Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 42, Number 2, 2016, Pages 188–201

GLOBAL EXISTENCE, ASYMPTOTIC BEHAVIOR AND BLOW-UP OF SOLUTIONS FOR MIXED PROBLEM FOR THE COUPLED WAVE EQUATIONS WITH NONLINEAR DAMPING AND SOURCE TERMS

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Abstract. The initial-boundary value problem for a system of nonlinear wave equations with a nonlinear damping and source terms is studied. It is proved the existence of the global solutions when the exponent of the damping term is greater than the exponent of source term and blow-up of the solutions with the positive initial energy infinite time is shown in otherwise case. Unlike the works conducted in this direction, in our work the corresponding exponents in source term are independent.

1. Introduction

Let Ω be bounded domain in R^n with smooth boundary Γ. We consider the mixed problem for a system of wave equations with nonlinear damping and source terms

$$\left.\begin{array}{l} u_{1_{tt}} - \Delta u_1 + m_1 u_1 + \alpha_1 \left|u_{1_t}\right|^{r_1 - 1} u_{1_t} = g_1(u_1, u_2) \\ u_{2_{tt}} - \Delta u_2 + m_2 u_2 + \alpha_2 \left|u_{2_t}\right|^{r_2 - 1} u_{2_t} = g_1(u_1, u_2) \end{array}\right\}, \quad x \in \Omega\,, \quad t > 0, \tag{1.1}$$

$$u_i\left(0, x\right) = \varphi_i\left(x\right), u_{it}\left(0, x\right) = \psi_i\left(x\right), x \in \Omega, \;\; i = 1, 2, \tag{1.2}$$

$$u_i\left(t, x\right) = 0, \;\; t > 0, \;\; x \in \Gamma, \;\;\; i = 1, 2, \tag{1.3}$$

where (u_1, u_2) is pair of real functions, $(t, x) \in R_+ \times \Omega$, $\alpha_j > 0, m_j > 0$, $j = 1, 2$. Concerning the functions $g_1(u_1, u_2)$ and $g_1(u_1, u_2)$, we assume that

$$g_1(u_1, u_2) = a_1 \left|u_1 + u_2\right|^{p_1 + p_2} (u_1 + u_2) + b_1 \left|u_1\right|^{p_1 - 1} \left|u_2\right|^{p_2 + 1} u_1,$$

$$g_2(u_1, u_2) = a_2 \left|u_1 + u_2\right|^{p_1 + p_2} (u_1 + u_2) + b_2 \left|u_1\right|^{p_1 + 1} \left|u_2\right|^{p_2 - 1} u_2,$$

where $a_i > 0$, $b_i > 0$, $p_i \geq 1$, $r_i \geq 1$, $i = 1, 2$ are constants.

Before stating our results, we first recall the existing results about the initial boundary value problem for the semi- linear wave equation

$$u_{tt} - \Delta u + h(u_t) = f(u), \tag{1.4}$$

where $h(u_t) = a \left|u_t\right|^{m-1} u_t$, $f(u) = b \left|u\right|^{p-1} u$, $a > 0$, $b > 0$, $m > 1$ and $p > 1$. There are numerous results about the global existence, asymptotic behavior and

2010 *Mathematics Subject Classification.* Primary: 35L05, 35L52, 35L76. Secondary: 35B40, 35B45.

Key words and phrases. Blow up, global existence, global nonexistence, system of wave equations, nonlinear dissipation.

blow-up of solutions for (1.4) [6-19]. The interaction between the damping term and the source term makes the problem more interesting. Levine [9, 10] firstly showed that the solutions with negative initial energy blow up in finite time for equation (1.4) with linear damping. Georgiev and Todorova introduced a new method and determined suitable relation between m and p, for which there is global existence or alternatively finite - time blow up (see [7]). They showed that if $p \leq m$ the solutions exist globally in time and blow up in finite time if $p > m$ and the initial energy is sufficiently negative.

Later, Levine and Serrin [12] and Levine, Park, and Serrin [11] generalized this result to an abstract setting and to unbounded domains. By combining the arguments in [11] and [12], Vitillaro extended these results to wave equation with nonlinear damping ($m > 1$) and when the initial energy is positive [18]. Messaoudi [13] improved the work [12] without imposing the condition that energy is sufficiently negative. For related results on a single wave equation, we refer the reader to [4,6,8,19,20] and the references therein.

Similar issues were also investigated for some systems of semilinear wave equations (see [2, 3, 21]).

In this paper, we consider problem (1.1)-(1.3) and investigate existence and nonexistence of global solutions.

Concerning blow up and nonexistence results in systems of wave equations, Agre and Rammaha [2], Said-Houari [15] studied the problem (1.1)-(1.3) in the case $p_1 = p_2$. They investigated the existence of local and global solutions and a question of the absence of global solutions.

At first we cite a theorem on local solvability. This theorem is proved by combining the Galerkin method and the fixed point method.

Note that here we have not mentioned such studies for nonlinear wave equations and systems with viscosity.

2. Preliminaries and main results

In this section we give our main results on the existence and non-existence of global solution. First of all, we give the theorem of local existence, which can be proved by the combined method of Galerkin and approximation method.

Theorem 2.1. *(Local existence). Suppose that* $p_1 \geq 0$, $p_2 \geq 0$ *and in additionally*

$$p_1 + p_2 \leq \frac{2}{n-2} \quad if \quad n > 2. \tag{2.1}$$

Then for any $\varphi_1(\cdot), \varphi_2(\cdot) \in \overset{0}{W}{}_2^1(\Omega)$, $\psi_1(\cdot), \psi_2(\cdot) \in L_2(\Omega)$ *there exists* $T > 0$ *such that the problem* (1.1)-(1.3) *has a unique local solution* $(u_1(t,x), u_2(t,x))$ *which satisfies*

$$u_i(\cdot) \in C\left([0,T)\,; \overset{0}{W}{}_2^1(\Omega)\right), u_{it}(\cdot) \in C\left([0,T)\,; L_2(\Omega)\right) \cap L_{r_i+1}([0,T] \times \Omega),\ i = 1, 2.$$

Moreover, at least one of the following statements holds true:

1) $\lim\limits_{t \to T-0} \sum\limits_{i=1}^{2} \left[\|u_{it}(t,\cdot)\|^2 + \|\nabla u_i(t,\cdot)\|^2 \right] = +\infty;$

2) $T = +\infty$.

The following theorem shows that the solution obtained in Theorem 2.1 is a global solution, if sum of exponents source term does not outclasses the exponent of damping term.

Theorem 2.2. *Assume that*

$$p_1 + p_2 + 1 \leq \min \{r_1, r_2\} . \tag{2.2}$$

Then the local solutions $\{u_1(\cdot), u_2(\cdot)\}$ *furnished in Theorem 2.1 are global solutions and* T *may be taken arbitrarily large.*

Next, let us discuss these problem under the assumption

$$p_1 + p_2 + 1 > \max \{r_1, r_2\} . \tag{2.3}$$

and

$$\frac{a_1(p_1+1)}{b_1} = \frac{a_2(p_2+1)}{b_2} . \tag{2.4}$$

In this case we will study the blow-up property to the Cauchy problem (1.1) - (1.3).

It easy to see that

$$\begin{aligned} G(u_1, u_2) &= \frac{p_1+1}{b_1} u_1 g_1(u_1, u_2) + \frac{p_2+1}{b_2} u_2 g_2(u_1, u_2) = \\ &= \lambda \left| u_1 + u_2 \right|^{p_1+p_2+2} + \ (p_1+p_2+2) \left| u_1 \right|^{p_1+1} \ \left| u_2 \right|^{p_2+1} \end{aligned} , \tag{2.5}$$

where

$$\lambda = \frac{a_1(p_1+1)}{b_1} = \frac{a_2(p_2+1)}{b_2}.$$

Moreover, a quick computation (see [15]) will show that there exist two positive constants $C_1 > 0$ and $C_2 > 0$ such that the following inequality holds

$$C_1(|u_1|^{p_1+p_2+2} + |u_2|^{p_1+p_2+2}) \leq G(u_1, u_2) \leq C_2(|u_1|^{p_1+p_2+2} + |u_2|^{p_1+p_2+2}) . \tag{2.6}$$

Let s be a number with $2 \leq s < +\infty$, if $n \leq 2$ and $2 \leq s \leq \frac{2}{n-2}$ if $n > 2$. Then there exists a constant B_s depending on n and s such that for every $v(\cdot) \in \overset{0}{W}{}_2^1(\Omega)$ the following estimate holds

$$\|v(\cdot)\|_s \leq B_s \|\nabla v\| \quad \text{(see [1])} \tag{2.7}$$

Now, we define the following energy functions associated with a solution $\{u_1(\cdot), u_2(\cdot)\}$ of the problem (1.1)-(1.3):

$$E_0(t) = \sum_{i=1}^{2} \frac{\lambda}{2a_i} \left[\|u_{it}(t, \cdot)\|^2 + \|\nabla u_i(t, \cdot)\|^2 \right] \tag{2.8}$$

and

$$E(t) = \sum_{i=1}^{2} \frac{\lambda}{2a_i} \left[\|u_{it}(t, \cdot)\|^2 + \|\nabla u_i(t, \cdot)\|^2 \right] - \frac{1}{p_1+p_2+2} \int_\Omega G(u_1, u_2) dx. \tag{2.9}$$

Multiplying equation (1.1) by $\frac{p_i+1}{b_i} u_{it}(t, x)$ and integrating over $[0, t] \times \Omega$. Then, integrating by parts, we get

$$E(t) - E(0) + \sum_{i=1}^{2} \frac{\lambda}{a_i} \int_0^t \int_\Omega |u_{it}(t, x)|^{r_i+1} dx dt = 0.$$

Thus, if the function $\{u_1(\cdot), u_2(\cdot)\}$ is a solution to the problem (1.1)-(1.3), then $E(t)$ is a non-increasing function for $t>0$ and

$$E'(t) = -\sum_{i=1}^{2} \frac{\lambda}{a_i} \int_\Omega |u_{it}(t,x)|^{r_i+1}\, dx. \tag{2.10}$$

It follows from (1.4) and (2.7) that

$$E(t) \geq \frac{1}{2}\sum_{i=1}^{2} \frac{\lambda}{a_i} \|\nabla u_i(t,\cdot)\|^2 - \frac{C_2}{p_1+p_2+2} \sum_{i=1}^{2} \|u_i(t,\cdot)\|_{p_1+p_2+2}^{p_1+p_2+2} \geq$$

$$\geq \frac{1}{2}\sum_{i=1}^{2} \frac{\lambda}{a_i} \|\nabla u_i(t,\cdot)\|^2 - \frac{B^{p_1+p_2+2}\, C_2}{p_1+p_2+2} \sum_{i=1}^{2} \|\nabla u_i(t,\cdot)\|^{p_1+p_2+2} \geq$$

$$\geq \frac{1}{2}\sum_{i=1}^{2} \frac{\lambda}{a_i} \|\nabla u_i(t,\cdot)\|^2 - \frac{B^{p_1+p_2+2}\, C_2 A}{\Lambda(p_1+p_2+2)} \left[\sum_{i=1}^{2} \frac{\lambda}{a_i} \|\nabla u_i(t,\cdot)\|^2\right]^{\frac{p_1+p_2+2}{2}} = g(\alpha), \tag{2.11}$$

where

$$A = a^{\frac{p_1+p_2+2}{2}}, \quad a = \max\{a_1, a_2\}, \; B = B_{(p_1+p_2+2)}, \; \Lambda = \lambda^{\frac{p_1+p_2+2}{2}},$$

$$\alpha = \left[\sum_{i=1}^{2} \frac{\lambda}{a_i} \|\nabla u(t,\cdot)\|^2\right]^{1/2}.$$

$$g(\alpha) = \frac{1}{2}\alpha^2 - \frac{B^{p_1+p_2+2}\, C_2 A}{\Lambda\,(p_1+p_2+2)} \alpha^{p_1+p_2+2}.$$

It is obvious that

$$g(0) = 0, \quad \lim_{\alpha\to+\infty} g(\alpha) = -\infty, g'(\alpha_1) = 0, \tag{2.12}$$

where $\alpha_1 = \left[\dfrac{\Lambda}{B^{p_1+p_2+2}\, C_2 A}\right]^{\frac{1}{p_1+p_2}}$.

Applying the idea of E.Vitillaro [18], we have the following lemma.

Lemma 2.1. *Assume that (2.3), (2.4) hold. Let*

$$E(0) < E_1, \tag{2.13}$$

and

$$\left[\sum_{i=1}^{2} \frac{\lambda}{a_i} \|\nabla \varphi_i(\cdot)\|^2\right]^{1/2} > \alpha_1, \tag{2.14}$$

where $E_1 = \left(\dfrac{1}{2} - \dfrac{1}{p_1+p_2+1}\right)\alpha_1^2$, *then there exists* $\alpha_2 > \alpha_1$ *such that*

$$\left[\sum_{i=1}^{2} \frac{\lambda}{a_i} \|\nabla u_i(t,\cdot)\|^2\right]^{1/2} \geq \alpha_2, \;\; t>0 \tag{2.15}$$

and

$$\int_\Omega G(u_1(t,\cdot), u_1(t,\cdot))dx \geq \frac{B^{p_1+p_2+2}\, C_2 A}{\Lambda} \alpha_2^{p_1+p_2+2}, \; t>0. \tag{2.16}$$

Proof. Let $E(0) < E_1$. Then by virtue of (2.12) there exists a number α_2 such that $\alpha_2 > \alpha_1$ and $g(\alpha_2) = E(0)$. On the other hand virtue of (2.10) $g(\alpha_0) \leq E(0) = g(\alpha_2)$, where $\alpha_0 = \left(\sum\limits_{i=1}^{2} \frac{\lambda}{a_i} \|\nabla\varphi_i(\cdot)\|^2\right)^{\frac{1}{2}}$. Since $g'(\alpha) > 0$ if $\alpha < \alpha_1$ and $g'(\alpha) < 0$ if $\alpha > \alpha_1$.

Considering (2.12) it follows that $\alpha_0 > \alpha_2$. Now, to establish (2.15), we suppose by contradiction that

$$\left(\sum_{i=1}^{2} \frac{\lambda}{a_i} \|\nabla u_i(t_0,\cdot)\|\right)^{\frac{1}{2}} < \alpha_2 \tag{2.17}$$

for some $t_0 > 0$. On the other hand because of the continuity of $\left(\sum\limits_{i=1}^{2} \frac{\lambda}{a_i} \|\nabla u_i(t,\cdot)\|\right)^{\frac{1}{2}}$ we chosen t_0 so that $\alpha_1 < \left(\sum\limits_{i=1}^{2} \frac{\lambda}{a_i} \|\nabla u_i(t_0,\cdot)\|\right)^{\frac{1}{2}} < \alpha_2$.

From (2.11), (2.17) we obtain that

$$E(t_0) \geq g\left(\left(\sum_{i=1}^{2} \|u_i(t_0,\cdot)\|^2\right)^{\frac{1}{2}}\right) > g(\alpha_2) = E(0).$$

In view of (2.10) it is impossible. Thus inequality (2.14) is valid.

On the other hand from (2.9), (2.10) we see that

$$\frac{1}{p_1+p_2+2}\int_\Omega G(u_1(t,\cdot),u_1(t,\cdot))dx \geq \sum_{k=1}^{n} \frac{\lambda}{2a_i} \|\nabla u_i(t,\cdot)\|^2 - E(t) \geq$$

$$\geq \frac{1}{2}\alpha_2^2 - g(\alpha_2) = \frac{B^{p_1+p_2+2}\ C_2 A}{\Lambda\,(p_1+p_2+2)}\alpha_2^{p_1+p_2+2}.$$

Theorem 2.3. *Assume that* (2.3), (2.4) *hold. If* $\varphi_i(\cdot) \in \overset{\circ}{W}{}_2^1(\Omega)$, $\psi_i(\cdot) \in L_2(\Omega)$, $i = 1, 2$, *then any solution of* (1.1)-(1.3) *with initial data satisfying* (2.13) *and*

$$B_p\left[\sum_{i=1}^{2} \frac{\lambda}{a_i} \|\nabla\varphi_i(\cdot)\|^2\right]^{\frac{p_1+p_2+2}{2}} > 1 \tag{2.18}$$

blows- up at a finite time.

3. Proof of Theorem 2.2

Let $\{u_1(t,x), u_2(t,x)\}$ be a weak solution to the initial-boundary value problem (1.1)-(1.3) defined on $[0,T)\times\Omega$ as furnished by Theorem 2.1. Then we easily see from (1.1) - (1.3) that

$$\sum_{i=1}^{2} \frac{\lambda}{2a_i}\left[\|u_{it}(t,\cdot)\|^2 + \|\nabla u_i(t,\cdot)\|^2\right] + \frac{1}{p_1+p_2+1}\int_\Omega G(u_1,u_2)dx+$$

$$+\sum_{i=1}^{2} \frac{\lambda}{2a_i}\int_0^t\int_\Omega |u_{it}(t,x)|^{r_i+1}\,dxdt = \sum_{i=1}^{2} \frac{\lambda}{2a_i}\left[\|\psi_i(\cdot)\|^2 + \|\nabla\varphi_i(\cdot)\|^2\right] +$$

$$+\frac{1}{p_1+p_2+1}\int_\Omega G(\varphi_1,\varphi_2)dx+$$

$$+\frac{2}{p_1+p_2+1}\int_0^t\int_\Omega \frac{\partial}{\partial t}G(u_1(t,x),u_2(t,x))dxdt. \tag{3.1}$$

From (2.5) we yield

$$J=\frac{2}{p_1+p_2+1}\int_0^t\int_\Omega \frac{\partial}{\partial t}G(u_1(t,x),u_2(t,x))dxdt=$$

$$=2\lambda\int_0^t\int_\Omega |u_1+u_2|^{p_1+p_2+1}(u_{1t}+u_{2t})dxdt+$$

$$+(p_1+1)\int_0^t\int_\Omega |u_1|^{p_1}|u_2|^{p_2+1}u_{1t}dxdt+$$

$$+(p_2+1)\int_0^t\int_\Omega |u_1|^{p_1+1}|u_2|^{p_2}u_{2t}dxdt= \quad J_1+J_2+J_3+J_4.$$

Each term can be estimated from above. Applying the Hölder inequality with exponents $q=\frac{r_1+1}{r_1}$, $q'=r_1+1$ we obtain that

$$J_1\le 2\lambda\int_0^t\int_\Omega |u_1+u_2|^{p_1+p_2+1}|u_{1t}|\,dxdt\le$$

$$\le 2\lambda\left(\int_0^t\int_\Omega |u_1+u_2|^{(p_1+p_2+1)\frac{r_1+1}{r_1}}dxdt\right)^{\frac{r_1}{r_1+1}}\cdot\left(\int_0^t\int_\Omega |u_{1t}|^{r_1+1}dxdt\right)^{\frac{1}{r_1+1}}.$$

Next applying the Young inequality with parameter $\eta=\varepsilon^{\frac{1}{r_1+1}}$ we obtain that

$$J_1\le 2\lambda\varepsilon^{-r_1}\int_0^t\int_\Omega |u_1+u_2|^{(p_1+p_2+1)\frac{r_1+1}{r_1}}dxdt\ +2\lambda\varepsilon\int_0^t\int_\Omega |u_{1t}|^{r_1+1}dxdt.$$

Since $p_1+p_2+1\le r_1$ from here we have that

$$\int_0^t\int_\Omega |u_1+u_2|^{(p_1+p_2+1)\frac{r_1+1}{r_1}}dxdt\ \le C_{11}+C_{12}\int_0^t\int_\Omega |u_1+u_2|^{p_1+p_2+2}dxdt$$

where $C_{11}=0$, $C_{12}=1$ if $p_1+p_2+1=r_1$,

$$C_{11}=\frac{[r_1-(p_1+p_2+1)]\ T\ mes\Omega}{(p_1+p_2+1)\,r_1},\quad C_{12}=\frac{(p_1+p_2+1)\,r_1}{(p_1+p_2+1)\,(r_1+1)}$$

if $p_1+p_2+1<r_1$. Thereby

$$J_1\le 2C_{11}\lambda\varepsilon^{-r_1}+2C_{12}\lambda\varepsilon^{-r_1}\int_0^t\int_\Omega |u_1+u_2|^{p_1+p_2+2}dxdt+$$

$$+2\lambda\varepsilon\int_0^t\int_\Omega |u_{1t}|^{r_1+1}dxdt.$$

Similarly we can prove that

$$J_2\le 2C_{21}\lambda\varepsilon^{-r_2}+2C_{22}\lambda\varepsilon^{-r_2}\int_0^t\int_\Omega |u_1+u_2|^{p_1+p_2+2}dxdt+$$

$$+2\lambda\varepsilon\int_0^t\int_\Omega |u_{2t}|^{r_2+1}dxdt.$$

Further applying the Hölder inequality with exponents $q = \frac{r_1+1}{r_1}$, $q' = r_1 + 1$ and Young inequality with parameter $\eta = \varepsilon^{\frac{1}{r_1+1}}$ we obtain that

$$J_3 \leq (p_1+1)\int_0^t \int_\Omega |u_1|^{p_1} |u_2|^{p_2+1} |u_{1t}|\, dxdt \leq$$

$$\leq (p_1+1)\left(\int_0^t \int_\Omega |u_1|^{p_1\frac{r_1+1}{r_1}} |u_2|^{(p_2+1)\frac{r_1+1}{r_1}}\, dxdt\right)^{\frac{r_1}{r_1+1}} \times$$

$$\times \left(\int_0^t \int_\Omega |u_{1t}|^{r_1+1}\, dxdt\right)^{\frac{1}{r_1+1}} \leq$$

$$\leq \frac{r_1(p_1+1)}{(r_1+1)\varepsilon^{r_1}} \int_0^t \int_\Omega |u_1|^{p_1\frac{r_1+1}{r_1}} |u_2|^{(p_2+1)\frac{r_1+1}{r_1}}\, dxdt +$$

$$+\frac{\varepsilon(p_1+1)}{r_1+1}\int_0^t \int_\Omega |u_{1t}|^{r_1+1}\, dxdt.$$

Finally applying the Hölder inequality with exponents

$$\rho = \frac{r_1(p_1+p_2+2)}{(p_2+1)\,(r_1+1)},$$

$$\rho' = \frac{r_1(p_1+p_2+2)}{r_1(p_1+p_2+2)\, - (p_2+1)(r_1+1)} = \frac{r_1(p_1+p_2+2)}{r_1p_1 - p_2 + r_1 - 1}$$

and Young inequality we obtain that

$$\int_0^t \int_\Omega |u_1|^{p_1\frac{r_1+1}{r_1}} |u_2|^{(p_2+1)\frac{r_1+1}{r_1}}\, dxdt \leq$$

$$\leq \left(\int_0^t \int_\Omega |u_1|^{p_1\frac{r+1}{r_1}\rho'}\, dxdt\right)^{\frac{1}{\rho'}} \left(\int_0^t \int_\Omega |u_2|^{p_1+p_2+2}\, dxdt\right)^{\frac{1}{\rho}} \leq$$

$$\leq \frac{1}{\rho'}\int_0^t \int_\Omega |u_1|^{p_1\frac{r_1+1}{r_1}\rho'}\, dxdt + \frac{1}{\rho}\int_0^t \int_\Omega |u_2|^{p_1+p_2+2}\, dxdt \leq$$

$$\leq \frac{1}{\rho'}\int_0^t \int_\Omega |u_1|^{p_1\frac{r_1+1}{r_1}\rho'}\, dxdt + \frac{1}{\rho}\int_0^t \int_\Omega |u_2|^{p_1+p_2+2}\, dxdt.$$

Since $p_1 + p_2 + 1 \leq r_1$, consequently

$$p_1\frac{r_1+1}{r_1}\rho' = p_1\frac{(r_1+1)(p_1+p_2+1)}{r_1(p_1+p_2+1) - (p_2+1)(r_1+1)} \leq p_1 + p_2 + 2.$$

So applying the Hölder inequality we have

$$\int_0^t \int_\Omega |u_1|^{p_1\frac{r+1_1}{r_1}\rho'}\, dxdt \leq C_{31} + C_{32}\int_0^t \int_\Omega |u_1|^{p_1+p_2+2}\, dxdt$$

where $C_{31} = 0,\ \ C_{32} = 1$ if $p_1 + p_2 + 1 = r_1$

$$C_{31} = \frac{[\,r_1 - (p_1+p_2+1)\ T]\ mes\Omega}{r_1p_1 - p_2 + r_1 - 1},\quad C_{32} = \frac{p_1\,(r_1+1)}{r_1p_1 - p_2 + r_1 - 1}$$

if $p_1 + p_2 + 1 < r_1$. Thereby

$$\sum_{i=1}^{2} \frac{\lambda}{2a_i}\left[\|u_{it}(t,\cdot)\|^2 + \|\nabla u_i(t,\cdot)\|^2\right] +$$

$$+\frac{1}{p_1+p_2+1}\int_\Omega G(u_1(t,x),u_2(t,x))dx+$$

$$+\sum_{i=1}^{2}(\frac{\lambda}{2a_i}-4\lambda\varepsilon)\int_0^t\int_\Omega |u_{it}(t,x)|^{r_i+1}\,dxdt\le\sum_{i=1}^{2}\frac{\lambda}{2a_i}\left[\|\psi_i(\cdot)\|^2+\|\nabla\varphi_i(\cdot)\|^2\right]+$$

$$+\frac{1}{p_1+p_2+1}\int_\Omega G(\varphi_1(x),\varphi_2(x))dx+C+C\int_0^t\int_\Omega G(u_1(t,x),u_2(t,x))dxdt. \quad (3.2)$$

By Gromwell's lemma, from (3.2) we find that

$$\sum_{i=1}^{2}\frac{\lambda}{2a_i}\left[\|u_{it}(t,\cdot)\|^2+\|\nabla u_i(t,\cdot)\|^2\right]+\frac{1}{p_1+p_2+1}\int_\Omega G(u_1(t,x),u_2(t,x))dx+$$

$$+\sum_{i=1}^{2}(\frac{\lambda}{2a_i}-4\lambda\varepsilon)\int_0^t\int_\Omega |u_{it}(t,x)|^{r_i+1}\,dxdt\le C.$$

By virtue of Theorem 2.1, $T>0$ is an arbitrary number.

4. Proof of theorem 2.3.

Let $E(0)<E_1$. Putting

$$H(t)=E_1-E(t),\ \ t\ge 0 \quad (4.1)$$

from (2.10) we have

$$H'(t)=-E'(t)=\sum_{i=1}^{2}\frac{\lambda}{a_i}\int_\Omega \left|u'_{it}(t,x)\right|^{r_i+1}dx\ge 0,\quad t\ge 0. \quad (4.2)$$

Thus we obtain from (2.5) and (2.15) that

$$0<H(0)<H(t),\ \ t\ge 0. \quad (4.3)$$

It follows from (4.1) the inequality

$$H(t)\le E_1-\sum_{i=1}^{2}\frac{\lambda}{2a_i}\|\nabla u_i(t,\cdot)\|^2+\frac{1}{p_1+p_2+2}\int_\Omega G(u_1,u_2)dx,\ \ t\ge 0.$$

By the inequality (2.15) we get

$$H(t)\le E_1-\frac{1}{2}\alpha_1^2+\frac{1}{p_1+p_2+2}\int_\Omega G(u_1,u_2)dx<\frac{1}{p_1+p_2+2}\int_\Omega G(u_1,u_2)dx,\ \ t\ge 0.$$

Introduce the notation

$$\Theta(t)=H^{1-\delta}(t)+\varepsilon\sum_{i=1}^{2}\frac{\lambda}{a_i}\int_\Omega u_i(t,x)u_{it}(t,x)dx,$$

where

$$\delta<\min\left\{\frac{p_1+p_2}{2(p_1+p_2+2)},\frac{p_1+p_2+1-r_1}{r_1(p_1+p_2+2)},\frac{p_1+p_2+1-r_2}{r_2(p_1+p_2+2)}\right\}. \quad (4.4)$$

Taking into account (1.1) we have

$$\Theta'(t) = (1-\delta)H^{-\delta}(t)H'(t) + \varepsilon \sum_{i=1}^{2} \frac{\lambda}{a_i} \int_\Omega |u_{it}(t,x)|^2 \, dx - \varepsilon \sum_{i=1}^{2} \frac{\lambda}{a_i} \|\nabla u_i(t,\cdot)\|^2 - $$
$$-\varepsilon \sum_{i=1}^{2} \frac{\lambda}{a_i} \int_\Omega |u_{it}(t,x)|^{r_i-1} u_{it}(t,x)u_i(t,x)dx + \varepsilon \int_\Omega G(u_1,u_2)dx \ . \tag{4.5}$$

Using the Hölder inequality with exponents $q = r_i + 1, \ q' = \dfrac{r_i+1}{r_i}$ we have

$$\left| \int_\Omega |u_{it}(t,x)|^{r_i-1} u_{it}(t,x)u_i(t,x)dx \right| \le$$
$$\le \left(\int_\Omega |u_{it}(t,x)|^{r_i+1} dx \right)^{r_i/(r_i+1)} \left(\int_\Omega |u_i(t,x)|^{r_i+1} dx \right)^{r_i/(r_i+1)} . \tag{4.6}$$

If in Young inequality

$$ab \le \frac{\eta^q}{q} a^q + \frac{1}{q'\eta^{q'}} b^{q'}, \ \ a, \, b \ge 0, \quad q > 1, \ \ q' = \frac{q}{q-1}$$

with the parameter $\eta > 0$, we take $q_i = \frac{r_i+1}{r_i}$, $q_i' = r_i + 1$, $\eta_i = \left[M_i H^{-\delta}(t)\right]^{\frac{r_i}{r_i+1}}$, $a_i = \left(\int_\Omega |u_{it}(t,x)|^{r_i+1} dx\right)^{r_i/(r_i+1)}$ and $b_i = \left(\int_\Omega |u_i(t,x)|^{r_i+1} dx\right)^{1/(r_i+1)}$, we have

$$\left| \varepsilon \int_\Omega |u_{it}(t,x)|^{r_i-1} u_{it}(t,x)u_i(t,x)dx \right| \le \varepsilon M_i H^{-\delta}(t) \int_\Omega |u_{it}(t,x)|^{r_i+1} dx +$$
$$+\varepsilon M_i^{-r_i} H^{\delta r_i}(t) \int_\Omega |u_i(t,x)|^{r_i+1} dx \tag{4.7}$$

and

$$\left| \varepsilon \sum_{i=1}^{2} \frac{\lambda}{a_i} \int_\Omega |u_{it}(t,x)|^{r_i-1} u_{it}(t,x)u_i(t,x)dx \right| \le$$
$$\le \varepsilon H^{-\delta}(t) \sum_{i=1}^{2} M_i \int_\Omega |u_{it}(t,x)|^{r_i+1} dx +$$
$$+\varepsilon \sum_{i=1}^{2} M_i^{-r_i} H^{\delta r_i}(t) \int_\Omega |u_i(t,x)|^{r_i+1} dx, \tag{4.8}$$

where $M_i = \frac{\lambda}{a_i} L$.

By virtue of (2.16) and (3.2), from (4.1) we get

$$\left| \varepsilon \int_B |u_{it}(t,x)|^{r_i-1} u_{it}(t,x)u_i(t,x)dx \right| \le \varepsilon L H^{-\delta}(t)H'(t) +$$
$$+ \sum_{i=1}^{2} \frac{\varepsilon}{M_i^{r_i}} \left(\frac{1}{p_1+p_2+2} \int_\Omega G(u_1,u_2)dx \right)^{r_i\delta} \int_\Omega |u_i(t,x)|^{r_i+1} dx. \tag{4.9}$$

Taking into the obvious inequality

$$\int_\Omega |u_i(t,x)|^{r_i+1} dx \le K_i \left(\int_\Omega |u_i(t,x)|^{p_1+p_2+2} dx \right)^{\frac{r_i+1}{p_1+p_2+2}}$$

where $K_i = |\Omega|^{\frac{p_1+p_2+2}{p_1+p_2+1-r_i}}$ from (4.9) we get

$$\left|\varepsilon \int_\Omega |u_{it}(t,x)|^{r_i-1} u_{it}(t,x)u_i(t,x)dx\right| \leq \varepsilon L H^{-\delta}(t)H'(t)+$$

$$+\varepsilon \sum_{i=1}^{2} \frac{K_i}{M_i^{r_i}(p_1+p_2+2)^{r_i\delta}} \left(\int_\Omega G(u_1,u_2)dx\right)^{r_i\delta+\frac{r_i+1}{p_1+p_2+2}}. \tag{4.10}$$

From (2.9) and (4.1) we have

$$-\sum_{i=1}^{2} \frac{\lambda}{a_i} \|\nabla u_i(t,\cdot)\|^2 = 2H(t) - 2E_1+$$

$$+\sum_{i=1}^{2} \frac{\lambda}{a_i} \int_\Omega |u_{it}(t,x)|^2 dx - \frac{2}{p_1+p_2+2} \int_\Omega G(u_1,u_2)dx \tag{4.11}$$

Namely (2.16) implies that

$$-\varepsilon E_1 \geq -\varepsilon E_1 \frac{\Lambda}{B^{p_1+p_2+2}\, C_2 A} \alpha_2^{-(p_1+p_2+2)} \int_\Omega G_1(u_1(t,\cdot),u_2(t,\cdot))dx. \tag{4.12}$$

On the other hand $g'(\alpha_1) = \alpha_1^2 - \dfrac{B^{p_1+p_2+1}}{\Lambda} C_2 A\, \alpha_1^{p_1+p_2+1} = 0$ therefore

$$\frac{p_1+p_2}{2(p_1+p_2+2)}\ \varepsilon\alpha_1^2 = \frac{p_1+p_2}{2(p_1+p_2+2)}\ \frac{B^{p_1+p_2+2} C_2 A}{\Lambda}\ \varepsilon\alpha_1^{p_1+p_2+2}, \quad t>0. \tag{4.13}$$

It follows, from(4.12)-(4.13) that

$$-\varepsilon E_1 \geq = -\frac{p_1+p_2}{2(p_1+p_2+2)}\ \varepsilon \left(\frac{\alpha_1}{\alpha_2}\right)^{p_1+p_2+2} \int_\Omega G(u_1(t,\cdot),u_2(t,\cdot))dx. \tag{4.14}$$

Using (4.11), (4.14) from (4.5) we have

$$\Theta'(t) \geq (1-\delta-\varepsilon L)H^{-\delta}(t)H'(t) + 2\varepsilon H(t) + 2\varepsilon \sum_{i=1}^{2} \frac{\lambda}{a_i} \int_\Omega |u_{it}(t,x)|^2 dx+$$

$$+\varepsilon\left[\frac{p_1+p_2}{p_1+p_2+2} - \frac{p_1+p_2}{2(p_1+p_2+2)} \alpha_1^{p_1+p_2+1} \alpha_2^{-(p_1+p_2+1)}\right] \int_\Omega G(u_1(t,\cdot),u_2(t,\cdot))dx- \tag{4.15}$$

$$-\varepsilon \sum_{i=1}^{2} \frac{K_i}{M_i^{r_i}(p_1+p_2+2)^{r_i\delta}} \left(\int_\Omega G(u_1,u_2)dx\right)^{r_i\delta+\frac{r_i+1}{p_1+p_2+2}}.$$

Since $\alpha_2 > \alpha_1$, so $\beta = \dfrac{p_1+p_2}{p_1+p_2+2} - \dfrac{p_1+p_2}{2(p_1+p_2+2)} \alpha_1^{p_1+p_2+1} \alpha_2^{-(p_1+p_2+1)} > 0$.

From (4.15) we have

$$\Theta'(t) \geq (1-\delta-\varepsilon L)H^{-\delta}(t)H'(t) + 2\varepsilon \sum_{i=1}^{2} \frac{\lambda}{a_i} \int_\Omega |u_{it}(t,x)|^2 dx + 2\varepsilon H(t)+$$

$$+\varepsilon\beta \int_\Omega G(u_1(t,\cdot),u_2(t,\cdot))dx-$$

$$-\varepsilon\sum_{i=1}^{2}\frac{K_i}{M_i^{r_i}\left(p_1+p_2+\right)^{r_i\delta}}\left(\int_\Omega G(u_1,u_2)dx\right)^{r_i\delta+\frac{r_i+1}{p_1+p_2+2}}. \tag{4.16}$$

In view of (4.4)

$$\nu=r_i\delta+\frac{r_i+1}{p_1+p_2+2}<1.$$

Then using the inequality

$$z^\nu\le z+1\le(1+\frac{1}{a}).(z+a),\quad a>0, \tag{4.17}$$

and (2.5) we obtain that

$$\left(\int_\Omega G(u_1,u_2)dx\right)^{r_i\delta+\frac{r_i+1}{p_1+p_2+2}}\le\left(1+\frac{1}{H(0)}\right)\left(\int_\Omega G(u_1,u_2)dx+H(0)\right)\le$$

$$\le\gamma\left(\int_\Omega G(u_1,u_2)dx+H(t)\right). \tag{4.18}$$

where $\gamma=1+\dfrac{1}{H(0)}$.

It follows from (4.16) and (4.18) that

$$\begin{aligned}\Theta'(t)\ge(1-\delta-\varepsilon L)H^{-\delta}(t)H'(t)+2\varepsilon\sum_{i=1}^{2}\tfrac{\lambda}{a_i}\textstyle\int_\Omega|u_{it}(t,x)|^2dx+\\+\varepsilon\left[2-\sum_{i=1}^{2}\frac{K_i\gamma a_i^{r_i}}{L^{r_i}\lambda^{r_i}\left(p_1+p_2+2\right)^{r_i\delta}}\right]H(t)+\\+\varepsilon\left[\beta-\sum_{i=1}^{2}\frac{K_i\gamma a_i^{r_i}}{L^{r_i}\lambda^{r_i}\left(p_1+p_2+2\right)^{r_i\delta}}\right]\textstyle\int_\Omega G(u_1,u_2)dx.\end{aligned} \tag{4.19}$$

Further we choose $L>\left(\dfrac{\min\{2,\beta\}}{\sum_{i=1}^{2}\frac{K_i\gamma a_i^{r_i}}{\lambda^{r_i}(p_1+p_2+2)^{r_i\delta}}}\right)^{\frac{1}{\max\{r_1,r_2\}}}$ and

$0<\varepsilon<\min\left\{\dfrac{1-\delta}{L},\ \dfrac{H^{1-\delta}(0)}{\left|\int_\Omega\varphi(t)\psi(x)dx\right|}\right\}$ then from (4.19) we get

$$\Theta(t)\ge\Theta(0)>0. \tag{4.20}$$

It is obvious that

$$\Theta^{\frac{1}{1-\delta}}(t)\le C_0\left[H(t)+\varepsilon^{\frac{1}{1-\delta}}\left|\sum_{i=1}^{2}\frac{\lambda}{a_i}\int_\Omega u_i(t,x)\ u_{it}(t,x)dx\right|^{\frac{1}{1-\delta}}\right]. \tag{4.21}$$

We have by Hölder inequalities for $\rho_1=p+1,\ \rho_2=\dfrac{p+1}{p-1},\ \rho_3=2$

$$\left|\int_\Omega u_i(t,x)\ u_{it}(t,x)dx\right|\le$$

$$\leq \left(\int_\Omega |u_i(t,x)|^{p+1}\,dx\right)^{\frac{1}{p+1}} \left(\int_\Omega |u_{it}(t,x)|^2\,dx\right)^{\frac{1}{2}} |\Omega|^{\frac{p-1}{2(p+1)}}. \tag{4.22}$$

Next, we have by Young's inequalities for $q = \dfrac{2(1-\delta)}{1-2\delta}$, $q' = 2(1-\delta)$

$$\left[\sum_{i=1}^{2}\left|\frac{\lambda}{a_i}\int_\Omega u_i(t,x)u_{it}(t,x)dx\right|\right]^{\frac{1}{1-\delta}} \leq$$

$$\leq \frac{1-2\delta}{2(1-\delta)}|\Omega|^{\frac{p_1+p_2+1)}{2(p_1+p_2+2)}}\left(\sum_{i=1}^{2}\int_\Omega |u_i(t,x)|^{p_1+p_2+2}\,dx\right)^{\frac{2}{(p_1+p_2+2)(1-2\delta)}} +$$

$$+\frac{\lambda}{2a(1-\delta)}|\Omega|^{\frac{p_1+p_2+1}{2(p_1+p_2+2)}}\sum_{i=1}^{2}\frac{\lambda}{a_i}\int_\Omega |u_{it}(t,x)|^2\,dx.$$

By (4.4), we have $v_1 = \dfrac{2}{(p_1+p_2+2)(1-2\delta)} < 1$. It follows from (4.17) that

$$\left(\sum_{i=1}^{2}\int_\Omega |u_i(t,x)|^{p_1+p_2+2}\,dx\right)^{\frac{2}{(p_1+p_2+2)(1-2\delta)}} \leq$$

$$\leq \gamma\left[\sum_{i=1}^{2}\int_\Omega |u_i(t,x)|^{p_1+p_2+2}\,dx + H(t)\right]. \tag{4.23}$$

By (4.22), (4.23) we obtain the inequality

$$\left[\sum_{i=1}^{2}\left|\frac{\lambda}{a_i}\int_\Omega u_i(t,x)u_{it}(t,x)dx\right|\right]^{\frac{1}{1-\delta}} \leq$$

$$\leq \gamma\frac{1-2\delta}{2(1-\delta)}|\Omega|^{\frac{p_1+p_2+1)}{2(p_1+p_2+2)}}\left(\sum_{i=1}^{2}\int_\Omega |u_i(t,x)|^{p_1+p_2+2}\,dx + H(t)\right) +$$

$$+\frac{\lambda}{2a(1-\delta)}|\Omega|^{\frac{p_1+p_2+1}{2(p_1+p_2+2)}}\sum_{i=1}^{2}\frac{\lambda}{a_i}\int_\Omega |u_{it}(t,x)|^2\,dx. \tag{4.24}$$

The inequalities (4.22) and (4.24) imply that

$$\begin{gathered}\Theta'(t) \geq (1-\delta-\varepsilon L)H^{-\delta}(t)H'(t) + 2\varepsilon\sum_{i=1}^{2}\int_\Omega |u_{it}(t,x)|^2\,dx + \\ +\varepsilon\theta_1 H(t) + \varepsilon\theta_2 C_1\sum_{i=1}^{2}\int_\Omega |u_i(t,x)|^{p_1+p_2+2}\,dx\ .\end{gathered} \tag{4.25}$$

$$\theta_1 = 2 - \sum_{i=1}^{2}\frac{K_i\gamma a_i^{r_i}}{L^{r_i}\lambda^{r_i}(p_1+p_2+2)^{r_i\delta}},$$

$$\theta_2 = \beta - \sum_{i=1}^{2}\frac{K_i\gamma a_i^{r_i}}{L^{r_i}\lambda^{r_i}(p_1+p_2+2)^{r_i\delta}}$$

Moreover, choosing $\varepsilon > 0$ sufficiently small from (4.25), (4.28) and (42.9) we can obtain the following differential inequality

$$\Theta'(t) \geq \omega\ \Theta^{\frac{1}{1-\delta}}(t) \tag{4.26}$$

where $\omega = \dfrac{1}{C_0} \min \left\{ \varepsilon\theta_1, \dfrac{2C_1\theta_2(1-\delta)}{(1-2\delta)\,|\Omega|^{\frac{p_1+p_2+1}{2(p_1+p_2+2)}}}, \dfrac{4a(1-\delta)}{\varepsilon^{\frac{\delta}{1-\delta}}\lambda\,|\Omega|^{\frac{p_1+p_2+1}{2(p_1+p_2+2)}}} \right\}$.

Integrating both sides of (4.26) over [0, t] yields that

$$\Theta^{\frac{\delta}{1-\delta}}(t) \geq \frac{\Theta^{\frac{\delta}{1-\delta}}(0)}{1 - \frac{\omega\delta}{1-\delta}\Theta^{\frac{\delta}{1-\delta}}(0)\,t}.$$

Noting that $\Theta(0) > 0$, then $\lim\limits_{t \to t_0 - 0} \Theta(t) = +\infty$, where

$$t_0 = \frac{1-\delta}{\omega\delta\,\Theta^{\frac{\delta}{1-\delta}}(0)}.$$

Namely, the solutions of the problem (1.1)-(1.3) blow-up in finite time.

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Received: June 16, 2016; Revised: August 21, 2016; Accepted: September 16, 2016