Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 42, Number 2, 2016, Pages 273–278

ON ALMOST COMPLEX STRUCTURES IN COVARIANT TENSOR BUNDLES

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Abstract. The main purpose of the present paper is to discuss relations between holomorphic covariant tensor fields and lifts on pure cross-sections in tensor bundles of types $(0,q)$. We prove that the complete lift of almost complex structure, when restricted to the pure cross-section determined by an almost holomorphic covariant $(0,q)$-tensor fields, is an almost complex structure on covariant tensor bundle.

1. Introduction

We suppose that (M_{2r},φ) is an almost complex manifold. Let $\mathbb{C}$ be a complex algebra and $\overset{*}{\omega}=(\overset{*}{\omega}_{v_1\ldots v_q})$, $v_1\ldots v_q=1,\ldots,r$ be a complex tensor field of type $(0,q)$ on holomorphic (analytic) complex manifold $\mathfrak{X}_r(\mathbb{C})$. Then the real model of $\overset{*}{\omega}$ is a tensor field $\omega=(\omega_{j_1\ldots j_q})$, $j_1\ldots j_q=1,\ldots,2r$ on M_{2r} such that

$$\omega(\varphi X_1,X_2,\ldots,X_q)=\omega(X_1,\varphi X_2,\ldots,X_q)=\ldots=\omega(X_1,X_2,\ldots,\varphi X_q)$$

for any $X_1,X_2,\ldots,X_q\in\Im_0^1(M_{2r})$. Such tensor fields are said to be *pure* with respect to φ. They were studied by many authors ([2, 3, 5, 6]). The covector field (1-form) is considered to be pure, by convention.

We denote by $\overset{*}{\Im}{}_q^0(M_{2r})$ the module of all pure tensor fields ω of type $(0,q)$ on M_{2r} with respect to the almost complex structure φ. We now fix a positive integer λ, where $1\le\lambda\le q$. If ω is any pure tensor fields of type $(0,q)$, then the tensor product of ω and φ with contraction $(\omega\overset{C}{\otimes}\varphi)(Y_1,Y_2,\ldots,Y_q)=\omega(Y_1,\ldots,\varphi Y_\lambda,\ldots,Y_q)=(\varphi_{j_\lambda}^{m_\lambda}\omega_{j_1\ldots m_\lambda\ldots j_q})$ is also pure tensor field. We shall prove only the case when $\varphi\in\overset{*}{\Im}{}_1^1(M_{2r})$ and $\omega\in\overset{*}{\Im}{}_2^0(M_{2r})$. In fact, we have

$$(\omega\overset{C}{\otimes}\varphi)(\varphi X,Y)=\omega(\varphi(\varphi X),Y))=\omega(\varphi X,\varphi Y))=(\omega\overset{C}{\otimes}\varphi)(X,\varphi Y)$$

for any $X,Y\in\Im_0^1(M_{2r})$. The product $\omega\overset{C}{\otimes}\varphi$ is also denoted by $\omega\circ\varphi$ and called the pure product.

Let now $\omega\in\overset{*}{\Im}{}_q^0(M_{2r})$. The Φ_φ-operator associated with φ and applied to ω is defined by [5], [6]

2010 *Mathematics Subject Classification.* 53C15, 55R10.

Key words and phrases. Almost complex structure, tensor bundle, pure cross-section, Nijenhuis tensor, holomorphic tensor field.

$$(\Phi_\varphi\omega)(X,Y_1,\ldots,Y_q)=(\varphi X)(\omega(Y_1,Y_2,\ldots,Y_q))-X(\omega(\varphi Y_1,Y_2,\ldots,Y_q))$$
$$+\sum_{\lambda=1}^{q}\omega(Y_1,Y_2,\ldots,\varphi(L_XY_\lambda),\ldots,Y_q)\ , \tag{1.1}$$

where $\Phi_\varphi\omega\in\Im_{q+1}^0(M_{2r})$ and L_X is the Lie derivation with respect to X.
Let on M_{2r} be given the integrable almost complex structure φ. For complex tensor field $\overset{*}{\omega}$ of type $(0,q)$ on $\mathfrak{X}_r(\mathbb{C})$ to be $\mathbb{C}-$holomorphic tensor field it is necessary and sufficient that $\Phi_\varphi\omega=0$ (see [4], p.57). Let now M_{2r} be a manifold with non-integrable almost complex structure φ. In this case, when $\Phi_\varphi\omega=0$, ω is said to be almost holomorphic.

2. Lifts on a cross-sections

Let M_n be a differentiable manifold of class C^∞ and finite dimension n. Then the set $T_q^0(M_n)=\bigcup\limits_{P\in M_n}T_q^0(P)$ is, by definition, the tensor bundle of type $(0,q)$ over M_n, where $\bigcup$ denotes the disjoint union of the tensor spaces $T_q^0(P)$ for all $P\in M_n$. For any point $\tilde{P}$ of $T_q^0(M_n)$ such that $\tilde{P}\in T_q^0(M_n)$, the surjective correspondence $\tilde{P}\to P$ determines the natural projection $\pi:T_q^0(M_n)\to M_n$. The projection π defines the natural differentiable manifold structure of $T_q^0(M_n)$, that is, $T_q^0(M_n)$ is a C^∞-manifold of dimension $n+n^q$. If x^j are local coordinates in a neighborhood U of $P\in M_n$, then a tensor t at P which is an element of $T_q^0(M_n)$ is expressible in the form $(x^j,t_{j_1\ldots j_q})$, where $t_{j_1\ldots j_q}$ are components of t with respect to natural base. We may consider $(x^j,t_{j_1\ldots j_q})=(x^j,x^{\bar{j}})=(x^J)$, $j=1,\ldots,n$, $\bar{j}=n+1,\ldots,n+n^q$, $J=1,\ldots,n+n^q$ as local coordinates in a neighborhood $\pi^{-1}(U)\subset T_q^0(M_n)$.
We denote by $\Im_q^p(M_n)$ the module of all tensor fields of type (p,q) on M_n. If $\alpha\in\Im_0^q(M_n)$, then it is regarded in a natural way (by contraction) as a function in $T_q^0(M_n)$, which we denote by $\imath\alpha$. If α has local expression

$$\alpha=\alpha^{j_1\ldots j_q}\partial_{j_1}\otimes\ldots\otimes\partial_{j_q}$$

in a coordinate neighborhood $U(x^j)\subset M_n$, then $\imath\alpha=\alpha(t)$ has the local expression

$$\imath\alpha=\alpha^{j_1\ldots j_q}t_{j_1\ldots j_q}$$

with respect to the coordinates $(x^j,x^{\bar{j}})$ in $\pi^{-1}(U)$.
Suppose that $A\in\Im_q^0(M_n)$. Then there is a unique vector field ${}^VA\in\Im_0^1(T_q^0(M_n))$ (vertical lift of A) such that for $\alpha\in\Im_0^q(M_n)$ [1]

$${}^VA(\imath\alpha)=\alpha(A)\circ\pi={}^V(\alpha(A)),$$

where ${}^V(\alpha(A))$ is the vertical lift of the function $\alpha(A)\in F(M_n)$.
If ${}^VA={}^VA^j\partial_j+{}^VA^{\bar{j}}\partial_{\bar{j}}$, then the vertical lift ${}^VA$ of A to $T_q^0(M_n)$ has components

$${}^VA=\begin{pmatrix}{}^VA^j\\ {}^VA^{\bar{j}}\end{pmatrix}=\begin{pmatrix}0\\ A_{j_1\ldots j_q}\end{pmatrix} \tag{2.1}$$

with respect to the coordinates $(x^j,x^{\bar{j}})$ in $T_q^0(M_n)$ [2].

Let L_V be the Lie derivation with respect to $V \in \Im_0^1(M_n)$. We define the complete lift ${}^cV = \bar{L}_V$ of V to $T_q^0(M_n)$ [1] by

$$ {}^cV(\imath\alpha) = \imath(L_V\alpha) $$

for $\alpha \in \Im_0^q(M_n)$. The vector field ${}^cV$ has components

$$ {}^cV = \begin{pmatrix} {}^cV^j \\ {}^cV^{\bar{j}} \end{pmatrix} = \begin{pmatrix} V^j \\ -\sum_{\mu=1}^{q} t_{j_1\ldots m\ldots j_q}\partial_{j_\mu}V^m \end{pmatrix} \tag{2.2} $$

with respect to the coordinates $(x^j, x^{\bar{j}})$ in $T_q^0(M_n)$ [2].

Suppose that there is given a tensor field $\omega \in \Im_q^0(M_n)$. Then the correspondence $x \to \omega_x$, ω_x being the value of ω at $x \in M_n$, determines a mapping $\sigma_\omega : M_n \to T_q^0(M_n)$, such that $\pi \circ \sigma_\omega = id_{M_n}$, and the n-dimensional submanifold $\sigma_\omega(M_n)$ of $T_q^0(M_n)$ is called the cross-section determined by ω. If the tensor field ω has the local component $\omega_{k_1\ldots k_q}(x^k)$, the cross-section $\sigma_\omega(M_n)$ is locally expressed by

$$ \begin{cases} x^k = x^k \\ x^{\bar{k}} = \omega_{k_1\ldots k_q}(x^k) \end{cases} \tag{2.3} $$

with respect to the coordinates $(x^k, x^{\bar{k}})$ in $T_q^0(M_n)$. Differentiating (2.3) by x^j, we see that n tangent vector fields $B_j (j = 1, \ldots, n)$ to $\sigma_\omega(M_n)$ have components

$$ (B_j^K) = \left(\frac{\partial x^K}{\partial x^j}\right) = \begin{pmatrix} \delta_j^k \\ \partial_j \omega_{k_1\ldots k_q} \end{pmatrix} \tag{2.4} $$

with respect to the natural frame $\{\partial_k, \partial_{\bar{k}}\}$ in $T_q^0(M_n)$.

On the other hand, the fibre $T_q^0(x) = \pi^{-1}(x)$ is locally expressed by

$$ \begin{cases} x^k \quad = const, \\ t_{k_1\ldots k_q} = t_{k_1\ldots k_q}, \end{cases} $$

$t_{k_1\ldots k_q}$ being consider as parameters. On differentiating with respect to $x^{\bar{j}} = t_{j_1\ldots j_q}$, we see that n^q tangent vector fields $C_{\bar{j}} (\bar{j} = 1, \ldots, n^q)$ to the fibre $T_q^0(x)$ have components

$$ (C_{\bar{j}}^K) = \left(\frac{\partial x^K}{\partial x^{\bar{j}}}\right) = \begin{pmatrix} 0 \\ \delta_{k_1}^{j_1} \ldots \delta_{k_q}^{j_q} \end{pmatrix} \tag{2.5} $$

with respect to the natural frame $\{\partial_k, \partial_{\bar{k}}\}$ in $T_q^0(M_n)$, where δ_i^j is the Kronecker symbol.

A vector field X along a cross-section $\sigma_\omega : M_n \to T_q^0(M_n)$ is mapping $X : M_n \to T\left(T_q^0(M_n)\right)$ $(T(T_q^p(M_n))$-tangent bundle over the manifold $T_q^0(M_n))$ such that $\tilde{\pi} \circ x = \sigma_\omega$, where $\tilde{\pi}$ is the projection $\tilde{\pi} : T\left(T_q^0(M_n)\right) \to T_q^0(M_n)$. Thus X assigns to each point $x \in M_n$ a tangent vector to $T_q^0(M_n)$ at $\sigma_\omega(x)$ and therefore $n + n^q$ local vector fields B_j and $C_{\bar{j}}$ in $\pi^{-1}(U) \subset T_q^0(M_n)$ are vector fields along $\sigma_\omega(M_n)$. They form a local family of frames $\{B_j, C_{\bar{j}}\}$ along $\sigma_\omega(M_n)$, which is called the adapted (B, C)- frame of $\sigma_\omega(M_n)$ in $\pi^{-1}(U)$. From ${}^cV = {}^cV^h\partial_h + {}^cV^{\bar{h}}\partial_{\bar{h}}$ and ${}^cV = {}^cV^jB_j + {}^cV^{\bar{j}}C_{\bar{j}}$, we easily obtain ${}^cV^k = {}^cV^jB_j^k + {}^cV^{\bar{j}}C_{\bar{j}}^k$, ${}^cV^{\bar{k}} = {}^cV^jB_j^{\bar{k}} + {}^cV^{\bar{j}}C_{\bar{j}}^{\bar{k}}$. Now, taking account of (2.2) on the cross-section $\sigma_\omega(M_n)$, and also (2.4) and

(2.5), we have ${}^{c}\tilde{V}^{k} = V^{k}$, ${}^{c}\tilde{V}^{\bar{k}} = -L_{V}\omega_{k_1\ldots k_q}$. Thus, the complete lift ${}^{c}V$ has along $\sigma_\omega(M_n)$ components of the form

$$ {}^{c}V = \begin{pmatrix} {}^{c}\tilde{V}^{k} \\ {}^{c}\tilde{V}^{\bar{k}} \end{pmatrix} = \begin{pmatrix} V^{k} \\ -L_{V}\omega_{k_1\ldots k_q} \end{pmatrix} $$

with respect to the adapted (B, C)- frame. From (2.1), (2.4) and (2.5), by using similar way the vertical lift ${}^{V}A$ also has components

$$ {}^{V}A = \begin{pmatrix} {}^{V}\tilde{A}^{k} \\ {}^{V}\tilde{A}^{\bar{k}} \end{pmatrix} = \begin{pmatrix} 0 \\ A_{k_1\ldots k_q} \end{pmatrix} $$

with respect to the adapted (B, C)- frame.

3. Lifts on a holomorphic pure cross-sections

Let $S \in \Im_2^1(M_{2r})$. Making use of the Jacobi matrix of the coordinate transformation in $T_q^0(M_{2r})$:

$$ \begin{cases} x^{j'} = x^{j'}(x^{j}), \\ x^{\bar{j}'} = t_{j'_1\ldots j'_q} = A_{j'_1}^{j_1} \ldots A_{j'_q}^{j_q} t_{j_1\ldots j_q} = A_{(j')}^{(j)} x^{\bar{j}}, \end{cases} $$

where

$$ A_{(j')}^{(j)} = A_{j'_1}^{j_1} \ldots A_{j'_q}^{j_q}, \quad A_{j'_1}^{j_1} = \frac{\partial x^{j_1}}{\partial x^{j'}}, $$

we can define a (1,1)-tensor field $\gamma S \in \Im_1^1(T_q^0(M_{2r}))$:

$$ \gamma S = ((\gamma S)_J^I) = \begin{pmatrix} (\gamma S)_j^i & (\gamma S)_{\bar{j}}^i \\ (\gamma S)_j^{\bar{i}} & (\gamma S)_{\bar{j}}^{\bar{i}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ t_{m i_2\ldots i_s} S_{j i_1}^{m} & 0 \end{pmatrix}, $$

where $S_{j i_1}^{m}$ are local components of S in M_{2r}. Clearly, we have $\gamma S({}^{V}A) = 0$ for any $A \in \Im_q^0(M_{2r})$. We can easily verify that the lift γS has along cross-section $\sigma_\omega(M_{2r})$ components

$$ \gamma S = ((\tilde{\gamma} S)_J^I) = \begin{pmatrix} 0 & 0 \\ \omega_{m i_2\ldots i_s} S_{j i_1}^{m} & 0 \end{pmatrix} \tag{3.1} $$

with respect to the adapted (B, C)-frame, where $\omega_{m i_2\ldots i_s}$ are local components of ω in M_{2r}.

Theorem 3.1. *Let $\omega \in \Im_q^0(M_{2r})$ be a pure tensor field with respect to φ and $\varphi^2 = -id_{M_{2r}}$. Then $\omega \circ \varphi \in Ker\,\Phi_\varphi$ if and only if $\omega \in Ker\,\Phi_\varphi$, where Φ_φ is defined by ( ) and $(\omega \circ \varphi)\ (Y_1, Y_2, \ldots, Y_q) = \omega(\varphi Y_1, Y_2, \ldots, Y_q)$.*

Proof. Taking account of (1.1) and the purity of ω, we have

$$ (\Phi_\varphi \omega)(X, Y_1, \ldots, Y_q) = \left(L_{\varphi X}\omega - L_X(\omega \circ \varphi)\right)(Y_1, \ldots, Y_q). \tag{3.2} $$

If we substitute $\omega \circ \varphi$ into ω and φX into X, then the equation (3.2) may be written as

$$ \begin{aligned} (\Phi_\varphi(\omega \circ \varphi))(\varphi X, Y_1, \ldots, Y_q) &= \left(L_{\varphi^2 X}(\omega \circ \varphi) - L_{\varphi X}\left(\omega \circ \varphi^2\right)\right)(Y_1, \ldots, Y_q) \\ &= -\left(L_X(\omega \circ \varphi) + L_{\varphi X}\omega\right)(Y_1, \ldots, Y_q) \\ &= -\left(\Phi_\varphi \omega\right)(X, Y_1, \ldots, Y_q) \end{aligned} $$

or

$$((\Phi_\varphi(\omega\circ\varphi))\circ\varphi)(X,Y_1,\dots,Y_q)=-(\Phi_\varphi\omega)(X,Y_1,\dots,Y_q),$$

from which by virtue of $\det\varphi\neq 0$, we see that $\Phi_\varphi(\omega\circ\varphi)=0$ if and only if $\Phi_\varphi\omega=0$. □

We also have

Theorem 3.2. *Let $\omega\in\Im_q^0(M_{2r})$ be a pure tensor field with respect to φ, and $\omega\in Ker\,\Phi_\varphi$. If $\varphi^2=-id_{M_{2r}}$, then $\omega\circ N_\varphi=0$, where*

$$(\omega\circ N_\varphi)(X,Y_1,Y_2,\dots,Y_s)=\omega(N_\varphi(X,Y_1),Y_2,\dots,Y_s),$$

N_φ is the Nijenhuis tensor of φ.

Proof. Since $(\Phi_\varphi\varphi)(X,Y)=-(L_{\varphi Y}\varphi)X+\varphi((L_Y\varphi)X)=N_\varphi(X,Y)$ (see [3]), the statement of theorem follows immediately from Theorem 3.1 and the following formula:

$$\begin{aligned}&(\Phi_\varphi(\omega\circ\varphi))(X,Y_1,Y_2,\dots Y_s)=(\Phi_\varphi\omega)(\varphi X,Y_1,Y_2,\dots Y_s)\\&+\omega((\Phi_\varphi\varphi)(X,Y_1),Y_2,\dots Y_s)=((\Phi_\varphi\omega)\circ\varphi)(X,Y_1,Y_2,\dots Y_s)\\&+\omega(N_\varphi(X,Y_1),Y_2,\dots Y_s).\end{aligned}$$

Let now $T_q^0(M_{2r})=\bigcup_{P\in M}T_q^0(P)$ be a tensor bundle of type $(0,q)$ with local coordinates $(x^i,x^{\bar i}=t_{i_1i_2\dots i_q})$, $i=1,\dots,2r$; $\bar i=2r+1,\dots,2r+(2r)^q$. It is well known that [2], the complete lift ${}^C\varphi$ to $T_q^0(M_{2r})$ with components

$$\begin{cases}{}^c\tilde\varphi_l^k=\varphi_l^k,\ {}^c\tilde\varphi_{\bar l}^k=0,\ {}^c\tilde\varphi_l^{\bar k}=-(\Phi_\varphi\omega)_{lk_1\dots k_q},\\ {}^c\tilde\varphi_{\bar l}^{\bar k}=\varphi_{k_1}^{l_1}\delta_{k_2}^{l_2}\dots\delta_{k_q}^{l_q}\ \ (x^{\bar k}=t_{k_1\dots k_q},\ x^{\bar l}=t_{l_1\dots l_q})\end{cases}$$

with respect to the adapted (B,C)- frame of $\sigma_\omega(M_{2r})$ satisfies the following equations

$$\begin{cases}({}^C\varphi)^2({}^CX)={}^C(\varphi^2)({}^CX)+\gamma N_\varphi({}^CX),\\ ({}^C\varphi)^2({}^VA)={}^C(\varphi^2)({}^VA),\quad {}^VA\in\Im_0^1(T_q^0(M_{2r}))\end{cases}\tag{3.3}$$

for any $X\in\Im_0^1(M_{2r})$ and $A\in\Im_q^0(M_{2r})$, where $\gamma N_\varphi\in\Im_1^1(T_q^0(M))$ has components (see (3.1))

$$\gamma N_\varphi=\begin{pmatrix}(\tilde\gamma N_\varphi)_j^i & (\tilde\gamma N_\varphi)_{\bar j}^i\\ (\tilde\gamma N_\varphi)_j^{\bar i} & (\tilde\gamma N_\varphi)_{\bar j}^{\bar i}\end{pmatrix}=\begin{pmatrix}0 & 0\\ (\omega\circ N)_{ji_1i_2\dots i_s} & 0\end{pmatrix}$$

with respect to the adapted $(B,C)-$frame. When φ is an almost complex structure on M_{2r}, a pure tensor field ω of type $(0,s)$ satisfying $\omega\in Ker\,\Phi_\varphi$ is said to be almost holomorphic (see the end of Introduction). □

Therefore from (3.3), Theorem 3.1 and Theorem 3.2 we obtain a following theorem:

Theorem 3.3. *Let M_{2r} be a C^∞- manifold with an almost complex structure φ. Then the complete lift ${}^C\varphi\in\Im_1^1\left(T_q^0(M_{2r})\right)$, when restricted to the pure cross-section determined by an almost holomorphic tensor field ω on M_{2r}, is an almost complex structure.*

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Received: October 10, 2016; Accepted: October 30, 2016