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ON FINDING RIGHT HAND SIDES OF EQUATIONS OF FLEXURAL-TORSIONAL VIBRATIONS OF A BAR

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Abstract. In the present paper, an inverse problem for equations connected with flexural-torsional vibrations of a bar is studied. This problem is reduced to an optimal control problem and is studied by the methods of optimal control theory.

1. Introduction

It is known that some problems of mathematical physics, mechanics, etc. are described by fourth order partial equations. A tuning fork, a bar vibrations equation, a rotary shaft, oscillating motions equation, plate vibrations equation and so on are among these equations (see [1,4,8,12]). Therefore, investigation of optimal control problems in processes described by these equations is urgent.

The principles of mathematical theory of control of vibrating elastic systems were laid in the papers of A.G. Butkovsky, A.E. Egorov, K.A. Lurie, T.K. Sirazetdinov and others (see [2,11]). Note that basic principles of optimal control for vibrating bars were developed in the paper [6]. The control connected with frexural-torsional vibrations of a bar has a great significance in dynamics of aircraft constructions. Therefore, the study of bar vibrations problems controls described by differential equations is necessary both from practical and theoretical point of view (see [8,11]). In recent years the problems of bar vibrations control are intensively studied (see [3,10,15]).

2. Problem statement

We consider a boundary value problem for equations of flexural-torsional vibrations a bar, described by the system of two differential equations in the domain $Q = \{0 < x < l,\, 0 < t < T\}$

$$\frac{\partial^2}{\partial x^2}\left(E\left(x\right)I\left(x\right)\frac{\partial^2 y}{\partial x^2}\right)+\rho\left(x\right)A\left(x\right)\frac{\partial^2 y}{\partial t^2}-\rho\left(x\right)A\left(x\right)e\left(x\right)\frac{\partial^2\theta}{\partial t^2}=v_1\left(x,\,t\right), \quad (2.1)$$

$$\frac{\partial^2}{\partial x^2}\left(E\left(x\right)C_w\left(x\right)\frac{\partial^2\theta}{\partial x^2}\right)-\frac{\partial^2}{\partial x^2}(G(x)C\left(x\right)\theta)-$$

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$$-\rho(x)A(x)e(x)\frac{\partial^2 y}{\partial t^2}+\rho(x)\left(I(x)+A(x)e^2(x)\right)\frac{\partial^2\theta}{\partial t^2}=v_2(x,t), \tag{2.2}$$

$$y|_{t=0}=\varphi_0(x), \left.\frac{\partial y}{\partial t}\right|_{t=0}=\varphi_1(x), \theta|_{t=0}=\tilde{\varphi}_0(x), \left.\frac{\partial\theta}{\partial t}\right|_{t=0}=\tilde{\varphi}_1(x), \tag{2.3}$$

$$y|_{x=0}=y|_{x=l}=0, \left.\frac{\partial y}{\partial x}\right|_{x=0}=\left.\frac{\partial y}{\partial x}\right|_{x=l}=0, \tag{2.4}$$

$$\theta|_{x=0}=\theta|_{x=l}=0, \left.\frac{\partial\theta}{\partial x}\right|_{x=0}=\left.\frac{\partial\theta}{\partial x}\right|_{x=l}=0, \tag{2.5}$$

where $l>0,\ T>0$ are the given numbers, $y(x,t)$ is the lateral displacement of the bar, $\theta(x,t)$ is the turning angle of the bar cross-section, $E(x)$ is the Young modulus, $I(x)$ is a polar inertia moment of the cross section with respect to its gravity center, $\rho(x)$ is the density of the bar material, $A(x)$ is the area cross section, $e(x)$ is the distance from the gravity center to the centre of torsion, $C_\omega(x)$ is the sectional moment of inertia of the cross-section, $G(x)$ a shear modulus, $C(x)$ is geometrical rigidity of free torsion, $E(x)C_\omega(x)$ is the rigidity of flexural torsion, $G(x)C(x)$ is the rigidity of free torsion, φ_0, φ_1, $\tilde{\varphi}_0$, $\tilde{\varphi}_1$ are the given functions, the functions $v_1(x,t)$ and $v_2(x,t)$ to be defined.

To determine $v(x,t)=(v_1(x,t), v_2(x,t))$ we give the additional conditions

$$y(d(t),t;v)=\varphi(t), \theta(d(t),t;v)=g(t), \tag{2.6}$$

where $x=d(t)$, $\varphi(t)$, $g(t)$, $t\in(0,T)$ are the given functions.

We reduce this problem to the following optimal control problem: it is required to find such a vector-function $v(x,t)=(v_1(x,t),\ v_2(x,t))\in L_2(Q)\times L_2(Q)$ that minimizes the functional

$$J(v)=\frac{1}{2}\int_0^T\left[(y(d(t),t;v)-\varphi(t))^2+(\theta(d(t),t;v)-g(t))^2\right]dt, \tag{2.7}$$

together with the solution of boundary value problem (2.1)-(2.5). The function $v(x,t)=(v_1(x,t),v_2(x,t))$ is said to be a control. We call problem (2.7), (2.1)-(2.5) a reduced problem.

Suppose that the data of problem (2.1)-(2.5) satisfy the following conditions:

1)$E(x)$, $I(x)$, $\rho(x)$, $A(x)$, $e(x)$, $C_\omega(x)$, $G(x)$, $C(x)$ are measurable, bounded and positive functions on the interval $[0,l]$.

2) $\varphi_0\in \overset{\circ}{W}{}_2^2(0,l)$, $\tilde{\varphi}_0\in \overset{0}{W}{}_2^2(0,l)$, $\varphi_1\in L_2(0,l)$, $\tilde{\varphi}_1\in L_2(0,l)$, $\varphi\in W_2^1(0,T)$, $g\in W_2^1(0,T)$, $d(t)\in KC^1(0,T)$.

Note that for each fixed vector-function $v(x,t)=(v_1(x,t),\ v_2(x,t))\in L_2(Q)\times L_2(Q)$, problem (2.1)-(2.5) has a unique generalized solution from the space $W_2^{2,1}(Q)$ [4, 9, 7].

3. On a property of the reduced problem (2.7), (2.1)-(2.5).

Show that $\inf\limits_{v\in L_2(Q)\times L_2(Q)} J(v)=0.$

This issue is equivalent to the issue on density in $L_2(0,T)\times L_2(0,T)$ of the image $L_2(Q)\times L_2(Q)$ under the mapping

$$(v_1,v_2)\to y(d(t),t;v_1,v_2)\times\theta(d(t),t;v_1,v_2).$$

For solving this issue, we use the Hahn-Banach theorem [5]. Let $\xi_0(t)$ and $\xi_1(t)$ be the given functions from $L_2(0,T)$ such that

$$\int_0^T y(d(t),t;v_1,v_2)\,\xi_0(t)\,dt=0,$$

$$\int_0^T \theta(d(t),t;v_1,v_2)\,\xi_1(t)\,dt=0, \forall v_1,v_2\in L_2(Q). \tag{3.1}$$

We want to know if this implies $\xi_0(t)=0$, $\xi_1(t)=0$.

Introduce the vector-function $(w_1(x,t),\ w_2(x,t))$ as the solution of the boundary value problem

$$\frac{\partial^2}{\partial x^2}\left(E(x)\,I(x)\,\frac{\partial^2 w_1}{\partial x^2}\right)+\rho(x)\,A(x)\,\frac{\partial^2 w_1}{\partial t^2}-$$

$$-\rho(x)\,A(x)\,e(x)\,\frac{\partial^2 w_2}{\partial t^2}=\xi_0(t)\,\delta(x-d(t)), \tag{3.2}$$

$$\frac{\partial^2}{\partial x^2}\left(E(x)\,C_w(x)\,\frac{\partial^2 w_2}{\partial x^2}\right)-G(x)\,C(x)\,\frac{\partial^2 w_2}{\partial x^2}-$$

$$-\rho(x)\,A(x)\,e(x)\,\frac{\partial^2 w_2}{\partial t^2}+\rho(x)\,(I(x)+ \tag{3.3}$$

$$+A(x)\,e^2(x))\,\frac{\partial^2 w_1}{\partial t^2}=\xi_1(t)\,\delta(x-d(t)),(x,t)\in Q,$$

$$w_1|_{t=T}=w_2|_{t=T}=0,\ \left.\frac{\partial w_1}{\partial t}\right|_{t=T}=\left.\frac{\partial w_2}{\partial t}\right|_{t=T}=0, 0\le x\le l, \tag{3.4}$$

$$w_1|_{x=0}=w_2|_{x=0}=0,\ \left.\frac{\partial w_1}{\partial x}\right|_{x=0}=\left.\frac{\partial w_2}{\partial x}\right|_{x=0}=0, 0\le t\le T, \tag{3.5}$$

$$w_1|_{x=l}=w_2|_{x=l}=0,\ \left.\frac{\partial w_1}{\partial x}\right|_{x=l}=\left.\frac{\partial w_2}{\partial x}\right|_{x=l}=0, 0\le t\le T, \tag{3.6}$$

where $\delta(x,t)$ is Dirac's delta function.

Note that this problem has a unique generalized solution in $W_2^{2,1}(Q)\times W_2^{2,1}(Q)$ [7].

By definition of the generalized solution of problem (2.1)-(2.5) we have: for $t=0$ the conditions $y(x,0;v_1,v_2)=\varphi_0(x)$, $\theta(x,0;v_1,v_2)=\tilde{\varphi}_0(x)$ and the following integral identities

$$\iint_Q\left(E(x)\,I(x)\,\frac{\partial^2 y}{\partial x^2}\frac{\partial^2 \mu_1}{\partial x^2}-\rho(x)\,A(x)\,\frac{\partial y}{\partial t}\frac{\partial \mu_1}{\partial t}+\rho(x)\,A(x)\,e(x)\,\frac{\partial \theta}{\partial t}\frac{\partial \mu_1}{\partial t}\right)dxdt-$$

$$-\int_0^l \rho(x)\,A(x)\,\varphi_1(x)\,\mu_1(x,0)\,dx+$$

$$+\int_0^l \rho(x)\,A(x)\,e(x)\,\tilde{\varphi}_1(x)\,\mu_1(x,0)\,dx=\iint_Q v_1\cdot\mu_1 dxdt, \tag{3.7}$$

$$\iint_Q\left(E(x)\,C_w(x)\,\frac{\partial^2 \theta}{\partial x^2}\frac{\partial^2 \mu_2}{\partial x^2}-G(x)\,C(x)\,\theta\frac{\partial^2 \mu_2}{\partial x^2}+\right.$$

$$\left.+\rho(x)\,A(x)\,e(x)\,\frac{\partial y}{\partial t}\frac{\partial \mu_2}{\partial t}-\rho(x)\left(I(x)+A(x)\,e^2(x)\right)\frac{\partial \theta}{\partial t}\frac{\partial \mu_2}{\partial t}\right)dxdt+$$

$$+\int_0^l \rho(x) A(x) e(x) \varphi_1(x) \mu_2(x,0)\, dx-$$

$$-\int_0^l \rho(x) \left(I(x) + A(x) e^2(x)\right) \tilde{\varphi}_1(x) \mu_2(x,0)\, dx =$$

$$= \iint_Q v_2 \cdot \mu_2 dx dt, (x,t) \in Q, \tag{3.8}$$

are fulfilled for arbitrary functions $\mu_1,\ \mu_2 \in W_2^{2,1}(Q)$,

$$\mu_1|_{t=T} = \mu_2|_{t=T} = 0, \tag{3.9}$$

$$\mu_1|_{x=0} = \mu_2|_{x=0} = 0,\ \left.\frac{\partial \mu_1}{\partial x}\right|_{x=0} = \left.\frac{\partial \mu_2}{\partial x}\right|_{x=0} = 0, \tag{3.10}$$

$$\mu_1|_{x=l} = \mu_2|_{x=l} = 0,\ \left.\frac{\partial \mu_1}{\partial x}\right|_{x=l} = \left.\frac{\partial \mu_2}{\partial x}\right|_{x=l} = 0. \tag{3.11}$$

From definition of the generalized solution of equation (3.2)-(3.6) we have: for $t = T$ the conditions $w_1(x,T) = 0$, $w_2(x,T) = 0$ and the integral identities

$$\iint_Q \left(E(x) I(x) \frac{\partial^2 w_1}{\partial x^2} \frac{\partial^2 g_1}{\partial x^2} - \rho(x) A(x) \frac{\partial w_1}{\partial t} \frac{\partial g_1}{\partial t} + \rho(x) A(x) e(x) \frac{\partial w_1}{\partial t} \frac{\partial g_1}{\partial t} \right) dx dt-$$

$$-\int_0^l \rho(x) A(x) \frac{\partial w_1(x,0)}{\partial t} g_1(x,0)\, dx+$$

$$+\int_0^l \rho(x) A(x) e(x) \frac{\partial w_2(x,0)}{\partial t} g_1(x,0)\, dx = \int_0^T \xi_0(t) g_1(d(t), t)\, dt, \tag{3.12}$$

$$\iint_Q \left(E(x) C_w(x) \frac{\partial^2 w_2}{\partial x^2} \frac{\partial^2 g_2}{\partial x^2} - G(x) C(x) \frac{\partial^2 w_2}{\partial x^2} g_2 + \right.$$

$$\left. +\rho(x) A(x) e(x) \frac{\partial w_1}{\partial t} \frac{\partial g_2}{\partial t} - \rho(x) \left(I(x) + A(x) e^2(x)\right) \frac{\partial w_2}{\partial t} \frac{\partial g_2}{\partial t} \right) dx dt-$$

$$+\int_0^l \rho(x) A(x) e(x) \frac{\partial w_1(x,0)}{\partial t} g_2(x,0)\, dx-$$

$$-\int_0^l \rho(x) \left(I(x) + A(x) e^2(x)\right) \frac{\partial w_2(x,0)}{\partial t} g_2(x,0)\, dx =$$

$$= \int_0^T \xi_1(t) g_2(d(t), t)\, dt, \tag{3.13}$$

are fulfilled for arbitrary functions $g_1,\ g_2 \in W_2^{2,1}(Q)$,

$$g_1|_{x=0} = g_2|_{x=0} = 0,\ g_1|_{x=l} = g_2|_{x=l} = 0, \tag{3.14}$$

$$\left.\frac{\partial g_1}{\partial x}\right|_{x=0} = \left.\frac{\partial g_2}{\partial x}\right|_{x=0} = 0,\ \left.\frac{\partial g_1}{\partial x}\right|_{x=l} = \left.\frac{\partial g_2}{\partial x}\right|_{x=l} = 0. \tag{3.15}$$

Now in identities (3.7) and (3.8), in place of the functions μ_1 and μ_2 we take $w_1(x,t)$ and $w_2(x,t)$, while in the identities (3.12) and (3.13), in place of the functions g_1 and g_2 we take $y(x,t;v_1,v_2)$ and $\theta(x,t;v_1,v_2)$, respectively. Then from (3.7) and (3.8) we subtract (3.12) and (3.13), respectively and sum the obtained expression.

Then we have:

$$-\int_0^l \rho(x) A(x) \varphi_1(x) w_1(x,0)\, dx + \int_0^l \rho(x) A(x) e(x) \tilde{\varphi}_1(x) w_1(x,0)\, dx +$$
$$+\int_0^l \rho(x) A(x) e(x) \varphi_1(x) w_2(x,0)\, dx -$$
$$-\int_0^l \rho(x) \left(I(x) + A(x) e^2(x)\right) \tilde{\varphi}_1(x) w_2(x,0)\, dx +$$
$$+\int_0^l \rho(x) A(x) \frac{\partial w_1(x,0)}{\partial t} \varphi_0(x)\, dx - \int_0^l \rho(x) A(x) e(x) \frac{\partial w_2(x,0)}{\partial t} \varphi_0(x)\, dx -$$
$$-\int_0^l \rho(x) A(x) e(x) \frac{\partial w_1(x,0)}{\partial t} \tilde{\varphi}_0(x)\, dx +$$
$$+\int_0^l \rho(x) \left(I(x) + A(x) e^2(x)\right) \frac{\partial w_2(x,0)}{\partial t} \tilde{\varphi}_0(x)\, dx = \iint_Q (v_1 w_1 + v_2 w_2)\, dxdt -$$
$$-\int_0^T \left(\xi_0(t) y(d(t), t; v_1, v_2) + \xi_1(t) \theta(d(t), t; v_1, v_2)\right) dt,$$
$$\forall v_1, v_2 \in L_2(Q).$$

Hence, from conditions (3.1) we have

$$-\int_0^l \rho(x) A(x) \varphi_1(x) w_1(x,0)\, dx + \int_0^l \rho(x) A(x) e(x) \tilde{\varphi}_1(x) w_1(x,0)\, dx +$$
$$+\int_0^l \rho(x) A(x) e(x) \varphi_1(x) w_2(x,0)\, dx -$$
$$-\int_0^l \rho(x) \left(I(x) + A(x) e^2(x)\right) \tilde{\varphi}_1(x) w_2(x,0)\, dx +$$
$$+\int_0^l \rho(x) A(x) \frac{\partial w_1(x,0)}{\partial t} \varphi_0(x)\, dx - \int_0^l \rho(x) A(x) e(x) \frac{\partial w_2(x,0)}{\partial t} \varphi_0(x)\, dx -$$
$$-\int_0^l \rho(x) A(x) e(x) \frac{\partial w_1(x,0)}{\partial t} \tilde{\varphi}_0(x)\, dx +$$
$$+\int_0^l \rho(x) \left(I(x) + A(x) e^2(x)\right) \frac{\partial w_2(x,0)}{\partial t} \tilde{\varphi}_0(x)\, dx -$$
$$-\iint_Q (v_1 w_1 + v_2 w_2)\, dx = 0, \forall v_1, v_2 \in L_2(Q) \times L_2(Q). \tag{3.16}$$

If we write this relation for arbitrary $v_1^1(x,t)$, $v_1^2(x,t)$ and $v_2^1(x,t)$, $v_2^2(x,t)$, then from the obtained two equalities it follows that

$$\iint_Q \left[\left(v_1^1(x,t) - v_1^2(x,t)\right) w_1(x,t) + \left(v_2^1(x,t) - v_2^2(x,t)\right) w_2(x,t)\right] dxdt = 0,$$
$$\forall v_1, v_2 \in L_2(Q) \times L_2(Q).$$

Then, hence and by the analog of Lagrangain lemma [14, p. 95] it follows that $w_1(x,t) = 0$, $w_2(x,t) = 0$ almost everywhere in Q. Since $w(x,t) =$

$(w_1(x,t), w_2(x,t))$ as the solution of problem (3.2)-(3.6) is a continuous vector-function on $\overline{Q}$, then $w_1(x,t) \equiv 0$, $w_2(x,t) \equiv 0$, $(x,t) \in Q$, therefore, from (3.2) and (3.3) it follows $\xi_0(t) = 0$ and $\xi_1(t) = 0$. Thus, we get $\inf\limits_{v \in L_2(Q)} J(v) = 0$.

4. Formula for increment of functional (2.7).

Now for a class of admissible controls we take a convex, closed set $U_{ad} \subset L_2(Q) \times L_2(Q)$ of vector-functions $v(x,t) = (v_1(x,t), v_2(x,t))$.
Introduce the following problem adjoint to the problem (2.1)-(2.5), (2.7).

$$\frac{\partial^2}{\partial x^2}\left(E(x) I(x) \frac{\partial^2 \psi_1}{\partial x^2}\right) + \rho(x) A(x) \frac{\partial^2 \psi_1}{\partial t^2} - \rho(x) A(x) e(x) \frac{\partial^2 \psi_2}{\partial t^2} = 0, \quad (4.1)$$

$$\frac{\partial^2}{\partial x^2}\left(E(x) C_w(x) \frac{\partial^2 \psi_2}{\partial x^2}\right) - G(x) C(x) \frac{\partial^2 \psi_2}{\partial x^2} - \rho(x) A(x) e(x) \frac{\partial^2 \psi_1}{\partial t^2} +$$

$$+\rho(x)\left(I(x) + A(x) e^2(x)\right) \frac{\partial^2 \psi_2}{\partial t^2} = 0, \quad (4.2)$$

$$\psi_1|_{t=T} = \psi_2|_{t=T} = 0, \ \left.\frac{\partial \psi_1}{\partial t}\right|_{t=T} = \left.\frac{\partial \psi_2}{\partial t}\right|_{t=T} = 0, \quad (4.3)$$

$$\psi_1|_{x=0} = \psi_1|_{x=l} = 0, \ \left.\frac{\partial \psi_1}{\partial x}\right|_{x=0} = \left.\frac{\partial \psi_1}{\partial x}\right|_{x=l} = 0, \quad (4.4)$$

$$\psi_2|_{x=0} = \psi_2|_{x=l} = 0, \ \left.\frac{\partial \psi_2}{\partial x}\right|_{x=0} = \left.\frac{\partial \psi_2}{\partial x}\right|_{x=l} = 0, \quad (4.5)$$

$$[\psi_1]_\Gamma = [\psi_2]_\Gamma = 0, \left[\frac{\partial \psi_1}{\partial x}\right]_\Gamma = \left[\frac{\partial \psi_2}{\partial x}\right]_\Gamma = 0, \quad (4.6)$$

$$\left[E(x) I(x) \frac{\partial^2 \psi_1}{\partial x^2}\right]_\Gamma = 0, \left[E(x) C_w(x) \frac{\partial^2 \psi_2}{\partial x^2}\right]_\Gamma = 0, \quad (4.7)$$

$$\left[\frac{\partial}{\partial x}\left(E(x) I(x) \frac{\partial^2 \psi_1}{\partial x^2}\right)\right]_\Gamma = -\left(y(d(t), t; v) - \varphi(t)\right),$$

$$\left[\frac{\partial}{\partial x}\left(E(x) C_w(x) \frac{\partial^2 \psi_2}{\partial x^2}\right)\right]_\Gamma = -\left(\theta(d(t), t; v) - g(t)\right). \quad (4.8)$$

where Γ is a line of $x = d(t)$, $t \in (0,T)$, that divides the domain Q into two parts Q_1 and Q_2, the symbol $[\omega(x,t)]_\Gamma$ means the difference between limit values of the function $\omega(x,t)$ in the sense of L_2 on Γ, i.e. the difference between the traces calculated when approaching Γ from the side of domains Q_1 and Q_2 [9. p.264].

Note that (4.6)-(4.8) is called the adjoint conditions.

Let $\varphi_0, \tilde{\varphi}_0 \in W_2^4(0,l) \bigcap \overset{0}{W}{}_2^2(0,l)$, $\varphi_1, \tilde{\varphi}_1 \in \overset{0}{W}{}_2^2(0,l)$. Then by virtue of the results of [4], problem (4.1)-(4.8) has a solution $(\psi_1(x,t), \psi_2(x,t))$, such that $\psi_1(x,t), \psi_2(x,t) \in W_2^{4,2}(Q_k)$, $k = 1,2$. Therefore, the following adjoint conditions are fulfilled by itself:

$$[\psi_1]_\Gamma = [\psi_2]_\Gamma = 0, \left[\frac{\partial \psi_1}{\partial x}\right]_\Gamma = \left[\frac{\partial \psi_2}{\partial x}\right]_\Gamma = 0.$$

We take the two admissible controls:$v_1(x,t)$, $v_2(x,t)$ and we assign them the increments $\delta v_1 \in L_2(Q)$ and $\delta v_2 \in L_2(Q)$ in such a way that $v_1(x,t) + \delta v_1(x,t)$,$v_2(x,t) + \delta v_2(x,t) \in U_{ad}$.
Find the increment of the functional (7)

$$\delta J(v) = J(v+\delta v) - J(v) = \frac{1}{2}\int_0^T \Big\{ [y(d(t),t;v_1+\delta v_1, v_2+\delta v_2) - \varphi(t)]^2 -$$
$$- [y(d(t),t;v_1,v_2) - \varphi(t)]^2 +$$
$$+ [\theta(d(t),t;v_1+\delta v_1, v_2+\delta v_2) - g(t)]^2 - [\theta(d(t),t;v_1,v_2) - g(t)]^2 \Big\} dt,$$

where

$$y(x,t;v_1+\delta v_1, v_2+\delta v_2) = y(x,t;v_1,v_2) + \delta y(x,t),$$
$$\theta(x,t;v_1+\delta v_1, v_2+\delta v_2) = \theta(x,t;v_1,v_2) + \delta\theta(x,t).$$

Hence it follows that

$$\delta J(v) = \int_0^T [y(d(t),t;v_1,v_2) - \varphi(t)]\,\delta y(d(t),t)\,dt +$$
$$+ \int_0^T [\theta(d(t),t;v_1,v_2) - g(t)]\,\delta\theta(d(t),t)\,dt + R, \tag{4.9}$$

where

$$R = \frac{1}{2}\int_0^T \left[(\delta y(d(t),t))^2 + (\delta\theta(d(t),t))^2 \right] dt,$$

while $(\delta y(x,t), \delta\theta(x,t)) \in W_2^{2,1}(Q) \times W_2^{2,1}(Q)$ is the generalized solution of the following boundary value problem:

$$\frac{\partial^2}{\partial x^2}\left(E(x)I(x)\frac{\partial^2 \delta y}{\partial x^2}\right) + \rho(x)A(x)\frac{\partial^2 \delta y}{\partial t^2} - \rho(x)A(x)e(x)\frac{\partial^2 \delta\theta}{\partial t^2} = \delta v_1, \tag{4.10}$$

$$\frac{\partial^2}{\partial x^2}\left(E(x)C_w(x)\frac{\partial^2 \delta\theta}{\partial x^2}\right) - \frac{\partial^2}{\partial x^2}(G(x)C(x)\delta\theta) - \rho(x)A(x)e(x)\frac{\partial^2 \delta y}{\partial t^2} +$$
$$+\rho(x)\left(I(x) + A(x)e^2(x)\right)\frac{\partial^2 \delta\theta}{\partial t^2} = \delta v_2, \tag{4.11}$$

$$\delta y(x,t)|_{t=0} = \delta\theta(x,t)|_{t=0} = 0, \quad \left.\frac{\partial \delta y(x,t)}{\partial t}\right|_{t=0} = \left.\frac{\partial \delta\theta(x,t)}{\partial t}\right|_{t=0} = 0, \tag{4.12}$$

$$\delta y(x,t)|_{x=0} = \delta\theta(x,t)|_{x=0} = 0, \quad \left.\frac{\partial \delta y(x,t)}{\partial x}\right|_{x=0} = \left.\frac{\partial \delta\theta(x,t)}{\partial x}\right|_{x=0} = 0, \tag{4.13}$$

$$\delta y(x,t)|_{x=l} = \delta\theta(x,t)|_{x=l} = 0, \quad \left.\frac{\partial \delta y(x,t)}{\partial x}\right|_{x=l} = \left.\frac{\partial \delta\theta(x,t)}{\partial x}\right|_{x=l} = 0,$$

i.e. for any function $\forall \eta_1,\ \eta_2 \in W_2^{2,1}(Q)$,

$$\eta_1|_{t=T} = \eta_2|_{t=T} = 0,$$
$$\eta_1|_{x=0} = \eta_2|_{x=0} = 0,\ \eta_1|_{x=l} = \eta_2|_{x=l} = 0,$$
$$\left.\frac{\partial \eta_1}{\partial x}\right|_{x=0} = \left.\frac{\partial \eta_2}{\partial x}\right|_{x=0} = 0,\ \left.\frac{\partial \eta_1}{\partial x}\right|_{x=l} = \left.\frac{\partial \eta_2}{\partial x}\right|_{x=l} = 0,$$

the integral identities

$$\iint_Q \left(E(x)\, I(x)\, \frac{\partial^2 \delta y}{\partial x^2}\frac{\partial^2 \eta_1}{\partial x^2} - \rho(x)\, A(x)\, \frac{\partial \delta y}{\partial t}\frac{\partial \eta_1}{\partial t} + \rho(x)\, A(x)\, e(x)\, \frac{\partial \delta\theta}{\partial t}\frac{\partial \eta_1}{\partial t}\right) dt-$$

$$-\iint_Q \delta v_1 \cdot \eta_1 dxdt = 0, \tag{4.14}$$

$$\iint_Q \left(E(x)\, C_w(x)\, \frac{\partial^2 \delta\theta}{\partial x^2}\frac{\partial^2 \eta_2}{\partial x^2} - GC\frac{\partial^2 \delta\theta}{\partial x^2}\eta_2 + \right.$$

$$\left. +\rho(x)\, A(x)\, e(x)\, \frac{\partial \delta y}{\partial t}\frac{\partial \eta_2}{\partial t} - \rho(x)\left(I(x) + A(x)\, e^2(x)\right)\frac{\partial \delta\theta}{\partial t}\frac{\partial \eta_2}{\partial t}\right) dxdt-$$

$$-\iint_Q \delta v_2 \cdot \eta_2 dxdt = 0 \tag{4.15}$$

are fulfilled. As the functions $\psi_1(x,t)$, $\psi_2(x,t)$ are the generalized solutions of problem (4.1)-(4.8), for any functions $g_1,\ g_2 \in W_2^{2,1}(Q)$,

$$g_1|_{t=0} = g_2|_{t=0} = 0,$$

$$g_1|_{x=0} = g_2|_{x=0} = 0,\ \left.\frac{\partial g_1}{\partial x}\right|_{x=0} = \left.\frac{\partial g_2}{\partial x}\right|_{x=0} = 0,$$

$$g_1|_{x=l} = g_2|_{x=l} = 0,\ \left.\frac{\partial g_1}{\partial x}\right|_{x=l} = \left.\frac{\partial g_2}{\partial x}\right|_{x=l} = 0,$$

the following integral identities are fulfilled

$$\iint_Q \left(E(x)\, I(x)\, \frac{\partial^2 \psi_1}{\partial x^2}\frac{\partial^2 g_1}{\partial x^2} - \rho(x)\, A(x)\, \frac{\partial \psi_1}{\partial t}\frac{\partial g_1}{\partial t} + \right.$$

$$\left. +\rho(x)\, A(x)\, e(x)\, \frac{\partial \psi_2}{\partial t}\frac{\partial g_1}{\partial t}\right) dxdt-$$

$$-\int_0^T \left[y(d(t), t; v_1, v_2) - \varphi(t)\right] g_1(d(t), t)\, dt = 0, \tag{4.16}$$

$$\iint_Q \left(E(x)\, C_w(x)\, \frac{\partial^2 \psi_2}{\partial x^2}\frac{\partial^2 g_2}{\partial x^2} - GC\frac{\partial^2 \psi_2}{\partial x^2} g_2 + \right.$$

$$\left. +\rho(x)\, A(x)\, e(x)\, \frac{\partial \psi_1}{\partial t}\frac{\partial g_2}{\partial t} - \rho(x)\left(I(x) + A(x)\, e^2(x)\right)\frac{\partial \psi_2}{\partial t}\frac{\partial g_2}{\partial t}\right) dxdt-$$

$$-\int_0^T \left[\theta(d(t), t; v_1, v_2) - g(t)\right] g_2(d(t), t)\, dt = 0. \tag{4.17}$$

In identities (4.14) and (4.15) instead of $\eta_1(x,t)$ and $\eta_2(x,t)$ we take $\psi_1(x,t)$ and $\psi_2(x,t)$, in the identities (4.16) and (4.17) instead of $g_1(x,t)$ and $g_2(x,t)$ we take $\delta y(x,t)$ and $\delta\theta(x,t)$ respectively, subtract the obtained relations and sum them.

Then we have

$$-\iint_Q \delta v_1 \cdot \psi_1 dxdt - \iint_Q \delta v_2 \cdot \psi_2 dxdt +$$

$$+\int_0^T \left[y(d(t), t; v_1,\ v_2) - \varphi(t)\right] \delta y(d(t), t)\, dxdt +$$

$$+\int_0^T [\theta(d(t),t;v_1,\ v_2) - g(t)]\,\delta\theta(d(t),t)\,dxdt = 0. \tag{4.18}$$

Therefore from formulas (4.9) and (4.18) it follows that

$$\delta J(v) = \iint_Q \delta v_1 \cdot \psi_1 dxdt + \iint_Q \delta v_2 \cdot \psi_2 dxdt + R. \tag{4.19}$$

5. Estimating the increment of the solution of problem (4.10)-(4.13) and the residual term R

Let show that for the generalized solution of boundary value problem (4.10)-(4.13) the following estimation is valid:

$$\|\delta y(x,t)\|^2_{W_2^{2,1}(Q)} + \|\delta\theta(x,t)\|^2_{W_2^{2,1}(Q)} \le c\left(\|\delta v_2\|^2_{L_2(Q)} + \|\delta v_1\|^2_{L_2(Q)}\right), \tag{5.1}$$

here and in the sequel, by c we will denote different constants independent of estimated values and on admissible controls.

For proving estimation (5.1), we apply the Faedo-Galerkin method.

Let $\{\omega_i(x)\}_{i=1}^{\infty}$ be a fundamental system in $\overset{0}{W_2^2}(0,l)$ and

$$\int_0^l \omega_i(x)\,\omega_k(x)\,dx = \begin{cases} 1, & i=k, \\ 0, & i\neq k. \end{cases}$$

We look for approximate solutions $\left(\delta y^N(x,t),\ \delta\theta^N(x,t)\right)$ of problem (4.10)-(4.13) in the form $\delta y^N(x,t) = \sum_{i=1}^N c_{1i}^N(t)\,\omega_i(x)$ and $\delta\theta^N(x,t) = \sum_{i=1}^N c_{2i}^N(t)\,\omega_i(x)$ from the following relations

$$\int_0^l E(x)\,I(x)\,\frac{\partial^2 \delta y^N}{\partial x^2}\frac{d^2\omega_p(x)}{dx^2}dx + \int_0^l \rho(x)\,A(x)\,\frac{\partial^2\delta y^N}{\partial t^2}\omega_p(x)\,dx -$$

$$-\int_0^l \rho(x)\,A(x)\,e(x)\,\frac{\partial^2\delta\theta^N}{\partial t^2}\omega_p(x)\,dx = \int_0^l \delta v_1(x,t)\,\omega_p(x)\,dx\,,\quad p=\overline{1,N}, \tag{5.2}$$

$$\int_0^l E(x)\,C_w(x)\,\frac{\partial^2\delta\theta^N}{\partial x^2}\frac{d^2\omega_p(x)}{\partial x^2}dx -$$

$$-\int_0^l G(x)\,C(x)\,\delta\theta^N\frac{d^2\omega_p(x)}{\partial x^2} - \int_0^l \rho(x)\,A(x)\,e(x)\,\frac{\partial^2\delta y^N}{\partial t^2}\omega_p(x)\,dx +$$

$$+\int_0^l \rho(x)\left(I(x) + A(x)\,e^2(x)\right)\frac{\partial^2\delta\theta^N}{\partial t^2}\omega_p(x)\,dx =$$

$$= \int_0^l \delta v_2(x,t)\,\omega_p(x)\,dx\,,\quad p=\overline{1,N}, \tag{5.3}$$

$$c_{1i}^N\big|_{t=0} = c_{2i}^N\big|_{t=0} = 0, \tag{5.4}$$

$$\left.\frac{dc_{1i}^N}{dt}\right|_{t=0} = \left.\frac{dc_{2i}^N}{dt}\right|_{t=0} = 0,\ \ i=\overline{1,N}. \tag{5.5}$$

Equalities (5.2) and (5.3) are the system of linear ordinary differential equations of second order for the unknowns $c_{1i}^N(t)$ and $c_{2i}^N(t)$, $i=\overline{1,N}$, solved with respect to $\frac{d^2c_{1i}^N}{dt^2}$ and $\frac{d^2c_{2i}^N}{dt^2}$. Under the conditions on the problem data, this system is uniquely solvable under initial conditions (5.4) and (5.5), moreover $\frac{d^2c_{1i}^N}{dt^2}, \frac{d^2c_{2i}^N}{dt^2} \in L_2(0,T)$, $i=\overline{1,N}$.

Multiplying each of the equalities (5.2) and (5.3) by its own $\frac{dc_{1p}^N}{dt}, \frac{dc_{2p}^N}{dt}$ and summing over p from 1 to N, we come to the equalities

$$\int_0^l E(x)\,I(x)\,\frac{\partial^2\delta y^N}{\partial x^2}\frac{\partial^3\delta y^N}{\partial x^2\partial t}dx+\int_0^l \rho(x)\,A(x)\,\frac{\partial^2\delta y^N}{\partial t^2}\frac{\partial\delta y^N}{\partial t}dx-$$

$$-\int_0^l \rho(x)\,A(x)\,e(x)\,\frac{\partial^2\delta\theta^N}{\partial t^2}\frac{\partial\delta y^N}{\partial t}dx=\int_0^l \delta v_1\frac{\partial\delta y^N}{\partial t}dx \tag{5.6}$$

$$\int_0^l E(x)\,C_w(x)\,\frac{\partial^2\delta\theta^N}{\partial x^2}\frac{\partial^3\delta\theta^N}{\partial x^2\partial t}dx-$$

$$-\int_0^l GC\delta\theta^N\frac{\partial^3\delta\theta^N}{\partial x^2\partial t}dx-\int_0^l \rho(x)\,A(x)\,e(x)\,\frac{\partial^2\delta y^N}{\partial t^2}\frac{\partial\delta\theta^N}{\partial t}dx+$$

$$+\int_0^l \rho(x)\left(I(x)+A(x)\,e^2(x)\right)\frac{\partial^2\delta\theta^N}{\partial t^2}\frac{\partial\delta\theta^N}{\partial t}dx=\int_0^l \delta v_2(x,t)\,\frac{\partial\delta\theta^N}{\partial t}dx\,. \tag{5.7}$$

Suppose that $G(x)\,,C(x)$ are independent of x.

Then from (5.6), (5.7) it follows that

$$\frac{1}{2}\frac{d}{dt}\int_0^l\Bigg[E(x)\,I(x)\left(\frac{\partial^2\delta y^N}{\partial x^2}\right)^2+\rho(x)\,A(x)\left(\frac{\partial\delta y^N}{\partial t}\right)^2+$$

$$+E(x)\,C_w(x)\left(\frac{\partial^2\delta\theta^N}{\partial x^2}\right)^2+GC\left(\frac{\partial\delta\theta^N}{\partial x}\right)^2+$$

$$+\rho(x)\left(I(x)+A(x)\,e^2(x)\right)\left(\frac{\partial\delta\theta^N}{\partial t}\right)^2-2\rho(x)\,A(x)\,e(x)\left(\frac{\partial\delta y^N}{\partial t}\frac{\partial\delta\theta^N}{\partial t}\right)\Bigg]dx=$$

$$=\int_0^l\left(\delta v_1\frac{\partial\delta y^N}{\partial t}+\delta v_2(x,t)\,\frac{\partial\delta\theta^N}{\partial t}\right)dx.$$

We integrate the last equality with respect to t from 0 to t :

$$\int_0^l\Bigg[E(x)\,I(x)\left(\frac{\partial^2\delta y^N}{\partial x^2}\right)^2+\rho(x)\,A(x)\left(\frac{\partial\delta y^N}{\partial t}\right)^2+$$

$$+E(x)\,C_w(x)\left(\frac{\partial^2\delta\theta^N}{\partial x^2}\right)+GC\left(\frac{\partial\delta\theta^N}{\partial x}\right)^2+ \tag{5.8}$$

$$+\rho(x)\left(I(x)+A(x)\,e^2(x)\right)\left(\frac{\partial\delta\theta^N}{\partial t}\right)^2-2\rho(x)\,A(x)\,e(x)\left(\frac{\partial\delta\theta^N}{\partial t}\frac{\partial\delta y^N}{\partial t}\right)\Bigg]dx=$$

$$=2\int_0^t\int_0^l\left(\delta v_1\frac{\partial\delta y^N}{\partial t}+\delta v_2\frac{\partial\delta\theta^N}{\partial t}\right)dxds.$$

In equality (5.8) we make some transformations

$$\int_0^l\Bigg[E(x)\,I(x)\left(\frac{\partial^2\delta y^N}{\partial x^2}\right)^2+\rho(x)\,A(x)\left(\frac{\partial\delta y^N}{\partial t}\right)^2+$$

$$+E(x)\,C_w(x)\left(\frac{\partial^2\delta\theta^N}{\partial x^2}\right)+GC\left(\frac{\partial\delta\theta^N}{\partial x}\right)^2+$$
$$+\rho(x)\left(I(x)+A(x)\,e^2(x)\right)\left(\frac{\partial\delta\theta^N}{\partial t}\right)^2-$$
$$\left.-\rho(x)\,A(x)\,e(x)\left(\left(\frac{\partial\delta y^N}{\partial t}\right)^2+\left(\frac{\partial\delta\theta^N}{\partial t}\right)^2\right)\right]dx\le$$
$$\le 2\int_0^t\int_0^l\left(\delta v_1\frac{\partial\delta y^N}{\partial t}+\delta v_2\frac{\partial\delta\theta^N}{\partial t}\right)dxds$$

or

$$\int_0^l E(x)\,I(x)\left(\frac{\partial^2\delta y^N}{\partial x^2}\right)^2dx+\int_0^l\rho(x)\,A(x)\,(1-e(x))\left(\frac{\partial\delta y^N}{\partial t}\right)^2dx+$$
$$+\int_0^l E(x)\,C_w(x)\left(\frac{\partial^2\delta\theta^N}{\partial x^2}\right)^2dx+\int_0^l GC\left(\frac{\partial\delta\theta^N}{\partial x}\right)^2dx+$$
$$+\int_0^l\rho(x)\left[I(x)+A(x)\,e^2(x)-A(x)\,e(x)\right]\left(\frac{\partial\delta\theta^N}{\partial t}\right)^2dx\le$$
$$\le\int_0^t\int_0^l(\delta v_1(x,s))^2+(\delta v_2(x,s))^2\,dxds+$$
$$+\int_0^t\int_0^l\left(\left(\frac{\partial\delta y^N}{\partial t}\right)^2+\left(\frac{\partial\delta\theta^N}{\partial t}\right)^2\right)dxds.$$

Assume that $1-e(x)\ge\alpha_0>0,\ I(x)+A(x)\,e(x)\,(e(x)-1)\ge\alpha_1>0$ $\forall x\in[0,l]$, where $\alpha_0,\,\alpha_1>0$ are the given numbers.

Since $E(x)\,,\ I(x)\,,\ A(x)\,,\ C_w(x)\,,\ \rho(x)$ are positive functions on the segment $[0,l]$, by equivalence of the norms in the space $\overset{0}{W}{}_2^2(0,l)$, from the last inequality by means of elementary transformations we get:

$$\int_0^l\left[\left(\delta y^N(x,t)\right)^2+\left(\frac{\partial\delta y^N(x,t)}{\partial t}\right)^2+\left(\frac{\partial\delta y^N(x,t)}{\partial x}\right)^2+\right.$$
$$+\left(\frac{\partial^2\delta y^N(x,t)}{\partial x^2}\right)^2+\left(\delta\theta^N(x,t)\right)^2+$$
$$\left.+\left(\frac{\partial\delta\theta^N(x,t)}{\partial t}\right)^2+\left(\frac{\partial\delta\theta^N(x,t)}{\partial x}\right)^2+\left(\frac{\partial^2\delta\theta^N(x,t)}{\partial x^2}\right)^2\right]\le$$
$$\le c\int_0^T\int_0^l\left((\delta v_1)^2+(\delta v_2)^2\right)dxdt+$$
$$+c\int_0^t\int_0^l\left[\left(\delta y^N(x,s)\right)^2+\left(\frac{\partial\delta y^N(x,s)}{\partial t}\right)^2+\left(\frac{\partial\delta y^N(x,s)}{\partial x}\right)^2+\right.$$
$$+\left(\frac{\partial^2\delta y^N(x,s)}{\partial x^2}\right)^2+\left(\delta\theta^N(x,s)\right)^2+$$

$$+\left(\frac{\partial\delta\theta^N(x,s)}{\partial t}\right)^2+\left(\frac{\partial\delta\theta^N(x,s)}{\partial x}\right)^2+\left(\frac{\partial^2\delta\theta^N(x,s)}{\partial x^2}\right)^2\Bigg]dxds.$$

Hence, applying the Gronwall lemma, we have:

$$\int_0^l\Bigg[\left(\delta y^N(x,t)\right)^2+\left(\frac{\partial\delta y^N(x,t)}{\partial t}\right)^2+\left(\frac{\partial\delta y^N(x,t)}{\partial x}\right)^2+\left(\frac{\partial^2\delta y^N(x,t)}{\partial x^2}\right)^2+$$

$$+\left(\delta\theta^N(x,t)\right)^2+dx+\left(\frac{\partial\delta\theta^N(x,t)}{\partial t}\right)^2+\left(\frac{\partial\delta\theta^N(x,t)}{\partial x}\right)^2+\left(\frac{\partial^2\delta\theta^N(x,t)}{\partial x^2}\right)^2\Bigg]dx\le$$

$$\le c\left(\|\delta v_1\|^2_{L_2(Q)}+\le\|\delta v_2\|^2_{L_2(Q)}\right),\forall t\in[0,T].$$

From the last inequality it follows

$$\int_0^T\int_0^l\Bigg[\left(\delta y^N(x,t)\right)^2+\left(\frac{\partial\delta y^N(x,t)}{\partial t}\right)^2+\left(\frac{\partial\delta y^N(x,t)}{\partial x}\right)^2+$$

$$+\left(\frac{\partial^2\delta y^N(x,t)}{\partial x^2}\right)^2+\left(\delta\theta^N(x,t)\right)^2+\left(\frac{\partial\delta\theta^N(x,t)}{\partial t}\right)^2+$$

$$+\left(\frac{\partial\delta\theta^N(x,t)}{\partial x}\right)^2+\left(\frac{\partial^2\delta\theta^N(x,t)}{\partial x^2}\right)^2\Bigg]dx\le c\left(\|\delta v_1\|^2_{L_2(Q)}+\|\delta v_2\|^2_{L_2(Q)}\right)$$

From the sequence $\left(\delta y^N,\ \delta\theta^N\right)$ we can choose a subsequence weakly convergent in $W_2^{2,1}(Q)\times W_2^{2,1}(Q)$ to some element $(\delta y,\ \delta\theta)\in W_2^{2,1}(Q)\times W_2^{2,1}(Q)$.

By virtue of weak lower semicontinuity of the norm in the Hilbert space, we get that for $\delta y(x,t)$ and $\delta\theta(x,t)$ the following estimation is valid

$$\|\delta y\|^2_{W_2^{2,1}(Q)}+\|\delta\theta\|^2_{W_2^{2,1}(Q)}\le c\left(\|\delta v_1\|^2_{L_2(Q)}+\|\delta v_2\|^2_{L_2(Q)}\right).$$

Since $W_2^{2,1}(Q)$ is boundedly imbedded in $L_2(\Gamma)$ [9, pp. 73-74], hence it follows that

$$\|\delta y(d(t),t)\|^2_{L_2(0,T)}\le c\|\delta y\|^2_{W_2^{2,1}(Q)}\le c\left(\|\delta v_1\|^2_{L_2(Q)}+\|\delta v_2\|^2_{L_2(Q)}\right),\tag{5.9}$$

$$\|\delta\theta(d(t),t)\|^2_{L_2(0,T)}\le c\|\delta\theta\|^2_{W_2^{2,1}(Q)}\le c\left(\|\delta v_1\|^2_{L_2(Q)}+\|\delta v_2\|^2_{L_2(Q)}\right).\tag{5.10}$$

As in [7, pp. 214-215], it is easy to show that $(\delta y(x,t),\delta\theta(x,t))$ is the generalized solution of problem (4.10)-(4.13).

From inequalities (5.9) and (5.10) it follows

$$R=\frac{1}{2}\int_0^T\left[(\delta y(d(t),t))^2+(\delta\theta(d(t),t))^2\right]dt\le$$

$$\le c\left(\|\delta v_1\|^2_{L_2(Q)}+\|\delta v_2\|^2_{L_2(Q)}\right).\tag{5.11}$$

6. Gradient of the functional and optimality condition

Thus, from (4.19) and (5.11) it follows that the gradient of the functional $J(v)$ equals

$$J'(v) = (\psi_1(x,t;v), \psi_2(x,t;v)).$$

Let $v^0(x,t) = (v_1^0(x,t), v_2^0(x,t))$ be an optimal control in problem (2.7), (2.1)-(2.5). As U_{ad} is a convex set in $L_2(Q)\times L_2(Q)$

$$\left\langle J'(v), v - v^0\right\rangle \geq 0, \forall v = (v_1, v_2) \in U_{ad},$$

here we get

Theorem 6.1. *For the control* $v^0(x,t) = \left(v_1^0(x,t), v_2^0(x,t)\right) \in U_{ad}$ *to be an optimal control in problem* (2.7), (2.1)-(2.5), *it is necessary and sufficient that*

$$\iint_Q \left(\psi_1\left(x,t;v^0\right)\left(v_1^1(x,t) - v_1^0(x,t)\right) + \right.$$

$$\left. +\psi_2\left(x,t;v^0\right)\left(v_2^1(x,t) - v_2^0(x,t)\right)\right) dxdt \geq 0, \quad \forall v = (v_1,\, v_2) \in U_{ad}. \qquad (6.1)$$

Example 6.1. We consider a boundary value problem for equations of flexural-torsional vibrations of a bar, described by the system of two differential equations in the domain $Q = \{0 < x < 1,\, 0 < t < 1\}$

$$\frac{\partial^4 y}{\partial x^4} + 4\frac{\partial^2 y}{\partial t^2} - 2\frac{\partial^2 \theta}{\partial t^2} = v_1(x,t),$$

$$\frac{\partial^4 \theta}{\partial x^4} - \frac{\partial^2 \theta}{\partial x^2} - 2\frac{\partial^2 y}{\partial t^2} + 3\frac{\partial^2 \theta}{\partial t^2} = v_2(x,t),$$

$$y|_{t=0} = 0, \quad \left.\frac{\partial y}{\partial t}\right|_{t=0} = x^2(1-x)^2, \; \theta|_{t=0} = x^2(1-x)^2, \quad \left.\frac{\partial \theta}{\partial t}\right|_{t=0} = 0,$$

$$y|_{x=0} = y|_{x=1} = 0, \quad \left.\frac{\partial y}{\partial x}\right|_{x=0} = \left.\frac{\partial y}{\partial x}\right|_{x=1} = 0, \; \theta|_{x=0} = \theta|_{x=1} = 0,$$

$$\left.\frac{\partial \theta}{\partial x}\right|_{x=0} = \left.\frac{\partial \theta}{\partial x}\right|_{x=1} = 0.$$

In the special case, the coefficients of equations (2.1)-(2.2) were taken in the from:

$$E = \frac{1}{2}, \;\; I = 2, \;\; \rho = 1, \;\; e = \frac{1}{2}, \; A = 4, \; C_w = 2, \; G = 1, \; C = 1.$$

Let $U_{ad} = L_2(Q)\times L_2(Q)$.

In order to determine $v(x,t) = (v_1(x,t), v_2(x,t))$, we give the additional conditions:

$$y\left((t-1)^2, t; v\right) = t(t-1)^4\left(-t^2+2t\right)^2, \theta\left((t-1)^2, t; v\right) = (t-1)^4\left(-t^2+2t\right)^2.$$

In this special case the functional (2.7) has the form

$$J(v) = \frac{1}{2}\int_0^T \left[\left(y\left((t-1)^2, t; v\right) - t(t-1)^4\left(-t^2+2t\right)^2\right)^2 + \right.$$

$$\left. + \left(\theta\left((t-1)^2, t; v\right) - (t-1)^4\left(-t^2+2t\right)^2\right)^2\right] dt.$$

On the example, as there are no constraint on controls, for $v = v^0(x,t) = (v_1^0(x,t), v_2^0(x,t)) = (24t, 22 + 12x - 12x^2)$

$$J'(v^0) = (\psi_1(x,t;v^0), \psi_2(x,t;v^0)) = (0,0).$$

Then

$$\min_{v \in L_2(Q) \times L_2(Q)} J(v) = 0.$$

In this case necessary and sufficient condition (6.1) is fulfilled by itself.

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