\right]$.

Definition 2.4. We say that $L \in \widetilde{Q}_\alpha[\nu_1, ..., \nu_m]$, $0 < \nu_1, ..., \nu_m < 2$, $0 < \alpha \leq 1$, if there exists a system of points $\{\zeta_i\} \in L$, $i = \overline{1, m}$, such that $L \in \widetilde{Q}[\nu_i]$ for any points $\zeta_i \in L$, $i = \overline{1, m}$, and $\Phi \in Lip\alpha$, $0 < \alpha \leq 1$, $z \in \overline{\Omega} \backslash \{\zeta_i\}$.

It is clear from Definition 2.3 (2.4), that each contour $L \in \widetilde{Q}_{\alpha}[\nu_1, ..., \nu_m]$, $0 < \nu_1, ..., \nu_m < 2$, $0 < \alpha \leq 1$, $i = \overline{1,m}$, may have "singularity" at the points $\{\zeta_i\}_{i=1}^m \in L$. If a contour L does not have such "singularity", i.e. if $\nu_i = 1$, $i = \overline{1,m}$, then it is written as $L \in \widetilde{Q}_{\alpha}$, $0 < \alpha \leq 1$.

Throughout this work, we will assume that the points $\{z_i\}_{i=1}^m \in L$ are defined in (1.1) and $\{\zeta_i\}_{i=1}^m \in L$ are defined in Definitions 2.2 coincides. Without the loss of generality, we also will assume that the points $\{z_i\}_{i=1}^m$ are ordered in the positive direction on the curve L.

In [26], it was shown the condition of "pay off" of singularity of curve and weight function at the points $\{z_i\}_{i=1}^m$:

Theorem B. *Let $p > 0$. Suppose that $L \in \widetilde{Q}_{\alpha}[\nu_1, ..., \nu_m]$, for some $0 < \nu_1, ..., \nu_m < 1$, $\frac{1}{2-\nu_i} \leq \alpha \leq 1$; $h(z)$ defined as in (1.1) and*

$$\gamma_i + 1 = \frac{1}{\alpha(2 - \nu_i)}, \tag{2.2}$$

for each points $\{z_i\}_{i=1}^m$. Then, for any $P_n \in \wp_n, n \in \mathbb{N}$, there exists $c_4 = c_4(L, p, \alpha) > 0$ such that

$$\|P_n\|_{\mathcal{L}_\infty} \leq c_4(n+1)^{\frac{1}{\alpha p}} \|P_n\|_{\mathcal{L}_p(h,L)}. \tag{2.3}$$

and consequently,

$$\|K_n\|_{\mathcal{L}_\infty} \leq c_4(n+1)^{\frac{1}{2\alpha}}. \tag{2.4}$$

Theorem B shows that, if the equality (2.2) is satisfied, then the growth of rate of the polynomials $P_n(z)$ $(K_n(z))$ on L does not depend on whether the weight function $h(z)$ and the boundary contour L have singularity or not. The condition (2.2) is called the condition of "interference of singularity" of weight function h and contour L at the "singular" points $\{z_i\}_{i=1}^m$.

Now, we assume the equality (2.2) does not hold for each singular points $\{z_i\}_{i=1}^m$. In [10], the case

$$\gamma_i + 1 < \frac{1}{\alpha(2 - \nu_i)}$$

was investigated.

In the present work, we investigate the case when

$$\gamma_i + 1 > \frac{1}{\alpha(2 - \nu_i)},$$

for each singular points $\{z_i\}_{i=1}^m \in L$ and obtain the following main results:

Theorem 2.1. *Let $p > 0$. Suppose that $L \in \widetilde{Q}_{\alpha}[\nu_1, \nu_2, ..., \nu_m]$, for some $0 < \nu_i < 1$ and $\frac{1}{2-\nu_i} \leq \alpha \leq 1$, $i = \overline{1,m}$; $h(z)$ are defined in (1.1) and*

$$\gamma_i + 1 > \frac{1}{\alpha(2 - \nu_i)} \tag{2.5}$$

for each points $\{z_i\}_{i=1}^m$. Then there exists $c_j = c_j(L, p, \gamma_i, \nu_i, \alpha) > 0$, $j = 5, 6$, such that, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, we have:

$$\max_{z \in L} \left(\prod_{i=1}^m |z - z_i|^{\mu_i} |P_n(z)| \right) \leq c_5 n^{\frac{1}{\alpha p}} \|P_n\|_{\mathcal{L}_p(h,L)}, \tag{2.6}$$

$$|P_n(z_i)| \leq c_6 n^{s_i} \|P_n\|_{\mathcal{L}_p(h,L)}, \tag{2.7}$$

where

$$\mu_i := \frac{1}{p}\left(\gamma_i + 1 - \frac{1}{\alpha}\right), \; s_i = \frac{\gamma_i + 1}{p}(2 - \nu_i), \; i = \overline{1,m}. \tag{2.8}$$

Corollary 2.2 *Under the conditions of Theorem 2.1, we have:*

$$\max_{z \in L}\left(\prod_{i=1}^{m} |z - z_i|^{\mu_i} |K_n(z)|\right) \leq c_5 n^{\frac{1}{2\alpha}},$$

$$|K_n(z_i)| \leq c_6 n^{s_i},$$

where μ_i and s_i *(for* $p = 2$*) defined as in (2.8).*

It follows from the conditions $\frac{1}{2} \leq \frac{1}{2-\nu_i} \leq \alpha \leq 1$, $i = \overline{1,m}$, the conditions (2.5) will be satisfied where $\gamma_i > 0$, $i = \overline{1,m}$. For that reason, we will call (2.5) algebraic zero conditions of the order $\eta_i := \alpha(2 - \nu_i)(1 + \gamma_i) - 1$ on each singular point $\{z_i\}_{i=1}^m \in L$.

For the curve $L \in C(1, \lambda, \nu_1)$, in case of one singular point $z_1 \in L$, we have:

Corollary 2.3 *If* $L \in C(1, \lambda, \nu_1)$ *and*

$$(\gamma_1 + 1)\,\nu_1 > 1,$$

is satisfies at the point z_1*, then we have*

$$|z - z_1|^{\mu_1} |P_n(z)| \leq c_5 \sqrt{n} \|P_n\|_{\mathcal{L}_2(h,L)}, \tag{2.9}$$

$$|P_n(z_1)| \leq c_6 n^{s_1} \|P_n\|_{\mathcal{L}_p(h,L)}, \tag{2.10}$$

where

$$\mu_1 := \frac{1}{2}\left(\gamma_1 + 1 - \frac{1}{\nu_1}\right), \; s_1 = \frac{1}{2}(1 + \gamma_1)\,\nu_1. \tag{2.11}$$

For $P_n \equiv K_n$, estimation (2.9) coincides from the result by P.K. Suetin, [37, Theorem 3]. Therefore, Theorem 2.1 generalizes the result [37, Th3] for $1 \leq \nu_1 \leq 2$ and extends the result to more general curves of the complex plane. Similar results for integral over an area are obtained in [3], [4].

Theorem 2.1 is true under the condition $0 < \nu_1 < 1$. For the analogous results corresponding to the case $1 \leq \nu_1 \leq 2$, we give a following definition.

Let S be rectifiable Jordan curve or arc and let $z = z(s), s \in [0, |S|]$, $|S| := mesS$, denote the natural representation of S.

Definition 2.5. [28, p.48] (see also [13]) We say that a Jordan curve or arc S called Dini-smooth (DS), if it has a parametrization $z = z(s), 0 \leq s \leq |S|$, such that $z'(s) \neq 0$, $0 \leq s \leq |S|$ and $\left|z'(s_2) - z'(s_1)\right| < g(s_2 - s_1)$, $s_1 < s_2$, where g is an increasing function for which

$$\int_0^1 \frac{g(x)}{x} dx < \infty.$$

Now, we shall define a new class of curves, which at the finite number points have exterior corners and interior cusps simultaneously.

Definition 2.6. We say that a Jordan curve $L \in PDS(\lambda_1, \lambda_2, ..., \lambda_m)$, $0 < \lambda_i \leq 2$, $i = \overline{1,m}$, if $L = \partial G$ consists of a union of finite number of Dini-smooth arcs $\{L_j\}_{j=0}^{m}$, connecting at the points $\{z_j\}_{j=0}^{m} \in L$ such that for every $z_i \in L$, $i = \overline{1,m}$, they have exterior (with respect to $\overline{G}$) angles $\lambda_i \pi$, $0 < \lambda_i \leq 2$, at the corner z_i.

In this case, we have the following:

Theorem 2.4. *Let $p > 0$. Suppose that $L \in PDS(\lambda_1, ..., \lambda_m)$, for some $0 < \lambda_i \leq 2$, $i = \overline{1,m}$; $h(z)$ defined as in (1.1) and*

$$\gamma_i + 1 > \frac{1}{\lambda_i}, \tag{2.12}$$

for each point $\{z_i\}_{i=1}^{m}$. Then there exists $c_j = c_j(L, p, \gamma_i, \lambda_i) > 0$, $j = 7, 8$, such that, for any $P_n \in \wp_n, n \in \mathbb{N}$, we have:

$$\max_{z \in L} \left(\prod_{i=1}^{m} |z - z_i|^{\mu_i} |P_n(z)| \right) \leq c_7 n^{\frac{1}{p}} \|P_n\|_{\mathcal{L}_p(h,L)}, \tag{2.13}$$

$$|P_n(z_i)| \leq c_8 n^{s_i} \|P_n\|_{\mathcal{L}_p(h,L)}, \tag{2.14}$$

where $\mu_i := \frac{1}{p}\left(\gamma_i + 1 - \frac{1}{\widetilde{\lambda}_i}\right)$, $s_i = \frac{\gamma_i+1}{p}\widetilde{\lambda}_i$, $i = \overline{1,m}$; $\widetilde{\lambda}_i := \begin{cases} \lambda_i, & \text{if } 0 < \lambda_i < 2, \\ 2, & \text{if } \lambda_i = 2. \end{cases}$

Corollary 2.5. *Under the conditions Theorem 2.4, we have:*

$$\max_{z \in L} \left(\prod_{i=1}^{m} |z - z_i|^{\mu_i} |K_n(z)| \right) \leq c_7 n^{\frac{1}{2}}, \tag{2.15}$$

$$|K_n(z_i)| \leq c_8 n^{s_i}, \tag{2.16}$$

for each points $\{z_i\}_{i=1}^{m}$.

Note that, $C(1, \alpha, \lambda_1) \subset PDS(\lambda_1)$ for each fixed $0 < \lambda_1 \leq 2$. In this, (2.15) and (2.16) coincides with (1.5) and (1.6). Thus, the Corollary 2.5 generalizes the corresponding result in [37].

Remark 2.2. a) The inequalities (2.1), (2.7) are sharp. For the polynomials $P_n^*(z) = 1 + 2z + ... + (n+1)z^n$, $h^*(z) = h_0(z)$ and $L := \{z : |z| = 1\}$, there exists a constant $c_9 = c_9(h_0) > 0$ such that:

$$\|P_n^*\|_{C(L)} \geq c_9 \sqrt{n} \|P_n^*\|_{\mathcal{L}_2(h^*,\ L)}.$$

b) The inequalities (2.6) and (2.9) are sharp in the sense that for the arbitrary polynomial $P_n \in \wp_n$, $L \in \widetilde{Q}_\alpha[\nu_1]$ and for arbitrary small ϵ, $0 < \epsilon < \mu_1$, the following is true:

$$|z - z_1|^{\mu_1 - \epsilon} |P_n(z)| \leq c_{10} n^{\frac{1}{p\alpha} + \epsilon} \|P_n\|_{\mathcal{L}_p(h,L)},$$

where

$$\mu_1 := \frac{1}{p}\left(\gamma_1 + 1 - \frac{1}{\alpha(2 - \nu_1)}\right).$$

In particular, for an arbitrary small ϵ^*, $0 < \epsilon^* < \mu_1^*$, there exists a contour L such that:

$$|z - z_1|^{\mu_1^* - \epsilon^*} |P_n(z)| \leq c_{11} n^{\frac{1}{2} + \epsilon^*} \|P_n\|_{\mathcal{L}_2(h,L)},$$

where

$$\mu_1^* := \frac{1}{2}\left(\gamma_1 + 1 - \frac{1}{\nu_1}\right).$$

3. Some auxiliary results

For $a > 0$ and $b > 0$, we shall use the notations "$a \preceq b$" (order inequality), if $a \leq cb$ and "$a \asymp b$" are equivalent to $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2 (independent of a and b) respectively.

The following definitions of the K-quasiconformal curves are well-known (see, for example, [11], [21, p.97] and [30]):

Definition 3.1. The Jordan arc (or curve) L is called $K-$ quasiconformal $(K \geq 1)$, if there is a $K-$quasiconformal mapping f of the region $D \supset L$ such that $f(L)$ is a line segment (or circle).

Let $F(L)$ denote the set of all sense preserving plane homeomorphisms f of the region $D \supset L$ such that $f(L)$ is a line segment (or circle) and lets define

$$K_L := \inf\left\{K(f) : f \in F(L)\right\},$$

where $K(f)$ is the maximal dilatation of a such mapping f. L is a quasiconformal curve, if $K_L < \infty$, and L is a $K-$quasiconformal curve, if $K_L \leq K$.

Remark 3.1. It is well-known that, if we are not interested with the coefficients of quasiconformality of the curve, the definitions of "quasicircle" and "quasiconformal curve" are identical. However, if we are also interested with the coefficients of quasiconformality of the given curve, then we will consider that if the curve L is $K-$ quasiconformal, then it is $\kappa-$quasicircle with $\kappa = \frac{K^2-1}{K^2+1}$.

By Remark 3.1, for simplicity, we will use both terms, depending on the situation.

For $z \in \mathbb{C}$ and $M \subset \mathbb{C}$, we set

$$d(z, M) = dist(z, M) := \inf\left\{|z - \zeta| : \ \zeta \in M\right\}.$$

For $\delta > 0$ and $z \in \mathbb{C}$ let us set: $B(z, \delta) := \{\zeta : |\zeta - z| < \delta\}$, $\Omega(z, \delta) := \Omega \cap B(z, \delta)$.

Lemma 3.1. [1] *Let L be a $K-$quasiconformal curve, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \preceq d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $j = 1, 2, 3$. Then*

a) *The statements $|z_1 - z_2| \preceq |z_1 - z_3|$ and $|w_1 - w_2| \preceq |w_1 - w_3|$ are equivalent, and similarly so are $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$.*

b) *If $|z_1 - z_2| \preceq |z_1 - z_3|$, then*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{\varepsilon} \preceq \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \preceq \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c},$$

where $\varepsilon = \varepsilon(L) < 1, c = c(L) > 1$, $0 < r_0 < 1$ are *constants, depending on L and $L_{r_0} := \{z = \psi(w) : |w| = r_0\}$.*

Corollary 3.2. *Under the assumptions of Lemma 3.1, if $z_3 \in L_{r_0} (z_3 \in L_{Rr_0})$, then*

$$|w_1 - w_2|^{K^2} \preceq |z_1 - z_2| \preceq |w_1 - w_2|^{K^{-2}}$$

Let $\{z_j\}_{j=1}^m$ be a fixed system of the points on L and the weight function $h(z)$ defined as (1.1).

Recall that for $0 < \delta_j < \delta_0 := \frac{1}{4}\min\{|z_i - z_j| : i, j = 1, 2, ..., m, i \neq j\}$,we put $\Omega(z_j, \delta_j) := \Omega \cap \{z : |z - z_j| \leq \delta_j\}$; $\delta := \min\limits_{1\leq j\leq m} \delta_j$, $\Omega(\delta) := \bigcup\limits_{j=1}^{m} \Omega(z_j, \delta)$, $\widehat{\Omega} := \Omega\backslash\Omega(\delta)$. Additionally, let $\Delta_j := \Phi(\Omega(z_j, \delta))$, $\Delta(\delta) := \bigcup\limits_{j=1}^{m} \Phi(\Omega(z_j, \delta))$, $\widehat{\Delta}(\delta) := \Delta\backslash\Delta(\delta)$.

Throughout this work, we will take $R = 1 + \frac{\varepsilon_0}{n+1}$, for some fixed $0 < \varepsilon_0 < 1$. Further, for any $t > 1$ and $j = \overline{1, m}$, we introduce:

$$\begin{aligned} w_j &: = \Phi(z_j), \varphi_j := \arg w_j,\ L_t^j := L_t \cap \overline{\Omega}_t^j;\ F_t^j := \Phi(L_t^j) \\ \Omega_t^j &: = \Psi(\Delta_{t,j}^{'}), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} \Delta_{t,1}^{'} &: = \left\{ w = te^{i\theta} : t > 1,\ \frac{\varphi_m + \varphi_1}{2} \leq \theta < \frac{\varphi_1 + \varphi_2}{2} \right\}, \\ \Delta_{t,m}^{'} &: = \left\{ w = te^{i\theta} : t > 1,\ \frac{\varphi_{m-1} + \varphi_m}{2} \leq \theta < \frac{\varphi_m + \varphi_1}{2} \right\}, \end{aligned}$$

and, for $j = \overline{2, m-1}$

$$\Delta_{t,j}^{'} := \left\{ w = te^{i\theta} : t > 1,\ \frac{\varphi_{j-1} + \varphi_j}{2} \leq \theta < \frac{\varphi_j + \varphi_{j+1}}{2} \right\}.$$

$L = \bigcup\limits_{j=1}^{m} L^j$; $L_t = \bigcup\limits_{j=1}^{m} L_t^j$.

We will use the well known estimation for the $\Psi^{'}$ (see, for example, [12, Th.2.8]):

$$\left|\Psi'(\tau)\right| \asymp \frac{d(\Psi(\tau), L)}{|\tau| - 1}. \tag{3.2}$$

The following lemma is a consequence of the results given in [28], [13, pp.32-36], and estimation (3.2) (see, for example, [12, Th.2.8]):

Lemma 3.3. *Let a Jordan curve $L \in PDS(\lambda_1, ..., \lambda_m)$, $0 < \lambda_j \leq 2$, $j = \overline{1, m}$. Then,*

i) for any $w \in \Delta_j$, $|\Psi(w) - \Psi(w_j)| \asymp |w - w_j|^{\lambda_j}$, $|\Psi'(w)| \asymp |w - w_j|^{\lambda_j - 1}$;

ii)for any $w \in \overline{\Delta}\ \Delta_j$, $|\Psi(w) - \Psi(w_j)| \asymp |w - w_j|$, $|\Psi'(w)| \asymp 1$.

Lemma 3.4. [7] *Let L be a rectifiable Jordan curve $h(z)$ defined as in (1.1). Then, for arbitrary $P_n(z) \in \wp_n$, any $R > 1$ and $n \in \mathbb{N}$ we have:*

$$\|P_n\|_{\mathcal{L}_p(h,L_R)} \leq R^{n+\frac{1+\gamma*}{p}} \|P_n\|_{\mathcal{L}_p(h,L)}, p > 0. \tag{3.3}$$

Remark 3.2. In case of $h(z) \equiv 1$, the estimation (3.3) has been proved in [17].

4. Proof of Theorems

Throughout proofs of all theorems, we will take $n \geq \left[\frac{\varepsilon_0}{R_0 - 1}\right]$, where ε_0, $0 < \varepsilon_0 < 1$, some fixed small constant. In addition, in case when $n = 0$, the number n, participating in the all inequalities below will be changed to $(n + 1)$.

4.1. Proof of Theorem 2.1.

Proof. Suppose that $L \in \widetilde{Q}_\alpha[\nu_1, ..., \nu_m]$, for some $0 < \nu_1, ..., \nu_m < 1$, $\frac{1}{2-\nu_i} \leq \alpha \leq 1, i = \overline{1,m}$, be given and $h(z)$ defined in (1.1). For any $R > 1$, let us define $R_1 := 1 + \frac{R-1}{2}$. Let $\{\xi_j\}$, $1 \leq j \leq m \leq n$, denote zeros of polynomial $P_n(z)$ lying in Ω. The Blashke function [39] with respect to the zeros of the polynomial $P_n(z)$ is defined as follows:

$$B_m(z) := \prod_{j=1}^{m} B^j(z) := \prod_{j=1}^{m} \frac{\Phi(z) - \Phi(\xi_j)}{1 - \overline{\Phi(\xi_j)}\Phi(z)}\ , \ z \in \Omega,$$

It is easy to see that the $B_m(\xi_j) = 0$ and $|B_m(z)| \equiv 1$ at $z \in L$. For any $p > 0$ and $w \in \Delta$ let us set:

$$g_n(w) := \prod_{j=1}^{m} \left[\frac{\Psi(w) - \Psi(w_j)}{w}\right]^{p\mu_j/2} \left[\frac{P_n(\Psi(w))}{w^{n+1}B_m(\Psi(w))}\right]^{p/2}, \ w = \Phi(z). \tag{4.1}$$

The function $g_n(w)$ is analytic in Δ, continuous on $\overline{\Delta}$, $g_n(\infty) = 0$ and does not have zeros in Δ. We take an arbitrary continuous branch of the $g_n(w)$ and for this branch, we maintain the same designation. Then, the Cauchy integral representation for the $g_n(z)$ is given by the formula

$$g_n(w) = -\frac{1}{2\pi i} \int\limits_{|\tau|=R_1} g_n(\tau) \frac{d\tau}{\tau - w}, |w| = R.$$

Therefore,

$$\begin{aligned}
&\left| \prod_{j=1}^{m} \left[\frac{\Psi(w) - \Psi(w_j)}{w}\right]^{p\mu_j/2} \left[\frac{P_n(\Psi(w))}{w^{n+1}B^j(\Psi(w))}\right]^{p/2} \right| \\
\leq\ & \frac{1}{2\pi} \int\limits_{|\tau|=R_1} \prod_{j=1}^{m} \left|\frac{\Psi(\tau) - \Psi(w_j)}{\tau}\right|^{p\mu_j/2} \left|\frac{P_n(\Psi(\tau))}{\tau^{n+1}B^j(\Psi(\tau))}\right|^{p/2} \frac{|d\tau|}{|\tau - w|},
\end{aligned}$$

or

$$\begin{aligned}
J_n \ :\ &= \prod_{j=1}^{m} \left[|\Psi(w) - \Psi(w_j)|\right]^{p\mu_j/2} |P_n(\Psi(w))|^{p/2} \qquad (4.2)\\
&\leq \frac{1}{2\pi} \prod_{j=1}^{m} \frac{\max\limits_{|w|=R} |w|^{p\mu_j/2} \left|w^{n+1}B^j(\Psi(w))\right|^{p/2}}{\min\limits_{|\tau|=R_1} |\tau|^{p\mu_j/2} \left|\tau^{n+1}B^j(\Psi(\tau))\right|^{p/2}} \\
&\times \int\limits_{|\tau|=R_1} \prod_{j=1}^{m} |\Psi(\tau) - \Psi(w_j)|^{p\mu_j^*/2} |P_n(\Psi(\tau))|^{p/2} \frac{|d\tau|}{|\tau - w|}.
\end{aligned}$$

Since $\left|B^j(\zeta)\right| = 1$, for $\zeta \in L$, then for arbitrary $\varepsilon, 0 < \varepsilon < \varepsilon_1$, there exists a circle $|w| = 1 + \frac{\varepsilon}{n}$, such that for any $j = 1, 2, ..., m$, the following inequalities are satisfied:

$$\left|B^j(\Psi(w)\right| > 1 - \varepsilon, \ \left|b^j(\Psi(w)\right| > 1 - \varepsilon.$$

Then

$$|B_m(\zeta)| > (1-\varepsilon)^m \succeq 1,$$

for $\varepsilon \leq n^{-1}$ and $\zeta \in L_{R_1}$. Further

$$|\Phi(\zeta)| = R_1 > 1, \ |\Phi(\zeta)|^{n+1} = R_1^{n+1} \succeq 1,$$

for $\zeta \in L_{R_1}$. On the other hand, we obtain

$$|w|^{p\mu_j/2} \preceq 1, \quad \left|w^{n+1}B_m(\Psi(w))\right|^{p/2} \preceq 1, \ z \in L_R.$$

According to this estimations, from (4.2), we have:

$$J_n \preceq \int\limits_{|\tau|=R_1} \prod_{j=1}^{m} |\Psi(\tau)-\Psi(w_j)|^{p\mu_j/2} \left|P_n\left(\Psi(\tau)\right)\right|^{p/2} \frac{|d\tau|}{|\tau-w|}.$$

Multiplying the numerator and determinator of the integrand by $h^{1/2}(\Psi(\tau)) \cdot |\Psi'(\tau)|^{1/2}$ and applying the Hölder inequality, we obtain:

$$J_n \preceq \left(\int\limits_{|\tau|=R_1} \prod_{j=1}^{m} |\Psi(\tau)-\Psi(w_j)|^{\gamma_j} \left|P_n\left(\Psi(\tau)\right)\right|^{p} \left|\Psi'(\tau)\right| |d\tau| \right)^{1/2} \tag{4.3}$$

$$\times \left(\int\limits_{|\tau|=R_1} \prod_{j=1}^{m} |\Psi(\tau)-\Psi(w_j)|^{p\mu_j-\gamma_j} \frac{|d\tau|}{|\Psi'(\tau)| \, |\tau-w|^2} \right)^{1/2} =: J_{n,1} \times J_{n,2},$$

By replacing the variable $\tau = \Phi(\zeta)$ and according to Lemma 3.4, we get

$$J_{n,1} \preceq \|P_n\|_{\mathcal{L}_p}^{p/2}. \tag{4.4}$$

Then, from (4.2)-(4.4), we get:

$$\prod_{j=1}^{m} |z-z_j|^{\mu_j} \left|P_n\left(z\right)\right| \tag{4.5}$$

$$\preceq \ \|P_n\|_{\mathcal{L}_p} \left(\int\limits_{|\tau|=R_1} \prod_{j=1}^{m} |\Psi(\tau)-\Psi(w_j)|^{p\mu_j-\gamma_j} \frac{(|\tau|-1)\,|d\tau|}{d(\Psi(\tau),L)\,|\tau-w|^2} \right)^{1/p}.$$

By denoting last integral as

$$\widetilde{J}_{n,m} := \left(\int\limits_{|\tau|=R_1} \prod_{j=1}^{m} |\Psi(\tau)-\Psi(w_j)|^{p\mu_j-\gamma_j} \frac{(|\tau|-1)\,|d\tau|}{d(\Psi(\tau),L)\,|\tau-w|^2} \right)^{1/p}, \tag{4.6}$$

we see that to prove the theorem it suffices to estimate the integral $\widetilde{J}_{n,m}$. Since the points $\{z_j\}_{j=1}^{m} \in L$ are distinct, according to notations (3.1), for arbitrary fixed $j, \ 1 \leq j \leq m$, we get:

$$\left(\widetilde{J}_{n,m}\right)^p \tag{4.7}$$

$$= \sum_{i=1}^{m} \int\limits_{F^i_{R_1}} \prod_{j=1}^{m} |\Psi(\tau)-\Psi(w_j)|^{p\mu_j-\gamma_j} \frac{(|\tau|-1)\,|d\tau|}{d(\Psi(\tau),L)\,|\tau-w|^2}$$

$$\asymp \sum_{i=1}^{m} \int\limits_{F^i_{R_1}} \prod_{j=1}^{m} |\Psi(\tau)-\Psi(w_j)|^{p\mu_j-\gamma_j} \frac{(|\tau|-1)\,|d\tau|}{d(\Psi(\tau),L)\,|\tau-w|^2} =: \sum_{i=1}^{m} \widetilde{J}^i_{n,j_o}(F^i_{R_1}),$$

where, for each subarc $l \subset F^i_R$, $\widetilde{J}^i_{n,j_0}(l)$ is denoted by

$$\widetilde{J}^i_{n,j_0}(l) := \int\limits_{l} |\Psi(\tau)-\Psi(w_{j_0})|^{p\mu_{j_0}-\gamma_{j_0}} \frac{(|\tau|-1)\,|d\tau|}{d(\Psi(\tau),L)\,|\tau-w|^2}. \tag{4.8}$$

It remains to estimate the integrals $\widetilde{J}^i_{n,j_0}(F^1_{R_1})$ for each $i=\overline{1,m}$. For simplicity of our next calculations, we assume that

$$m=1,\ j_0=1,\ \mu:=\mu_1;\ s^*:=s^*_1,\ \gamma:=\gamma_1,\ \nu:=\nu_1.\ R=1+\frac{\varepsilon_0}{n+1}. \tag{4.9}$$

In this situation, the integral $\widetilde{J}^i_{n,j}(L^1_{R_1})$ can be written as:

$$\widetilde{J}^1_{n,1}(F^1_{R_1}) := \int\limits_{F^1_{R_1}} |\Psi(\tau)-\Psi(w_1)|^{p\mu-\gamma} \frac{(|\tau|-1)\,|d\tau|}{d(\Psi(\tau),L)\,|\tau-w|^2}. \tag{4.10}$$

Under this assumptions, $L \in \widetilde{Q}_\alpha[\nu]$, for some $0<\nu<1,\ 0<\alpha\le 1$. Then, according to [22], $\psi \in Lip\,\nu$ and there exists the number $\delta, 0<\delta<\delta_0<diam\overline{G}$, such that

$$\Phi \in Lip\frac{1}{2-\nu},\ z\in\overline{\Omega(z_1,\delta)}. \tag{4.11}$$

We accept the notation

$$\begin{aligned} L^1_{R_1,1} &: = L^1_{R_1}\cap\Omega(z_1,\delta), L^1_{R_1,2} := L_{R_1}\backslash L^1_{R_1,1}; F^1_{R_1,i} := \Phi(L^1_{R_1,i}); \\ L^1_1 &: = L^1\cap B(z_1,\delta), L^1_2 := L^1\backslash L^1_1;\ F^1_i := \Phi(L^1_i),\ i=1,2. \end{aligned} \tag{4.12}$$

Taking into consideration these notation, from (4.10), we have:

$$\begin{aligned} \widetilde{J}^1_{n,1}(F^1_{R_1}) &= \int\limits_{F^1_{R_1}} |\Psi(\tau)-\Psi(w_1)|^{p\mu-\gamma} \frac{(|\tau|-1)\,|d\tau|}{d(\Psi(\tau),L)\,|\tau-w|^2} \\ &= \sum_{i=1}^{3} \int\limits_{F^1_{R_1,i}} |\Psi(\tau)-\Psi(w_1)|^{p\mu-\gamma} \frac{(|\tau|-1)\,|d\tau|}{d(\Psi(\tau),L)\,|\tau-w|^2} \\ &= : \sum_{i=1}^{2} \widetilde{J}^1_{n,1}(L^1_{R_1,i}), \end{aligned} \tag{4.13}$$

where

$$\widetilde{J}^1_{n,1}(F^1_{R_1,i}) := \int\limits_{F^1_{R_1,i}} |\Psi(\tau)-\Psi(w_1)|^{p\mu-\gamma} \frac{(|\tau|-1)\,|d\tau|}{d(\Psi(\tau),L)\,|\tau-w|^2},\ i=1,2. \tag{4.14}$$

We consider the individual cases.

1) Let $z \in L^1_{R,1}$. Let us denote $\widetilde{\zeta} \in L^1_{R_1,1}$ such that $d(\zeta, L) = \left|\zeta - \widetilde{\zeta}\right|$ and $\widetilde{w} := \Phi(\widetilde{\zeta})$;

$$\begin{aligned} L^{1,1}_{R_1,j} &: = \left\{\zeta \in L^1_{R_1,j} : |\zeta - z_1| \leq c_1 \left|\zeta - \widetilde{\zeta}\right|\right\}, \\ L^{1,2}_{R_1,j} &: = \left\{\zeta \in L^1_{R_1,j} : c_1 \left|\zeta - \widetilde{\zeta}\right| < |\zeta - z_1| < \delta\right\}, \\ F^{1,i}_{R_1,i} &: = \Phi(L^{1,i}_{R_1,i}),\ i,j = 1,2. \end{aligned} \tag{4.15}$$

1.1) Then

$$\widetilde{J}^1_{n,1}(F^1_{R_1,1}) = \widetilde{J}^1_{n,1}(F^{1,1}_{R_1,1}) + \widetilde{J}^1_{n,1}(F^{1,2}_{R_1,1}),$$

According to Lemma 3.1 and (3.2), for $\widetilde{J}^1_{n,1}(F^{1,1}_{R_1,1})$ we have:

$$\begin{aligned} &\widetilde{J}^1_{n,1}(F^{1,1}_{R_1,1}) \\ &\preceq (R_1 - 1) \int\limits_{F^{1,1}_{R_1,1}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(\widetilde{w})|^{1-p\mu+\gamma} |\tau - w|^2} \\ &\preceq (R_1 - 1) \int\limits_{F^{1,1}_{R_1,1}} \frac{|d\tau|}{|\tau - \widetilde{w}|^{(1-p\mu+\gamma)(2-\nu)} |\tau - w|^2} \\ &\preceq (R_1 - 1)^{1-(1-p\mu+\gamma)(2-\nu)} \int\limits_{F^{1,1}_{R_1,1}} \frac{|d\tau|}{|\tau - w|^2} \\ &\preceq (R_1 - 1)^{1-(1-p\mu+\gamma)(2-\nu)} \cdot \frac{1}{R - R_1} \preceq n^{\frac{1}{\alpha}}. \end{aligned} \tag{4.16}$$

Now, lets estimate the integral $\widetilde{J}^1_{n,1}(F^{1,2}_{R_1,1})$. According to Lemma 3.1, for $\zeta \in \widetilde{J}^1_{n,1}(F^{1,2}_{R_1,1})$ we have $R - R_1 < |\tau - w_1| \preceq 1$. We set $\varepsilon_0 := |\tau| - 1$. In this case, we take the discs centered at the point w_1, and radius $2^s\varepsilon_0$, $s = 1, 2, ...N$, where we choose a number N such that the circle is $Q_N = \left\{\tau : |\tau - w_1| = 2^N\varepsilon_0\right\}$, that satisfies the conditions $Q_N \cap \{t : |t| = R\} \neq \varnothing$, $Q_{N+1} \cap \{t : |t| = R_1\} = \varnothing$. Then, setting $F^s_{R_1,1} := F^{1,1}_{R_1,1} \cap \left\{t : 2^{s-1}\varepsilon_o \leq |t - w_1| \leq 2^s\varepsilon_o\right\}$, we have:

$$\widetilde{J}^1_{n,1}(F^{1,2}_{R_1,1}) \tag{4.17}$$

$$= \int\limits_{F^{1,2}_{R_1,1}} |\Psi(\tau)-\Psi(w_1)|^{p\mu-\gamma} \frac{(|\tau|-1)\,|d\tau|}{|\Psi(\tau)-\Psi(\widetilde{w})|\,|\tau-w|^2}$$

$$= \int\limits_{F^{1,2}_{R_1,1}} \left|\frac{\Psi(\tau)-\Psi(w_1)}{\Psi(\tau)-\Psi(\widetilde{w})}\right| \frac{1}{|\Psi(\tau)-\Psi(w_1)|^{1-p\mu+\gamma}} \frac{(|\tau|-1)\,|d\tau|}{|\tau-w|^2}$$

$$\preceq \sum_{s=1}^{\infty} \int\limits_{F^{s}_{R_1,1}} \left[\frac{|\tau-w_1|}{|\tau|-1}\right]^{\varepsilon} \frac{|d\tau|}{|\tau-w_1|^{(1-p\mu+\gamma)(2-\nu)}} \frac{(|\tau|-1)\,|d\tau|}{|\tau-w|^2}$$

$$\preceq \sum_{s=1}^{\infty} \left(\frac{2^s\varepsilon_o}{\varepsilon_o}\right)^{\varepsilon} \frac{\varepsilon_o}{(2^{s-1}\varepsilon_o)^{\frac{1}{\alpha}}} \int\limits_{F^{s}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^2}$$

$$\preceq 2^{\frac{1}{\alpha}} \varepsilon_o^{1-\frac{1}{\alpha}} \sum_{s=1}^{\infty} \left(\frac{2^{\varepsilon}}{2^{\frac{1}{\alpha}}}\right)^{s-1} \int\limits_{F^{s}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^2} \preceq n\cdot \varepsilon_o^{1-\frac{1}{\alpha}} \sum_{s=1}^{\infty} \left(\frac{2^{\varepsilon}}{2^{\frac{1}{\alpha}}}\right)^{s-1}$$

$$= n\cdot n^{\frac{1}{\alpha}-1} \sum_{s=1}^{\infty} \left(\frac{2^{\varepsilon}}{2^{\frac{1}{\alpha}}}\right)^{s-1} \preceq n^{\frac{1}{\alpha}},$$

where $\varepsilon=\varepsilon(L)<1$ defined from Lemma 3.1.

1.2) For any $\zeta \in L^1_{R_1,2}$, $\delta < |\zeta - z_1| < \delta_0$ and, from (4.14), we obtain:

$$\widetilde{J}^1_{n,1}(F^1_{R_1,2}) \tag{4.18}$$

$$= \int\limits_{F^{1}_{R_1,2}} |\Psi(\tau)-\Psi(w_1)|^{p\mu-\gamma} \frac{(|\tau|-1)\,|d\tau|}{|\Psi(\tau)-\Psi(\widetilde{w})|\,|\tau-w|^2}$$

$$\preceq (\delta_0)^{p\mu-\gamma} \int\limits_{F^{1}_{R_1,2}} \frac{(|\tau|-1)\,|d\tau|}{|\tau-\widetilde{w}|^{\frac{1}{\alpha}}\,|\tau-w|^2}$$

$$\preceq \frac{1}{n}\cdot\frac{1}{(R_1-1)^{\frac{1}{\alpha}}} \int\limits_{F^{1}_{R_1,2}} \frac{|d\tau|}{|\tau-w|^2} \preceq n^{\frac{1}{\alpha}}$$

2) Let $z \in L^1_{R,2}$.

2.1) Under the notations (4.15), we have

$$\widetilde{J}^1_{n,1}(F^1_{R_1,1}) = \widetilde{J}^1_{n,1}(F^{1,1}_{R_1,1}) + \widetilde{J}^1_{n,1}(F^{1,2}_{R_1,1}),$$

and analogously to the estimations (4.16) and (4.17), it is easy to obtain the following:

$$\widetilde{J}^1_{n,1}(F^{1,1}_{R_1,1}) \preceq n^{\frac{1}{\alpha}};\ \widetilde{J}^1_{n,1}(F^{1,2}_{R_1,1}) \preceq n^{\frac{1}{\alpha}}. \tag{4.19}$$

2.2) According to notations (4.15), integral can be written as follows:

$$\widetilde{J}^1_{n,1}(F^1_{R_1,2}) = \widetilde{J}^1_{n,1}(F^{1,1}_{R_1,2}) + \widetilde{J}^1_{n,1}(F^{1,2}_{R_1,2}).$$

Then, according to Lemma 3.1 and (3.2), for $\widetilde{J}^1_{n,1}(F^{1,1}_{R_1,2})$ we have:

$$\begin{aligned}
\widetilde{J}^1_{n,1}(F^{1,1}_{R_1,2}) = & \int\limits_{F^{1,1}_{R_1,2}} |\Psi(\tau)-\Psi(w_1)|^{p\mu-\gamma} \frac{(|\tau|-1)\,|d\tau|}{|\Psi(\tau)-\Psi(\widetilde{w})|\,|\tau-w|^2} \\
\preceq & \ (R_1-1) \int\limits_{F^{1,1}_{R_1,2}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(\widetilde{w})|^{1-p\mu+\gamma}\,|\tau-w|^2} \qquad (4.20) \\
\preceq & \ (R_1-1) \int\limits_{F^{1,1}_{R_1,2}} \frac{|d\tau|}{|\tau-\widetilde{w}|^{(1-p\mu+\gamma)(2-\nu)}\,|\tau-w|^2} \\
\preceq & \ (R_1-1)^{1-(1-p\mu+\gamma)(2-\nu)} \int\limits_{F^{1,1}_{R_1,2}} \frac{|d\tau|}{|\tau-w|^2} \\
\preceq & \ (R_1-1)^{1-(1-p\mu+\gamma)(2-\nu)} \cdot \frac{1}{R-R_1} \preceq n^{\frac{1}{\alpha}}.
\end{aligned}$$

Analogously to (4.18), we get:

$$\begin{aligned}
\widetilde{J}^1_{n,1}(F^{1,2}_{R_1,2}) = & \int\limits_{F^{1,2}_{R_1,2}} |\Psi(\tau)-\Psi(w_1)|^{p\mu-\gamma} \frac{(|\tau|-1)\,|d\tau|}{|\Psi(\tau)-\Psi(\widetilde{w})|\,|\tau-w|^2} \\
\preceq & \ (\delta_0)^{p\mu-\gamma} \int\limits_{F^{1,2}_{R_1,2}} \frac{(|\tau|-1)\,|d\tau|}{|\tau-\widetilde{w}|^{\frac{1}{\alpha}}\,|\tau-w|^2} \qquad (4.21) \\
\preceq & \ \frac{1}{n}\cdot\frac{1}{(R_1-1)^{\frac{1}{\alpha}}} \int\limits_{F^{1}_{R_1,2}} \frac{|d\tau|}{|\tau-w|^2} \\
\preceq & \ n^{\frac{1}{\alpha}-1} \int\limits_{F^{1}_{R_1,2}} \frac{|d\tau|}{|\tau-w|^2} \preceq n\cdot n^{\frac{1}{\alpha}-1} = n^{\frac{1}{\alpha}}.
\end{aligned}$$

Combining estimations (4.5), (4.10), (4.13)-(4.21) and according to notations (4.9), for arbitrary $z\in L_R$, we obtain:

$$|z-z_1|^{\mu}\,|P_n(z)| \ \preceq n^{\frac{1}{\alpha p}}\cdot \|P_n\|_{\mathcal{L}_p}, \ p>0. \qquad (4.22)$$

The estimation (4.22) satisfied on L_R. We show that it is also carried out on L. For $R>1$, let $w=\varphi_R(z)$ denote the univalent conformal mapping of G_R onto B normalized by $\varphi_R(0)=0,\ \varphi_R'(0)>0$, and let $\{\zeta_j\}, 1\le j\le m\le n$, zeros of $P_n(z)$, lying on G_R. Let

$$B_{m,R}(z) := \prod_{j=1}^{m} B_{j,R}(z) = \prod_{j=1}^{m} \frac{\varphi_R(z)-\varphi_R(\zeta_j)}{1-\overline{\varphi_R(\zeta_j)}\varphi_R(z)} \qquad (4.23)$$

denote a Blashke function with respect to zeros $\{\zeta_j\}$, $1\le j\le m\le n$, of $P_n(z)$. Clearly,

$$|B_{m,R}(z)|\equiv 1, z\in L_R;\ \ |B_{m,R}(z)|<1, z\in G_R. \qquad (4.24)$$

For any $\mu > 0$ and $z \in G_R$, let us set:

$$H_n(z) := \left[\frac{P_n(z)}{B_{m,R}(z)}\right]^{1/\mu}.$$

The function $H_n(z)$ is analytic in G_R, continuous on $\overline{G}_R$ and does not have zeros in G_R. We take an arbitrary continuous branch of the $H_n(w)$. Then, applying maximal modulus principle to $[H_n(z)]^{1/\mu}(z-z_1)$, we have:

$$\left|\left[\frac{P_n(z)}{B_{m,R}(z)}\right]^{1/\mu}(z-z_1)\right| \tag{4.25}$$

$$\begin{aligned}
&\leq \max_{\zeta\in\overline{G_R}}\left|\left[\frac{P_n(\zeta)}{B_{m,R}(\zeta)}\right]^{1/\mu}(\zeta-z_1)\right| \leq \max_{\zeta\in L_R}|P_n(\zeta)|^{1/\mu}|\zeta-z_1| \\
&\preceq \left(n^{\frac{1}{\alpha p}}\cdot\|P_n\|_{\mathcal{L}_p}\right)^{1/\mu}, \ z\in L,
\end{aligned}$$

and, therefore we find:

$$|(z-z_1)^{\mu}P_n(z)| \preceq n^{\frac{1}{\alpha p}}\cdot\|P_n\|_{\mathcal{L}_p}, \ z\in L. \tag{4.26}$$

Since the system of points $\{z_j\}_{j=1}^m$ are isolated, according to assumption (4.9), we get:

$$\max_{z\in L}\left(\prod_{j=1}^{m}[|z-z_j|]^{\mu_j}|P_n(z)|\right) \preceq n^{\frac{1}{\alpha p}}\cdot\|P_n\|_{\mathcal{L}_p}, \ p>0, \tag{4.27}$$

and we complete the proof of estimation (2.6).

Now, we prove the estimation (2.7). For each $R>1$, $p>0$ and $z\in G_R$, let us set

$$T_n(z) := \left[\frac{P_n(z)}{B_{m,R}(z)}\right]^{p/2}, \tag{4.28}$$

where $B_{m,R}(z)$ is a Blashke function defined in (4.23). The function $T_n(z)$ is analytic in G_R, continuous on $\overline{G}_R$ and does not have zeros in G_R. We take an arbitrary continuous branch of the $T_n(z)$ and for this branch we maintain the same designation. Then, the Cauchy integral representation for the $T_n(z)$ in G_R gives

$$T_n(z) = \frac{1}{2\pi i}\int_{L_R} T_n(\zeta)\frac{d\zeta}{\zeta-z}, \ z\in G_R,$$

or

$$\left|\left[\frac{P_n(z)}{B_{m,R}(z)}\right]^{p/2}\right| \leq \frac{1}{2\pi}\int_{L_R}\left|\frac{P_n(\zeta)}{B_{m,R}(\zeta)}\right|^{p/2}\frac{|d\zeta|}{|\zeta-z|} \leq \int_{L_R}|P_n(\zeta)|^{p/2}\frac{|d\zeta|}{|\zeta-z|},$$

since $|B_{m,R}(\zeta)|=1$, for $\zeta\in L_R$. Lets now $z\in L$. Multiplying the numerator and determinator of the integrand by $h^{1/2}(\zeta)$, by the Hölder inequality, we obtain

$$\left|\frac{P_n(z)}{B_{m,R}(z)}\right|^{p/2} \leq \frac{1}{2\pi}\left(\int_{L_R} h(\zeta)|P_n(\zeta)|^p|d\zeta|\right)^{1/2}$$

$$\times\left(\int\limits_{L_R}\frac{|d\zeta|}{\prod\limits_{j=1}^{m}|\zeta-z_j|^{\gamma_j}|\zeta-z|^2}\right)^{1/2}=:\frac{1}{2\pi}J_{n,1}(L_R)\times J_{n,2}(L_R).$$

Then, since $|B_{m,R}(z)|<1$, for $z\in L$, from Lemma 3.4, we have:

$$|P_n(z)|\preceq\left(J_{n,1}(L_R)\cdot J_{n,2}(L_R)\right)^{2/p}\preceq\|P_n\|_p\cdot\left(J_{n,2}(L_R)\right)^{2/p},\ z\in L. \tag{4.29}$$

By using notations (3.1), for the integral $J_{n,2}$, we obtain

$$\begin{aligned}(J_{n,2}(L_R))^2 &= \sum_{i=1}^{m}\int\limits_{L_R^i}\frac{|d\zeta|}{\prod\limits_{j=1}^{m}|\zeta-z_j|^{\gamma_j}|\zeta-z|^2} \\ &\asymp \sum_{i=1}^{m}\int\limits_{L_R^i}\frac{|d\zeta|}{|\zeta-z_i|^{\gamma_i}|\zeta-z|^2}=:\sum_{i=1}^{m}J_{n,2}^i(L_R^i),\end{aligned} \tag{4.30}$$

where

$$J_{n,2}^i(L_R^i):=\int\limits_{L_R^i}\frac{|d\zeta|}{|\zeta-z_i|^{\gamma_i}|\zeta-z|^2},\ i=\overline{1,m}, \tag{4.31}$$

since the points $\{z_j\}_{j=1}^m\in L$ are distinct. Therefore, it remains to estimate the integrals $J_{n,2}^i(L_R^i)$ for each $i=\overline{1,m}$. setting $z=z_1$, and assume that $m=1$, under the notations (4.12), we have:

$$|P_n(z_1)|\preceq\|P_n\|_{\mathcal{L}_p}\int\limits_{L_R^i}\frac{|d\zeta|}{|\zeta-z_i|^{2+\gamma_1}}.$$

$$=\ \|P_n\|_{\mathcal{L}_p}\left[\int\limits_{L_{R,1}^1}\frac{|d\zeta|}{|\zeta-z_1|^{2+\gamma_1}}+\int\limits_{L_{R,2}^1}\frac{|d\zeta|}{|\zeta-z_1|^{2+\gamma_1}}\right]. \tag{4.32}$$

By applying (3.2), we obtain:

$$\int\limits_{L_{R,1}^1}\frac{|d\zeta|}{|\zeta-z_1|^{2+\gamma_1}}=\int\limits_{F_{R,1}^1}\frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{2+\gamma_1}(|\tau|-1)} \tag{4.33}$$

$$\preceq\int\limits_{F_{R,1}^1}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{1+\gamma_1}(|\tau|-1)}\preceq n\int\limits_{F_{R,1}^1}\frac{|d\tau|}{|\tau-w_1|^{(\gamma_1+1)(2-\nu_1)}}$$

$$\preceq\ n^{(\gamma_1+1)(2-\nu_1)};$$

$$\int\limits_{L_{R,2}^1}\frac{|d\zeta|}{|\zeta-z_1|^{2+\gamma_1}}\preceq(\delta)^{-2-\gamma_1}mesL_{R_1,1}^1\preceq 1. \tag{4.34}$$

Then, from (4.32), we get:

$$|P_n(z_1)|\preceq n^{\frac{(\gamma_1+1)(2-\nu_1)}{p}}\|P_n\|_{\mathcal{L}_p},$$

and, according to our assumption $m = 1$, we complete the proof of estimation (2.7). □

4.2. Proof of Remark 2.2.

Proof. a) Let $L := \{z : |z| = 1\}$, $h^*(z) \equiv 1$ and $P_n^*(z) = \sum_{j=0}^{n}(j+1)z^j$. Then, $L \in \widetilde{Q}_1$;

$$|P_n^*(z)| \le \sum_{j=0}^{n} \left|(j+1)z^j\right| = \frac{(n+1)(n+2)}{2}, |z| = 1.$$

On the other hand,

$$|P_n^*(1)| = \frac{(n+1)(n+2)}{2}.$$

Therefore,

$$\|P_n^*\|_{\mathcal{L}_\infty} = \frac{(n+1)(n+2)}{2}; \ \|P_n^*\|_{\mathcal{L}_2(1,L)} = \sqrt{\frac{(n+1)(n+2)(2n+3)}{3}\pi}.$$

Then,

$$\|P_n^*\|_{\mathcal{L}_\infty} = \sqrt{\frac{3(n+1)(n+2)}{4\pi(2n+3)}}\,\|P_n^*\|_{\mathcal{L}_2(1,L)} \ge \sqrt{\frac{3}{8\pi}} \cdot \sqrt{n}\,\|P_n^*\|_{\mathcal{L}_2(1,L)}\,.$$

b) Verified directly using the scheme of the proof. □

4.3. Proof of Corollary 2.3.

Proof. If $L \in C(1, \alpha, \lambda_1)$, then the curve $L = \partial G$ has an interior (with respect to $\overline{G}$) $(2-\lambda_1)-$ angle at the z_1. Then, according to [22], $\psi \in Lip\frac{1}{2-\lambda_1}$, and so, by [22], $\Phi \in Lip\frac{1}{\lambda_1}$. Therefore, $L \in \widetilde{Q}_\alpha\left[\frac{1}{\lambda_1}\right]$ (2.1). In this case, for $p = 2$, from (2.2) and (2.4), we obtain the proof. □

Theorem 2.4 is proved analogously to the proof of Theorem 2.1, with using Lemma 3.3.

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