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GLOBAL BIFURCATION FROM PRINCIPAL EIGENVALUES FOR NONLINEAR FOURTH ORDER EIGENVALUE PROBLEM WITH INDEFINITE WEIGHT

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Abstract. In this paper we consider the nonlinear fourth order Sturmian system with sign-changing weight function. Under some natural hypotheses on nonlinear term we show the existence of an unbounded continua of nontrivial positive and negative solutions bifurcating from points of the line of trivial solutions corresponding to the positive and negative principal eigenvalues.

1. Introduction

We consider the following nonlinear fourth order eigenvalue problem

$$\ell(u) \equiv (p(t)u'')'' + (q(t)u')' = \mu r(t)u + g(t, u, u', u'', u''', \lambda), \ t \in (0,1), \tag{1.1}$$

$$\begin{array}{l} u'(0)\cos\alpha - (pu'')(0)\sin\alpha = 0, \\ u(0)\cos\beta + Tu(0)\sin\beta = 0, \\ u'(1)\cos\gamma + (pu'')(1)\sin\gamma = 0, \\ u(1)\cos\delta - Tu(1)\sin\delta = 0, \end{array} \tag{1.2}$$

where $\mu \in \mathbb{R}$ is a spectral parameter, $Ty \equiv (pu'')' - qu'$, $p(x)$ is positive and twice continuously differentiable function on $[0,1]$, $q(x)$ is non-negative and continuously differentiable function on $[0,1]$, $r(x)$ is continuous and sign-changing weight function on $[0,1]$, α, β, γ, δ are real constants, and $0 \le \alpha$, β, γ, $\delta \le \frac{\pi}{2}$ (with the exception of the cases $\alpha = \gamma = 0$, $\beta = \delta = \pi/2$ and $\alpha = \beta = \gamma = \delta = \pi/2$ for $q(x) \equiv 0$, and $\beta = \delta = \pi/2$ for $q \not\equiv 0$). The nonlinear term g is continuous function on $[0,1] \times \mathbb{R}^5$ and satisfies the condition:

$$g(t, u, s, v, w, \mu) = o(|u| + |s| + |v| + |w|) \tag{1.3}$$

near $(u, s, v, w) = (0, 0, 0, 0)$ uniformly in $t \in [0,1]$ and $\mu \in \Lambda$, for every bounded interval $\Lambda \subset \mathbb{R}$.

It is known that bifurcation of solutions of nonlinear Sturm-Liouville problems were studied many authors (see [1, 7, 9, 12, 21, 23-24, 26]). Note that the eigenfunction of corresponding linear Sturm-Liouville problems is characterized by the fact that it has only simple nodal zeros and the number of zeros of the eigenfunction is equal to the serial number of the corresponding eigenvalue increased by

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1. In [23] Rabinowitz to exploit these nodal properties an appropriate family of sets is introduced and the existence of various unbounded continua of solutions contained in these sets is proved. In future, the global results for nonlinearizable Sturm-Liouville problems were obtained by Berestycki [9], Schmitt and Smith [26], Aliyev [1], Rynne [24], Dai [12], Aliyev and Mamedova [7]. These papers prove the existence of global continua of nontrivial solutions in $\mathbb{R} \times C^1$ corresponding to the usual nodal properties and emanating from bifurcation intervals (in $\mathbb{R} \times \{0\}$ which we identify with $\mathbb{R}$) surrounding the eigenvalues of the linear problem.

Note that for the fourth order nonlinear eigenvalue problems nodal properties need not be preserved, so one cannot consider the alternative of Rabinowitz [23]. Only Lazer and McKenna [19], Rynne [25], Ma and Thompson [20] obtained results similar to above results for the second order nonlinear Sturm-Liouville problems. In [18, 24] the authors considered the linearizable problems with constant coefficients for simple boundary conditions. In a recent works of Aliyev [2-5] is consider nonlinear fourth order Sturmian system with positive weight function. A family of sets to exploit nodal properties and some other properties corresponding linear eigenvalue problem obtained from (1.1)-(1.2) by setting $g \equiv 0$ is introduced and the existence of global continua of solutions of linearizable and nonlinearizable problems contained in these sets is proved.

In the present paper we show that linear eigenvalue problem obtained from (1.1)-(1.2) by setting $g \equiv 0$ has two simple principal eigenvalues μ_1^+ and μ_1^- and the corresponding eigenfunctions are not vanish in $(0, 1)$. Moreover, for the nonlinear problem we prove the existence of two global continua of nontrivial positive and negative solutions bifurcating from each bifurcation points of the line of trivial solutions corresponding to the principal eigenvalues.

2. Preliminaries

Consider the following linear eigenvalue problem

$$\ell u = \mu r(t) u \text{ in } (0,1), \ u \in \text{B.C.}, \tag{2.1}$$

where by B.C. we denote the set of boundary conditions (1.2).

Note that by the hypotheses on coefficients of boundary conditions (1.2) it follows that the system (2.1) is a completely regular Sturmian system in a sense of Janczewsky [16] with indefinite weight function. Since r is a sign-changing continuous weight function on $[0, 1]$ it follows that for each $\nu \in \{+, -\}$ we have $\text{meas}\{t \in (0, 1) : \nu r(t) > 0\} \neq 0$.

In the proof of the following theorem an important role played the recent results obtained by Aliyev [4-5] (see also [6, 18]).

Theorem 2.1. *The spectral problem* (2.1) *has two sequences of real eigenvalues*

$$0 < \mu_1^+ \le \mu_2^+ \le ... \le \mu_k^+ \mapsto +\infty,$$

and

$$0 > \mu_1^- \ge \mu_2^- \ge ... \ge \mu_k^- \mapsto -\infty$$

and no other eigenvalues. Moreover, μ_1^+ and μ_1^- are simple principal eigenvalues, i.e. the corresponding eigenfunctions $v_1^+(t)$ and $v_1^-(t)$ are positive in $(0, 1)$.

Proof. It is well known that the differential operator L defined by $Lu = \ell u$ for all $u \in D(L) = \{u \in H = L^2(0,1) : u \in W^{2,4}(0,1), \ell u \in H, u \in \text{B.C.}\}$ is a densely defined self-adjoint operator on H whose spectrum contains only the eigenvalues $0 < \lambda_1 < \lambda_2 < \ ... \ < \lambda_k < ...$ (see [8, 16]). Hence the operator $L : D(L) \subset H \to H$ is positive definitely and has compact resolvent in H. Then there exists $L^{-1} : H \to H$ and is a compact self-adjoint operator in Hilbert space H. Let $R : H \to H$ denote the multiplication operator induced by the function r. Consequently, the problem (2.1) can be rewritten in the following equivalent form:

$$\mathcal{L}u = \lambda u \ \ (\mathcal{L} = L^{-1}R \text{ and } \lambda = \mu^{-1}). \tag{2.2}$$

Then by [14; Proposition 1.7] the eigenvalue problem (2.2) has two sequences of real eigenvalues

$$\lambda_1^- \le \lambda_2^- \le ... \le \lambda_k^- \mapsto 0$$

and

$$\lambda_1^+ \ge \lambda_2^+ \ge ... \ge \lambda_k^+ \mapsto 0,$$

and no other eigenvalues. For each $k \in \mathbb{N}$

$$\lambda_k^+ = \inf_{F_{k-1}} \sup \left\{(\mathcal{L}u, u) : \ ||u|| = 1, \ u \perp \mathcal{F}_{k-1}\right\},$$

where the infimum is taken over all subspaces $\mathcal{F}_{k-1}$ of H with dimension $k-1$, where $||\cdot||$ is the norm in H. A similar formula holds for λ_k^-:

$$\lambda_k^- = \sup_{F_{k-1}} \inf \left\{(\mathcal{L}u, u) : \ ||u|| = 1, \ u \perp \mathcal{F}_{k-1}\right\}.$$

Consequently, it is follows that the spectral problem (2.1) has two sequences of real eigenvalues

$$0 < \mu_1^+ \le \mu_2^+ \le ... \le \mu_k^+ \mapsto +\infty,$$

and

$$0 > \mu_1^- \ge \mu_2^- \ge ... \ge \mu_k^- \mapsto -\infty$$

and no other eigenvalues.

Note that

$$\lambda_1^+ = \sup\{(\mathcal{L}u, u) : \ u \in H, \ ||u|| = 1\} > 0,$$

$$\lambda_1^- = \inf\{(\mathcal{L}u, u) : \ u \in H, \ ||u|| = 1\} < 0.$$

Since $\mathcal{L}$ is a self-adjoint operator in H it follows that all eigenvalues μ_k^ν, $k \in \mathbb{N}$, $\nu \in \{+, -\}$ of problem (2.1) are semi-simple, i.e. the algebraic multiplicity of this eigenvalues coincide with the geometric multiplicities.

For all $v \in D(L)$ we define the following functional

$$\mathcal{Q}_\mu(v) = (Lv, v) - \mu \int_0^1 rv^2 dt.$$

We show that if there is a nonnegative eigenfunction corresponding to an eigenvalue μ of problem (2.1), then

$$\mathcal{Q}_\mu(v) \ge 0 \text{ for all } v \in D(L).$$

Indeed, let u is a nonnegative eigenfunction corresponding to the eigenvalue μ. Then u is an eigenfunction corresponding to the eigenvalue η of the following spectral problem

$$\ell u - \mu r(t)u = \eta u \text{ in } (0,1), \ u \in \text{B.C.}. \tag{2.3}$$

Obviously, the eigenvalues and eigenfunctions of problem (2.3) correspond to the eigenvalues and eigenfunctions of operator $B_\mu : D(L) \to H$ defined by

$$B_\mu u(t) = Lu(t) - \mu r(t)u(t).$$

By [6; Theorem 1] (see also [4, 5, 18]) for each $\mu \in \mathbb{R}$ the eigenvalues of the completely Sturmian system (2.3) with nonzero potential are real and simple and form an infinitely increasing sequence

$$\eta_1(\mu) < \eta_2(\mu) < \ \dots \ < \eta_k(\mu) < \ \dots \ .$$

Moreover, the eigenfunction $v_{k,\mu}$ corresponding to the eigenvalue $\eta_k(\mu)$ has $k-1$ simple zeros in the interval $(0,1)$. Hence u is not orthogonal to $v_{1,\mu}$ and so, since eigenfunctions of the operator B_μ corresponding to distinct eigenvalues are orthogonal, u must be an eigenfunction corresponding to $\eta_1(\mu)$, i.e. $\eta_1(\mu) = 0$. By the spectral theorem (see for example [17]) $(B_\mu v, v) \geq \eta_1(\mu)(v,v) = 0$ for all $v \in D(B_\mu)$, i.e. $\mathcal{Q}_\mu(v) \geq 0$ for all $v \in D(L)$.

It follows from [15, 22] that we can define μ_1^+ as follows

$$\mu_1^+ = \inf\{R(v) : v \ \in D(L), \int_0^1 rv^2 dt > 0\},$$

where $R(v)$ is a Rayleigh quotient

$$R(v) = \frac{(Lv,v)}{\int_0^1 rv^2 dt} = \frac{\int_0^1 \{pv''^2 + qv'^2\}dt + N(v)}{\int_0^1 rv^2 dt},$$

$$N(v) = v'^2(0)\cot\alpha + v^2(0)\cot\beta + v'^2(1)\cot\gamma + v^2(1)\cot\delta.$$

Obviously,

$$\mathcal{Q}_{\mu_1^+}(v) = (Lv,v) - \mu_1^+ \int_0^1 rv^2 dt \geq 0 \ \text{ for all } \ v \in D(L).$$

Moreover, by spectral theorem we have $(Lv,v) \geq \lambda_1(v,v)$ for all $v \in D(L)$. Hence, if $v \in D(L)$ and $\int_0^1 rv^2 dt > 0$, then we have

$$R(v) = \frac{(Lv,v)}{\int_0^1 rv^2 dt} \geq \frac{\lambda_1(v,v)}{\int_0^1 rv^2 dt} \geq \frac{\lambda_1(v,v)}{r_1\int_0^1 v^2 dt} = \frac{\lambda_1}{r_1},$$

which implies that

$$\mu_1^+ \geq \frac{\lambda_1}{r_1} > 0,$$

where $r_1 = \max\limits_{t\in[0,1]} r(t)$.

Note that if $\lambda > \mu_1^+$ then it follows by the definition of μ_1^+ that there exists $v \in D(L)$ such that

$$\frac{(Lv, v)}{\int\limits_0^1 rv^2 dt} < \lambda.$$

Hence, we have

$$\mathcal{Q}_\lambda(v) < 0.$$

Then it follows from the above consideration that if $\lambda > \mu_1^+$, then λ is not an eigenvalue of problem (2.1) with corresponding nonnegative eigenfunction.

We define the operator $B_1 : D(L) \to H$ by

$$B_1 u(t) = Lu(t) - \mu_1^+ r(t) u(t).$$

It obvious that μ_1^+ is an eigenvalue of problem (2.1) with corresponding eigenfunction v_1 if and only if 0 is an eigenvalue of operator B_1 (or of (2.3) for $\mu = \mu_1^+$) with corresponding eigenfunction v_1. The least eigenvalue of operator B_1 is given by the following

$$\eta_1 = \inf\{(Lv, v) - \mu_1^+ \int\limits_0^1 rv^2 dt : \ v \in D(L)\} = \inf\{Q_{\mu_1^+}(v) : v \in D(L)\}.$$

Since $Q_{\mu_1^+}(v) \geq 0$ for all $v \in D(L)$ it follows that $\eta_1 \geq 0$. On the other hand by the definition of μ_1^+ there exists a sequence $\{v_m\}_{m=1}^\infty \subset D(L)$ such that

$$\int\limits_0^1 rv_m^2 dt = 1 \ \text{ and } \ \lim_{m\to\infty} R(v_m) = \mu_1^+,$$

from where implies that $\lim\limits_{m\to\infty} Q_{\mu_1^+}(v_m) = 0$, and consequently, $\eta_1 \leq 0$. Hence η_1 is the least eigenvalue of problem (2.3) for $\mu = \mu_1^+$ and by [6; Theorem 1] η_1 is simple and the corresponding eigenfunction is not vanish in the interval $(0, 1)$.

Let $I_r^\nu = \{t \in (0, 1) : \mu r(t) > 0\}$, $\nu \in \{+, -\}$. It is clear that $I_r^- = -I_{-r}^+$. Then we have meas$\{I_{-r}^+\} =$ meas$\{I_r^-\} > 0$. The problem (2.1) can be rewritten in the following equivalent form

$$(\ell u)(t) = \hat{\mu}\, \hat{r}(t) u(t), \ t \in (0, 1), \ \ u \in \text{B.C.}, \tag{2.4}$$

where $\hat{\mu} = -\mu$ and $\hat{r}(x) = -r(x)$, $x \in \bar{I}$. Then by above the first positive eigenvalue $\hat{\mu}_1^+$ is simple and the corresponding eigenfunction $\hat{v}_1^+$ is not vanish in $(0, 1)$. Note that $\mu_1^- = -\hat{\mu}_1^+$ is a first negative eigenvalue of problem (2.1) and corresponding eigenfunction $v_1^- = \hat{v}_1^+$ is not vanish in $(0, 1)$. The proof of theorem is complete.

Remark 2.1. It follows from the proof of Theorem 2.1 that μ_1^+ and μ_1^- are positive and negative principal eigenvalues of problem (2.1), respectively.

3. Global bifurcation of positive and negative solutions of problem (1.1)-(1.2)

Let E be the Banach space of all continuously three times differentiable functions on $[0,1]$ which satisfy the conditions B.C. and is equipped with its usual norm $||u||_3 = ||u||_0 + ||u'||_0 + ||u''||_0 + ||u'''||_0$, where $||u||_0 = \max\limits_{t\in[0,1]} |u(t)|$.

Let

$$S = S_1 \cup S_2,$$

where

$$S_1 = \{u \in E : u^{(i)}(t) \neq 0,\ Tu(t) \neq 0,\ t \in (0,1),\ i = 0,\ 1,\ 2\,\}$$

and

$S_2 = \{u \in E :$ there exists $i_0 \in \{0,\ 1,\ 2\}$ and $t_0 \in (0,1)$ such that $u^{(i_0)}(t_0) = 0$, or $Tu(t_0) = 0$ and if $u(t_0)u''(t_0) = 0$, then $u'(t)Tu(t) < 0$ in a neighborhood of t_0, and if $u'(t_0)Tu(t_0) = 0$, then $u(t)u''(t) < 0$ in a neighborhood of $t_0\}$.

Note that if $u \in S$ then the Jacobian $J = \rho^3 \cos\psi \sin\psi$ (see [4, 5, 6, 8]) of the Prüfer-type transformation

$$\begin{cases} y(x) = \rho(x)\sin\psi(x)\cos\theta(x), \\ y'(x) = \rho(x)\cos\psi(x)\sin\varphi(x), \\ (py'')(x) = \rho(x)\cos\psi(x)\cos\varphi(x), \\ Ty(x) = \rho(x)\sin\psi(x)\sin\theta(x), \end{cases} \tag{3.1}$$

does not vanish in $(0,1)$.

For each $u \in S$ we define $\rho(u,t)$, $\theta(u,t)$, $\varphi(y,t)$, $w(u,t)$ to be continuous functions on $[0,1]$ satisfying

$$\rho(u,t) = u^2(t) + u'^2(t) + (p(t)u''(t))^2 + (Tu(t))^2,$$

$$\theta(u,t) = \operatorname{arctg}\frac{Tu(t)}{u(t)},\ \ \theta(u,0) = \beta - \pi/2\,,$$

$$\varphi(u,t) = \operatorname{arctg}\frac{u'(t)}{(pu'')(t)},\ \ \varphi(u,0) = \alpha\,,$$

$$w(u,t) = \operatorname{ctg}\psi(u,t) = \frac{u'(t)\cos\theta(u,t)}{u(t)\sin\varphi(u,t)},\ \ w(u,0) = \frac{u'(0)\sin\beta}{u(0)\sin\alpha}\,,$$

where if $u(0)u'(0) > 0$ or $u(0) = 0$ or $u'(0) = 0$ and $u(0)u''(0) > 0$, then $\psi(u,t) \in (0,\pi/2)$ for $t \in (0,1)$; if $u(0)u'(0) < 0$ or $u'(0) = 0$ and $u(0)u''(0) < 0$ or $u'(0) = u''(0) = 0$, then $\psi(u,t) \in (\pi/2,\pi)$ for $t \in (0,1)$; $\beta = \pi/2$ in the case $\psi(u,0) = 0$ and $\alpha = 0$ in the case $\psi(u,0) = \pi/2$.

It is apparent that $\rho,\ \theta,\ \varphi, w : S \times [0,1] \to \mathbb{R}$ are continuous.

Remark 3.1. By (3.1) for each $u \in S$ the function $w(u,t)$ can de determined one of the following relations

a) $w(y,x) = \operatorname{ctg}\psi(y,x) = \dfrac{(py'')(x)\cos\theta(y,x)}{y(x)\cos\varphi(y,x)},\ \ w(y,0) = \dfrac{(py'')(0)\sin\beta}{y(0)\cos\alpha}\,,$

b) $w(y,x) = \operatorname{ctg}\psi(y,x) = \dfrac{(py'')(x)\sin\theta(y,x)}{Ty(x)\cos\varphi(y,x)},\ \ w(y,0) = -\dfrac{(py'')(0)\cos\beta}{Ty(0)\cos\alpha}\,,$

c) $w(y,x) = \operatorname{ctg}\psi(y,x) = \dfrac{y'(x)\sin\theta(y,x)}{Ty(x)\sin\varphi(y,x)},\ \ w(y,0) = -\dfrac{y'(0)\cos\beta}{Ty(0)\sin\alpha}\,.$

For each $\nu \in \{+\,,\,-\}$ let by S_1^ν denote the subset of $u \in S$ such that

1) $\theta(u,1) = \pi/2 - \delta$, where $\delta = \pi/2$ in the case $\psi(u,1) = 0\,$;

2) $\varphi(u,1)=2\pi-\gamma$ or $\varphi(u,1)=\pi-\gamma$ in the case $\psi(u,0)\in[0,\pi/2)$; $\varphi(u,1)=\pi-\gamma$ in the case $\psi(y,0)\in[\pi/2,\pi)$, where $\gamma=0$ in the case $\psi(y,l)=\pi/2$;

3) for fixed u, as t increases from 0 to 1, the function $\theta(u,t)$ ($\varphi(u,t)$) strictly increasing takes values of $m\pi/2$, $m\in\mathbb{Z}$ ($s\pi$, $s\in\mathbb{Z}$); as t decreases from 1 to 0, the function $\theta(u,t)$ ($\varphi(u,t)$), strictly decreasing takes values of $m\pi/2$, $m\in\mathbb{Z}$ ($s\pi$, $s\in\mathbb{Z}$);

4) the function $\nu u(t)$ is positive in a deleted neighborhood of $t=0$.

By [4; Theorem 4.4], [6; Theorem 1], [18; Theorem 1] and the proof of Theorem 2.1 we have $u_1^+\in S_1^+$, $u_1^-\in S_1^-$, i.e the sets S_1^+, S_1^- and S_1 are nonempty. It follows immediately from the definition of these sets that they are disjoint and open in E. Moreover, by [4; Lemma 2.2] if $u(t)\in\partial S_1^\nu\cap C^4[0,1]$, $\nu\in\{+,-\}$, then $u(t)$ has at least one zero of multiplicity 4 in $(0,1)$.

Without loss of generality, we assume that u_1^+ and u_1^- lies in S_1^+ and $||u_1^\pm||_3=1$.

We denote by $\mathfrak{L}$ the closure in $\mathbb{R}\times E$ of the set of nontrivial solutions of problem (1.1)-(1.2). Let $\mathcal{S}^\nu=\mathbb{R}\times S_1^\nu$, $\nu\in\{+,-\}$ and $\mathcal{S}=\mathbb{R}\times S_1$.

Theorem 3.1 *For each $k\in\{1,-1\}$ and each $\nu\in\{+,-\}$ there exists a continuum $\mathfrak{L}_k^\nu$ of solutions of problem* (1.1)-(1.2) *in $\mathcal{S}^\nu\cup\{(\mu_1^{\mathrm{sgn}k},0)\}$ which meets $(\mu_1^{\mathrm{sgn}k},0)$ and ∞ in $\mathbb{R}\times E$.*

Proof. Note first that if (μ,u) is a solution of problem (1.1)-(1.2) and u has a zero of multiplicity 4, then the growth estimate on g near this zero and linearity of ℓ and ru implies that $u\equiv 0$ on $[0,1]$. Therefore, in particular, any solution (μ,u) of (1.1)-(1.2) with $u\in S_1^\nu$ has $u\equiv 0$.

Since $\mu=0$ is not eigenvalue of the linear problem (2.1) it follows that the problem (1.1)-(1.2) is equivalent to the following integral equation

$$u(t)=\mu\int_0^1 G(t,s)r(s)u(s)ds+\int_0^1 G(t,s)g(s,u(s),u'(s),u''(s),u'''(s),\mu)ds, \quad (3.2)$$

where $G(t,s)$ is a Green's function of differential expression ℓ with respect to the $B.C.$.

Define $\mathcal{B}:E\to E$ by

$$(\mathcal{B}u)(t)=\int_0^1 G(t,s)r(s)u(s)ds$$

and $\mathcal{G}:\mathbb{R}\times E\to E$ by

$$\mathcal{G}(\mu,u)(t)=\int_0^1 G(t,s)g(s,u(s),u'(s),u''(s),u'''(s),\mu)ds\,.$$

It is easily seen that he operator $\mathcal{B}$ is compact in E and the operator $\mathcal{G}:\mathbb{R}\times E\to E$ is a completely continuous. Thus problem (3.3) can be rewritten in the following equivalent form

$$u=\mu\mathcal{B}u+\mathcal{G}(\mu,u), \quad (3.3)$$

which is of the form $(0,1)$ from [23]. The principal eigenvalues μ_1^+ and μ_1^- of L are the characteristic values of $\mathcal{B}$ and are simple. Hence all the conditions of

Theorem 1.3 from [23] are satisfied and there exists a continuum $\mathfrak{L}_{\mu_1^{\mathrm{sgn}k}} \equiv \mathfrak{L}_k$ as in Theorem 1.3 in [23]. It follows from [23; Lemma 1.24] that if $(\mu, u) \in \mathfrak{L}_k$ and is in a small neighborhood of a point $(\mu_1^{\mathrm{sgn}k}, 0)$, then

$$u = \tau u_1^{\mathrm{sgn}k} + w, \text{ where } w = o(|\tau|) \text{ as } \tau \to 0. \tag{3.4}$$

Since S_1 open subset in E and $u_1^{\mathrm{sgn}k} \in S_1$, then we have the following relations

$$(\mu, u) \in \mathcal{S} \quad \text{and} \quad (\mathfrak{L}_k \backslash \{(\mu_1^{\mathrm{sgn}k}, 0)\}) \cap B_\xi(\mu_1^{\mathrm{sgn}k}) \subset \mathcal{S}, \tag{3.5}$$

for all small positive ξ, where $B_\xi(\mu_1^{\mathrm{sgn}k})$ is a open ball in $\mathbb{R} \times E$ of radius ξ centered at $(\mu_1^{\mathrm{sgn}k}, 0)$. It follows by the relation

$$(\mathfrak{L}_k \backslash \{(\mu_1^{\mathrm{sgn}k}, 0)\}) \cap \partial\mathcal{S} = \emptyset \tag{3.6}$$

that

$$\mathfrak{L}_k \subset \mathcal{S} \cup \{(\mu_1^{\mathrm{sgn}k}, 0)\}.$$

Now using Dancer's construction (see [13]) we decompose $\mathfrak{L}_k$, $k \in \{1, -1\}$ into two subcontinua $\mathfrak{L}_k^+$ and $\mathfrak{L}_k^-$ with meets $(\mu_1^{\mathrm{sgn}k}, 0)$ in $\mathcal{S}^+ \cup \{(\mu_1^{\mathrm{sgn}k}, 0)\}$ and $\mathcal{S}^- \cup \{(\mu_1^{\mathrm{sgn}k}, 0)\}$, respectively. Note that if $\tau \in \mathbb{R}^\nu \backslash \{0\}$ then $\tau u_1^{\mathrm{sgn}k} \in S_1^\nu$, where $\mathbb{R}^\nu = \{\varkappa \in \mathbb{R} : 0 \le \nu\varkappa \le +\infty\}$. Hence by (3.4) and (3.5) it follows that the following inclusions

$$(\mathfrak{L}_k^+ \backslash \{(\mu_1^{\mathrm{sgn}k}, 0)\}) \cap B_\xi(\mu_1^{\mathrm{sgn}k}) \subset \mathcal{S}^+$$

and

$$(\mathfrak{L}_k^- \backslash \{(\mu_1^{\mathrm{sgn}k}, 0)\}) \cap B_\xi(\mu_1^{\mathrm{sgn}k}) \subset \mathcal{S}^-,$$

for all small positive ξ. Moreover, by the relation (3.6) for each $\nu \in \{+, -\}$ we have

$$(\mathfrak{L}_k^\nu \backslash \{(\mu_1^{\mathrm{sgn}k}, 0)\}) \cap \partial\mathcal{S}^\nu = \emptyset,$$

which implies that

$$(\mathfrak{L}_k^\nu \backslash \{(\mu_1^{\mathrm{sgn}k}, 0)\}) \subset \mathcal{S}^\nu.$$

This means that $\mathfrak{L}_k^\nu \backslash \{(\mu_1^{\mathrm{sgn}k}, 0)\}$ cannot leave $\mathcal{S}^\nu$ outside of a neighborhood of $(\mu_1^{\mathrm{sgn}k}, 0)$. Note also that $\mathcal{S}^+ \cap \mathcal{S}^- = \emptyset$. Then it follows from [13; Theorem 2] that for each $k \in \{1, -1\}$ both the sets $\mathfrak{L}_k^+$ and $\mathfrak{L}_k^-$ are unbounded in $\mathbb{R} \times E$. The proof of Theorem 3.1 is complete.

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