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ON THE FRACTIONAL ORDER WEIGHTED HARDY INEQUALITY FOR MONOTONE FUNCTIONS

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Abstract. In this paper, using a new approach we study one dimensional fractional order weighted Hardy type inequality

$$\int_0^\infty u(t)^p \nu(t) t^{n-1} dt \le C \int_0^\infty \Big(\int_0^t \big(u(x) - u(t)\big)^p \omega(t-x) x^{n-1} dx\Big) t^{n-1} dt.$$

for monotone decreasing functions $u : (0,\infty) \to (0,\infty)$, $u(\infty) = 0$ and the weight $\nu(t) = \int_t^\infty \omega(s) s^{n-1} ds$.

1. Introduction

This paper relates to the difference analogue of the weighted Hardy inequality. It is estimated a weighted Lebesgue norm of monotone decreasing function through a weighted norm of its first difference. This topic is actual in view of the interpolation theory, boundedness of maximal function in Lorentz space, and compactness problem for the non smooth domains (see, [19, 12, 14, 13, 5])

In this regard, a difference inequality

$$\int_0^\infty \frac{|u(t)|^p}{x^\delta} dx \le C \int_0^\infty \int_o^\infty \frac{|u(x) - u(t)|^p}{|x - t|^{\delta+1}}$$

holds with a constant $C = \frac{1}{2}\left(1 + \frac{p}{|\delta - 1|}\right)^p$, where u is a locally integrable function on $[0,\infty)$ such that $\lim\limits_{x\to+\infty} \frac{1}{x}\int_0^x u(t)dt = 0$ for $0 < \delta < 1$ and $\lim\limits_{x\to+0} \frac{1}{x}\int_0^x u(t)dt = 0$ for $1 < \delta < p$ (see [10, 7, 21]). In [8], this was extended to the so called "slowly varying weight" case:

$$\int_0^\infty \Big|\frac{u(t)}{t^{\theta_2} b_2(t)}\Big|^p dt \le C \int_0^\infty \int_0^\infty \Big|\frac{u(t) - u(s)}{|t-s|^{\theta_1 + \frac{1}{p}} b_1(|t-s|)}\Big|^p dx dt, \tag{1.1}$$

where C does not depend on $u, u \in L^{1,loc}(0,\infty)$ satisfies $\lim\limits_{t\to+0} \frac{1}{t}\int_0^t u(s)ds = 0$ $\left(\lim\limits_{x\to+\infty} \frac{1}{x}\int_0^x u(t)dt = 0\right)$ and $p > 1$, $\theta_1 \in (0,1) \setminus \{\frac{1}{p}\}$, $\theta_2 \in \mathbb{R}$, $b_1(t)$, $b_2(t)$ are slowly varying functions; it is asserted that inequality (1.1) holds if and only if $\theta_2 = \theta_1 \underset{(<)}{>} \frac{1}{p}$ and $b_1(t) \le C_1 b_2(t)$. For a function u satisfying

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$\lim_{x\to\infty} \frac{1}{x}\int_0^x u(t)dt = 0$, the inequality

$$\int_0^\infty w_0(x)|u(x)|^p dx \le C\int_0^\infty \Big(\int_0^t \frac{|u(x)-u(t)|^p}{|t-x|^\beta} W(x)dx\Big)dt, \qquad (1.2)$$

where the integration is taken not all over the axe t, was proved under the sufficiency condition

$$\sup_{t>0}\Big(\int_0^x w_0(x)dx\Big)\Big(\int_x^\infty w_1^{1-p'}(t)dt\Big)^{p-1} < \infty \qquad (1.3)$$

where $\beta \ge 0$, $p > 1$, and the function $W(x) = x^{\beta-1}w_0(x) + x^{\beta-1-p}w_1(x)$. In [5, 6], the case of related weights

$$\nu(t) = \int_t^\infty \omega(s)ds < \infty, \;\; t > 0 \qquad (1.4)$$

was considered, where a sufficiency condition

$$\nu(h) \le C_1\nu(2h), \quad C_1 \in (1,2), \quad h > 0, \qquad (1.5)$$

was proved; for the results on base of difference inequalities exposed above, we refer [10, 9, 13].

This paper concerns to the (1.2) type difference inequality in the class of monotone decreasing functions. Using a new approach, we prove separate necessity and sufficiency theorems of Arino Muckenhoupt's type for the difference inequality to hold. The method of proof of sufficiency is close to [15].

Note that, the weighted Hardy inequality for monotone decreasing functions $u(x)$ was considered by Arino-Muckenhoupt [1], D.W.Boyd [3], E.T.Sawer [17], and V.D. Stepanov [18]. In [16], such type results was proved for the Riemann-Liouville operator in variable Lebesgue space. Bosa and Soria [4] have proved an interesting result that, the modular form variable exponent boundedness of the Hardy operator restricted to the no increasing functions take place only if the exponent is a constant.

We say, a positive function $y(t)$ is almost increasing (decreasing) on $0,\infty$ if there exists a $C > 1$ such that for any $0 < t_1 < t_2 < \infty$ it holds $y(t_1) \le Cy(t_2)\Big(Cy(t_1) \ge y(t_2)\Big)$. In the paper, by $C_1, C_2, C_3, ...$ we denote different constants which value is not essential for the purposes of the paper.

2. Main results

Theorem 2.1. *Let* $n \ge 1$, ω *be nonnegative measurable function on* $(0,\infty)$ *and*

$$\nu(t) = \int_t^\infty s^{n-1}\omega(s)ds \in L^1(0,a), \quad a > 0 \qquad (2.1)$$

be such that $x^n\nu(x)$ *is almost increasing. Let* $u = u(t)$ *be a decreasing function such that* $\lim_{t\to\infty} u(t) = 0$.

Then for the inequality

$$\int_0^\infty u(t)^p\nu(t)t^{n-1}dt \le C_1\int_0^\infty \Big(\int_0^t \big(u(x)-u(t)\big)^p\omega(t-x)x^{n-1}dx\Big)t^{n-1}dt. \quad (2.2)$$

to hold it suffices that

$$\int_0^x \nu(t)t^{n-1}dt \le C_2 x^n \nu(x), \quad x>0. \tag{2.3}$$

Theorem 2.2. *Let* $n \ge 1$, $p > 2n-1$, ω *be a nonnegative measurable function on* $(0,\infty)$ *and*

$$\nu(t) = \int_t^\infty s^{n-1}\omega(s)ds \in L^1(0,a), \quad a>0 \tag{2.4}$$

be such that the function $x^{2n-1-p}\int_0^x s^{p-n}\nu(s)ds$ *is almost increasing.*
Then for the inequality (2.2) *to hold it is necessary that*

$$\int_0^x \nu(t)t^{p-n}dt \le C_2 x^{p-n+1}\nu(x), \quad x>0. \tag{2.5}$$

Proof of Theorem 2.1. Let $\alpha>0$ denote $\Omega_\alpha = \{x>0: \; u(x)>\alpha\}$. The set of points Ω_α is an interval $(0, x(\alpha))$, since the function u is decreasing. It is clear that $x(2\alpha) \le x(\alpha)$. Denote

$$\nu(\Omega_\alpha) = \int_{\Omega_\alpha} \nu(x)x^{n-1}dx = \int_0^{x(\alpha)} \nu(t)t^{n-1}dt$$

Insert a number $0<\delta<1$ that will be specified later.
There are two possibilities :

$$\text{a)} x(2\alpha) < \delta x(\alpha); \qquad \text{b)}\, x(2\alpha) \ge \delta x(\alpha).$$

Due to [2] (Lemma 2, for increasing functions) and [11] (Theorem 2.5 for almost increasing functions), it follows from (2.5) that there exist an $0<\varepsilon<1$ such that the function $\frac{t^n\nu(t)}{t^\varepsilon}$ is almost increasing. From this we infer: there are $C>1$, $\varepsilon \in (0,1)$ such that it holds the inequality

$$(\delta t)^{n-\varepsilon}\nu(\delta t) \le C t^{n-\varepsilon}\nu(t), \quad t>0, \quad 0<\delta<1.$$

Therefore,

$$\nu(\delta t) \le C\delta^{\varepsilon-n}\nu(t).$$

Integrating this inequality we get,

$$\int_0^{\delta h} \nu(s)s^{n-1}ds \le C\delta^\varepsilon \int_0^h \nu(s)s^{n-1}ds, \qquad h>0. \tag{2.6}$$

Choosing δ, $C\delta^\varepsilon = \frac{1}{2^{p+1}}$, inserting $h = x(\alpha)$, and applying that

$$x(2\alpha) \le \delta x(\alpha)$$

from (2.6) it follows that

$$\begin{aligned} \nu(\Omega_{2\alpha}) &= \int_0^{x(2\alpha)} \nu(s)s^{n-1}ds \le \int_0^{\delta x(\alpha)} \nu(s)s^{n-1}ds \\ &\le C\delta^\varepsilon \int_0^{x(\alpha)} \nu(s)s^{n-1}ds \le \frac{1}{2^{p+1}}\nu(\Omega_\alpha). \end{aligned} \tag{2.7}$$

In the case b) apply the change of variables $t-x=\beta$ passing from variable t to β

$$\int_0^{x(2\alpha)} x^{n-1}dx\left(\int_{x(\alpha)}^{\infty}\omega(t-x)t^{n-1}dt\right)$$
$$=\int_0^{x(2\alpha)} x^{n-1}dx\left(\int_{x(\alpha)-x}^{\infty}\omega(\beta)\big(x+\beta\big)^{n-1}d\beta\right).$$

Applying Fubini's theorem and $(x+\beta)^{n-1}\geq\beta^{n-1}$, we get

$$=\int_{x(\alpha)-x(2\alpha)}^{x(\alpha)}\omega(\beta)\Big(\int_{x(\alpha)-\beta}^{x(2\alpha)}x^{n-1}(x+\beta)^{n-1}dx\Big)d\beta$$
$$+\int_{x(\alpha)}^{\infty}\omega(\beta)\Big(\int_0^{x(2\alpha)}x^{n-1}(x+\beta)^{n-1}dx\Big)d\beta$$
$$\geq\int_{x(\alpha)}^{\infty}\omega(\beta)\Big(\int_0^{x(2\alpha)}x^{n-1}(x+\beta)^{n-1}dx\Big)d\beta$$
$$\geq\frac{(x(2\alpha))^n}{n}\int_{x(\alpha)}^{\infty}\omega(\beta)\beta^{n-1}d\beta$$
$$=\frac{(x(2\alpha))^n}{n}\nu\big(x(\alpha)\big)$$
$$\geq\frac{\delta^n(x(\alpha))^n}{n}\nu\big(x(\alpha)\big)$$

Applying (2.5) and using $\nu(\Omega_{2\alpha})\leq\nu(\Omega_\alpha)$ we obtain that

$$\frac{\delta^n}{nC}\int_0^{x(\alpha)}\nu(t)t^{n-1}dt=\frac{\delta^n\nu(\Omega_\alpha)}{nC}>\frac{\delta^n\nu(\Omega_{2\alpha})}{nC}.$$

Hence

$$\nu(\Omega_{2\alpha})\leq\frac{nC}{\delta^n}\int_0^{x(2\alpha)}x^{n-1}dx\left(\int_{x(\alpha)}^{\infty}\omega(t-x)t^{n-1}dt\right).\tag{2.8}$$

On other hand,

$$\int_0^{x(2\alpha)}x^{n-1}dx\left(\int_{x(\alpha)}^{\infty}\omega(t-x)t^{n-1}dt\right)\leq\iint\limits_{\{x<t:\,u(x)-u(t)>\alpha\}}\omega(t-x)x^{n-1}t^{n-1}dxdt.$$

Thus

$$\nu(\Omega_{2\alpha})\leq\frac{nC}{\delta^n}\iint\limits_{\{x<t,\,u(x)-u(t)>\alpha\}}\omega(t-x)x^{n-1}t^{n-1}dxdt\tag{2.9}$$

Combining (2.7) and (2.9) (the cases a) and b) respectively) we get

$$\nu(\Omega_{2\alpha})=\frac{1}{2^{p+1}}\nu(\Omega_\alpha)+\frac{nC}{\delta^n}\iint\limits_{\{x<t,\,u(x)-u(t)>\alpha\}}\omega(t-x)x^{n-1}t^{n-1}dxdt.\tag{2.10}$$

Integrating (2.10), we get

$$\begin{aligned}\int_0^\infty \nu(\Omega_{2\alpha})d\alpha^p &\le \frac{1}{2^{p+1}}\int_0^\infty \nu(\Omega_\alpha)d\alpha^p\\ &+\frac{nC}{\delta^n}\int_0^\infty\Big(\iint\limits_{\{x<t,\,u(x)-u(t)>\alpha\}}\omega(t-x)x^{n-1}t^{n-1}dxdt\Big)d\alpha^p.\end{aligned}\tag{2.11}$$

By Fubini's theorem the second summand equals

$$\frac{nC}{\delta^n}\int_0^\infty\Big(\int_0^t\big(u(x)-u(t)\big)^p\omega(t-x)x^{n-1}dx\Big)t^{n-1}dt$$

Using this and the expressions

$$\int_0^\infty \nu(\Omega_\alpha)d\alpha^p=\int_0^\infty u(t)^p t^{n-1}dt$$

and

$$\int_0^\infty \nu(\Omega_{2\alpha})d\alpha^p=\frac{1}{2^p}\int_0^\infty u(t)^p t^{n-1}dt$$

it follows from (2.11) that

$$\int_0^\infty u(t)^p\nu(t)t^{n-1}dt\le\frac{2^{p+1}nC}{\delta^n}\int_0^\infty t^{n-1}\Big(\int_0^t\big(u(x)-u(t)\big)^p\omega(t-x)x^{n-1}dx\Big)dt.$$

This proves Theorem 2.1.

Proof of Theorem 2.2. Set the function

$$u_0(x)=\begin{cases}1, & \text{if } 0<x\le\frac{h}{2};\\ 2-\frac{2x}{h}, & \text{if } \frac{h}{2}<x<h;\\ 0, & \text{if } x\ge h,\end{cases}$$

into the inequality (2.2), where $h>0$. Notice the inequality

$$|u_0(x)-u_0(y)|\le\min\Big\{\frac{2}{h}|x-y|,1\Big\}\tag{2.12}$$

for any $x>0$, $y>0$, and $u_0(x)-u_0(y)=0$ if $x>h,\ y>h$. Using (2.2) and (2.12) it easily seen that

$$\begin{aligned}&\int_0^{\frac{h}{2}}\nu(t)t^{n-1}dt\le C\int_0^\infty\Big(\int_0^t\omega(t-x)\Big(u_0(x)-u_0(t)\Big)^p x^{n-1}dx\Big)t^{n-1}dt\\ &=C\int_0^\infty\Big(\int_x^\infty\omega(t-x)\Big(u_0(x)-u_0(t)\Big)^p t^{n-1}dt\Big)x^{n-1}dx\\ &=C\int_0^\infty\Big(\int_0^\infty\omega(s)\Big(u_0(x)-u_0(x+s)\Big)^p(x+s)^{n-1}ds\Big)x^{n-1}dx\\ &=C\int_0^h\Big(\int_0^\infty\omega(s)\Big(u_0(x)-u_0(x+s)\Big)^p(x+s)^{n-1}ds\Big)x^{n-1}dx\\ &+C\int_h^\infty\Big(\int_0^\infty\omega(s)\Big(u_0(x)-u_0(x+s)\Big)^p(x+s)^{n-1}ds\Big)x^{n-1}dx\end{aligned}$$

Since $u_0(x) = u_0(x+s) = 0$ for $x > h$ the second summand is zero. Therefore,

$$
\begin{aligned}
& C\int_0^h \Big(\int_0^\infty \omega(s)\Big(u_0(x)-u_0(x+s)\Big)^p (x+s)^{n-1}ds\Big)x^{n-1}dx \\
& = C\int_0^\infty \Big(\int_0^h \Big(u_0(x)-u_0(x+s)\Big)^p (x+s)^{n-1}x^{n-1}dx\Big)\omega(s)ds \\
& = C\int_0^{\frac{h}{2}} \Big(\int_0^h \Big(u_0(x)-u_0(x+s)\Big)^p (x+s)^{n-1}x^{n-1}dx\Big)\omega(s)ds \\
& + C\int_{\frac{h}{2}}^\infty \Big(\int_0^h \Big(u_0(x)-u_0(x+s)\Big)^p (x+s)^{n-1}x^{n-1}dx\Big)\omega(s)ds \\
& \le C\int_0^{\frac{h}{2}} \Big(\int_0^h \Big(\frac{2s}{h}\Big)^p (x+s)^{n-1}x^{n-1}dx\Big)\omega(s)ds \\
& + C\int_{\frac{h}{2}}^\infty \Big(\int_0^h (x+s)^{n-1}x^{n-1}dx\Big)\omega(s)ds \\
& \le C_1\Big[h^{2n-1-p}\int_0^{\frac{h}{2}}\omega(s)s^p ds + h^{2n-1}\int_{\frac{h}{2}}^\infty \omega(s)ds\Big],
\end{aligned}
$$

where $C_1 > 0$ does not depend on h.
Inserting h in place $\frac{h}{2}$ we get

$$
\int_0^h \nu(t)t^{n-1}dt \le C_2\Big[h^{2n-1-p}\int_0^h \omega(s)s^p ds + h^{2n-1}\int_h^\infty \omega(s)ds\Big], \quad h > 0 \tag{2.13}
$$

Inserting $\omega(t) = -t^{1-n}\nu'(t)$ and integrating by parts in this inequality we get:

$$
\begin{aligned}
& \int_0^h \nu(t)t^{n-1}dt \le -Ch^{2n-1-p}\int_0^h t^p\nu'(t)dt - Ch^{2n-1}\int_h^\infty t^{1-n}\nu'(t)dt, \\
& \quad (\text{since} \quad t^{1-n} \le h^{1-n} \quad \text{in the second integral}) \\
& \le -Ch^{2n-1-p}\int_0^h t^{p+1-n}\nu'(t) + Ch^n\nu(h) \\
& = -Ch^{2n-1-p}t^{p+1-n}\nu(t)\Big|_0^h + (p+1-n)Ch^{2n-1-p}\int_0^h t^{p-n}\nu(t)dt + Ch^n\nu(h) \\
& = (p+1-n)C\,h^{2n-1-p}\int_0^h t^{p-n}\nu(t)dt
\end{aligned}
\tag{2.14}
$$

Set $G(h) = \int_0^h t^{p-n}\nu(t)dt$. Integrating by parts, it follows from (2.14) that

$$\int_0^h G'(t)t^{2n-1-p}dt \le (p+1-n)Ch^{2n-1-p}G(h),$$

$$t^{2n-1-p}G(t)\Big|_0^h - (2n-1-p)\int_0^h t^{2n-2-p}G(t)dt \le C(p+1-n)\,h^{2n-1-p}G(h)$$

or

$$\int_0^h t^{2n-2-p}G(t)dt \le \frac{C(p+1-n)-1}{p-2n+1}\,h^{2n-1-p}G(h). \tag{2.15}$$

It follows from (2.14) and Bari-Stechkin type theorem [2] that the function $\frac{G(t)}{t^{p-2n+1+\varepsilon}}$ is almost increasing: there are $\varepsilon > 0$ and $C_2 > 0$ depending on C, p such that:

$$\frac{G(\delta t)}{(\delta t)^{p-2n+1+\varepsilon}} \le C_2\frac{G(t)}{t^{p-2n+1+\varepsilon}}, \quad 0<\delta<1, \quad t>0.$$

Therefore,

$$G(\delta t) \le C_2\delta^{p-2n+1+\varepsilon}G(t) \quad t>0. \tag{2.16}$$

Choose δ such that $C_2\delta^{p-2n+1+\varepsilon} \le \frac{1}{2}$. Then it follows from (2.16) that

$$\int_0^{\delta h} t^{p-n}\nu(t)dt \le \frac{1}{2}\int_0^h t^{p-n}\nu(t)dt$$

$$= \frac{1}{2}\int_0^{\delta h} \nu(t)t^{p-n}dt + \frac{1}{2}\int_{\delta h}^h \nu(t)t^{p-n}dt$$

or

$$\int_0^{\delta h} t^{p-n}\nu(t)dt \le \int_{\delta h}^h t^{p-n}\nu(t)dt.$$

Since $\nu(t)$ is decreasing, one get from here

$$\int_0^{\delta h} t^{p-n}\nu(t)dt \le \frac{(1-\delta^{p-n+1})}{(p-n+1)\delta^{p-n+1}}(\delta h)^{p-n+1}\nu(\delta h)$$

or

$$\int_0^x t^{p-n}\nu(t)dt \le C_5x^{p-n+1}\nu(x), \quad x>0, \tag{2.17}$$

where $C_5 = \frac{(1-\delta^{p-n+1})}{(p-n+1)\delta^{p-n+1}}$.

Theorem 2.2 has been proved.

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