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ON LAURENT COEFFICIENTS OF CAUCHY TYPE INTEGRALS OF FINITE COMPLEX MEASURES

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Abstract. Boundary values of Cauchy type integrals of finite complex measures given on an annulus, generally speaking, are not Lebesgue integrable, and therefore at expansion of Cauchy type integrals in Laurent series, the expansion coefficients cannot be expressed by boundary values using the Lebesgue integral. In this paper, using the notions of A-integration and N-integration, we get a formula for calculating the Laurent expansion coefficients of Cauchy type integrals of finite complex measures.

1. Introduction

Let $T_1 = \{z \in C : \ |\, z - a\, | = R\}$, $T_2 = \{z \in C : \ |\, z - a\, | = r\}$ are the concentric circles with the centers at the point $a \in C$, $0 < r < R$, and a finite complex measure ν is given on the set $T = T_1 \bigcup T_2$. The function

$$F\left(z\right) = \frac{1}{2\pi i}\int_T \frac{d\nu\left(\tau\right)}{\tau - z} = \frac{1}{2\pi i}\int_{T_1} \frac{d\nu_1\left(\tau\right)}{\tau - z} + \frac{1}{2\pi i}\int_{T_2} \frac{d\nu_2\left(\tau\right)}{\tau - z},$$

$z \in G = \{z \in C : \ r < |z - a| < R\}$, are called Cauchy type integrals of the measure ν on the annulus G, where ν_1 and ν_2 are the restrictions of the measure ν, respectively, on the sets T_1 and T_2. It is known that (see [13]) the function $F\left(z\right)$ is analytical on the domain G and expanded in Laurent series:

$$F\left(z\right) = \sum_{k=-\infty}^{\infty} a_k \left(z - a\right)^k, z \in G, \tag{1.1}$$

and the expansion coefficients a_k, $k \in Z$ are determined by the equalities

$$a_k = \frac{1}{2\pi i}\int_{T_1} \left(\tau - a\right)^{-k-1} d\nu_1\left(\tau\right), \ k \in Z_+,$$

$$a_k = -\frac{1}{2\pi i}\int_{T_2} \left(\tau - a\right)^{-k-1} d\nu_2\left(\tau\right), \ k \in Z \backslash Z_+. \tag{1.2}$$

Smirnoff (see [16]) proved that the analytic functions $F\left(z\right)$ have finite non-tangential boundary values $F\left(\tau\right)$ for almost all points $\tau \in T$.

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It follows from a theorem of Zygmund (see [12, ch.5, C, §3°]) that, if a measure ν is absolutely continuous: $d\nu(\tau) = f(\tau)\, d\tau$ and $f \in L\log L(T)$, then the equality

$$F(z) = \frac{1}{2\pi i}\int_{T_1} \frac{F(\tau)}{\tau - z} d\tau - \frac{1}{2\pi i}\int_{T_2} \frac{F(\tau)}{\tau - z} d\tau, z \in G \tag{1.3}$$

is holds. Therefore for the Laurent coefficients a_k, $k \in Z$ the following equalities are valid:

$$a_k = \frac{1}{2\pi i}\int_{T_1} (\tau - a)^{-k-1} F(\tau)\, d\tau = \frac{1}{2\pi i}\int_{T_2} (\tau - a)^{-k-1} F(\tau)\, d\tau, k \in Z. \tag{1.4}$$

If $f \in L(T)$ and $f \notin L\log L(T)$, it can happen that the boundary values $F(\tau)$ is not Lebesgue integrable on T, and therefore the equalities (1.3), (1.4) are not satisfied in this case. It follows from P.L.Ul'yanov's work [20] that the boundary values $F(\tau)$ is A-integrability on T in the case $f \in L(T)$. In the case when the measure ν is not absolutely continuous, the boundary values $F(\tau)$ do not satisfy the condition $\lambda\, m\{\tau \in T: \ |F^+(\tau)| > \lambda\} = o(1)$ as $\lambda \to +\infty$, and therefore do not integrable in the sense of A-integration. In [4], the author introduced the notion of N-integration and proved that the boundary values $F(\tau)$ of Cauchy type integrals of finite complex measures is N-integrability on T.

In the present paper, using the notion of A-integration and N-integration, we obtain the analogue of formula (1.4) for finite complex measures.

2. On Laurent coefficients of Cauchy type integrals of absolutely continuous measures

For a measurable complex function f on an interval $[a,b] \subset R$ we set

$$[f(x)]_n = [f(x)]^n = f(x) \ \text{ for } \ |f(x)| \le n$$

$$[f(x)]_n = n \cdot sgn f(x)\,, \ [f(x)]^n = 0 \text{ for } \ |f(x)| > n, \ n \in N,$$

where $sgn z = \frac{z}{|z|}$ for $z \neq 0$ and $sgn 0 = 0$.

In 1929, Titchmarsh [18] introduced the notion of Q- and Q'-integrals.

Definition 2.1. Let f be a measurable complex function, defined on an interval $[a,b]$. If the finite limit $\lim\limits_{n\to\infty} \int_a^b [f(x)]_n\, dx$ ($\lim\limits_{n\to\infty} \int_a^b [f(x)]^n\, dx$, respectively) exist, then f is said to be Q-integrable (Q'-integrable, respectively) on $[a,b]$, that is, $f \in Q[a,b]$ ($f \in Q'[a,b]$), and the value of this limit is referred to as the Q-integral (Q'-integral) of this function and is denoted by

$$(Q)\int_a^b f(x)\, dx \left((Q')\int_a^b f(x)\, dx\right).$$

In the same paper Titchmarsh established that, when studying the properties of trigonometric series conjugate to Fourier series of Lebesgue integrable functions, Q-integration leads to a series of natural results. A very uncomfortable fact impeding the application of Q-integrals and Q'-integrals studying diverse problems of function theory is the absence of the additivity property, that is, the Q-integrability (Q'-integrability) of two functions does not imply the Q-integrability

(Q'-integrability) of their sums. If one adds the condition

$$\lambda\, m\left\{ x\in[a,b]:\ |f(x)|>\lambda \right\} = o(1)\,,\lambda\to+\infty, \tag{2.1}$$

where m stands for the Lebesgue measure, to the definition of Q-integrability (Q'-integrability) of a function f on the interval $[a,b]$, then the Q-integral and Q'-integral coincide ($Q[a,b]=Q'[a,\,b]$), and these integrals become additive (see [18]).

Definition 2.2. Suppose that the function f satisfy the condition (2.1) and is Q'-integrable (or Q-integrable). That the function f is said to be A-integrable on $[a,b]$, $f\in A\,[a,b]$, and denoted by

$$(A)\int_a^b f(x)\,dx = \left(Q'\right)\int_a^b f(x)\,dx = (Q)\int_a^b f(x)\,dx.$$

Let the function f be given on the circle $T_1=\{z\in C:\ |\,z-a\,|=R\}$. If the function $f^*(\theta) = e^{i\theta}f\left(a+R\cdot e^{i\theta}\right)$, $\theta\in[0,\,2\pi]$ is A-integrable on the interval $[0,\,2\pi]$, then the function f is said to be A-integrable on T_1 and is denoted by

$$(A)\int_{T_1} f(\tau)\,d\tau = (A)\int_0^{2\pi} ie^{i\theta}f\left(a+R\cdot e^{i\theta}\right)d\theta.$$

The properties of A- and Q-integrals were investigated in [4-6, 9-11, 18], and for the applications of A- and Q-integrals in the theory of functions of real and complex variables we refer the reader to [1-4, 7, 8, 14, 15, 17-21].

We need the following theorems proved in [7], [14] and [20].

Theorem 2.1. [7]. *Let $T_0=\{z\in C:\ |\,z\,|=1\}$, $f\in L(T_0)$ and $F^+(z) = \frac{1}{2\pi i}\int_{T_0}\frac{f(\tau)d\tau}{\tau-z}$, $z\in D^+=\{z\in C:\ |z|<1\}$ be the Cauchy type integral of the function f. Then for the Taylor coefficients a'_k, $k\in Z_+$, of $F^+(z)$ by expanding Taylor series $F^+(z)=\sum_{k=0}^{\infty}a'_kz^k$ holds the equality*

$$a'_k = \frac{1}{2\pi i}(A)\int_{T_0}\tau^{-k-1}F^+(\tau)\,d\tau,\ k\in Z_+,$$

where $F^+(\tau)$ are the non-tangential boundary values of $F^+(z)$ as $z\to\tau\in T_0$.

Theorem 2.2. [20]. *Let $f\in L(T_0)$ and $F^+(z)$, $z\in D^+$ be the Cauchy type integral of the function f. Then*

$$(A)\int_{T_0}\tau^kF^+(\tau)\,d\tau = 0,\ k\in Z_+,$$

where $F^+(\tau)$ are the non-tangential boundary values of $F^+(z)$ as $z\to\tau\in T_0$.

Theorem 2.3. [7]. *Let $f\in L(T_0)$ and $F^-(z) = \frac{1}{2\pi i}\int_{T_0}\frac{f(\tau)d\tau}{\tau-z}$, $z\in D^-=\{z\in C:\ |z|>1\}$ be the Cauchy type integral of the function f. Then for the Taylor coefficients b'_k, $k\in N$, of $F^-(z)$ by expanding Taylor series $F^-(z)=\sum_{k=1}^{\infty}\frac{b'_k}{z^k}$ holds the equality*

$$b'_k = \frac{1}{2\pi i}(A)\int_{T_0}\tau^{k-1}F^-(\tau)\,d\tau,\ k\in N,$$

where $F^-(\tau)$ are the non-tangential boundary values of $F^-(z)$ as $z\to\tau\in T_0$.

Theorem 2.4. [14]. *Let $f \in L(T_0)$ and $F^-(z)$, $z \in D^-$ be the Cauchy type integral of the function f. Then*

$$(A)\int_{T_0} \tau^{-k} F^-(\tau)\, d\tau = 0,\ k \in N,$$

where $F^-(\tau)$ are the non-tangential boundary values of $F^-(z)$ as $z \to \tau \in T_0$.

Theorem 2.5. *Let $f \in L(T)$ and $F(z) = \frac{1}{2\pi i}\int_T \frac{f(\tau)d\tau}{\tau - z} = \frac{1}{2\pi i}\int_{T_1} \frac{f(\tau)d\tau}{\tau - z} + \frac{1}{2\pi i}\int_{T_2} \frac{f(\tau)d\tau}{\tau - z}$, $z \in G$ be the Cauchy type integral of the function f. Then for the Laurent coefficients a_k, $k \in Z$, of $F(z)$ by expanding Laurent series* (1.1) *holds the equalities*

$$a_k = \frac{1}{2\pi i}(A)\int_{T_1} (\tau - a)^{-k-1} F(\tau)\, d\tau$$

$$= \frac{1}{2\pi i}(A)\int_{T_2} (\tau - a)^{-k-1} F(\tau) d\tau, k \in Z, \tag{2.2}$$

where $F(\tau)$ are the non-tangential boundary values of $F(z)$ as $z \to \tau \in T$.

Proof. We denote by

$$F_1(z) = \frac{1}{2\pi i}\int_{T_1} \frac{f(\tau)\, d\tau}{\tau - z}, \{z \in C :\ |z - a| < R\},$$

$$F_2(z) = \frac{1}{2\pi i}\int_{T_2} \frac{f(\tau)\, d\tau}{\tau - z}, \{z \in C :\ |z - a| > r\}.$$

It follows from the theorems 2.1, 2.2, 2.3 and 2.4 that

$$a_k = \frac{1}{2\pi i}(A)\int_{T_1} (\tau - a)^{-k-1} F_1(\tau)\, d\tau,\ k \in Z_+, \tag{2.3}$$

$$(A)\int_{T_1} (\tau - a)^{-k-1} F_1(\tau)\, d\tau = 0,\ k \in Z\backslash Z_+, \tag{2.4}$$

$$a_k = \frac{1}{2\pi i}(A)\int_{T_2} (\tau - a)^{-k-1} F_2(\tau)\, d\tau,\ k \in Z\backslash Z_+. \tag{2.5}$$

$$(A)\int_{T_2} (\tau - a)^{-k-1} F_2(\tau)\, d\tau = 0,\ k \in Z_+. \tag{2.6}$$

Since the function $F_1(z)$ is an analytical in the bounded domain $\{z \in C :\ |z - a| < R\}$ and the function $F_2(z)$ is an analytical in the unbounded domain $\{z \in C :\ |z - a| > r\}$, then we have

$$\int_{T_1} (\tau - a)^{-k-1} F_2(\tau)\, d\tau = 0,\ k \in Z_+, \tag{2.7}$$

$$a_k = \frac{1}{2\pi i}\int_{T_1} (\tau - a)^{-k-1} F_2(\tau)\, d\tau,\ k \in Z\backslash Z_+, \tag{2.8}$$

$$\int_{T_2} (\tau - a)^{-k-1} F_1(\tau)\, d\tau = 0,\ k \in Z\backslash Z_+. \tag{2.9}$$

$$a_k = \frac{1}{2\pi i}\int_{T_2} (\tau - a)^{-k-1} F_1(\tau)\, d\tau,\ k \in Z_+. \tag{2.10}$$

Therefore, it follows from the equations (2.3)-(2.10) and from the additivity of A-integral (see [18]) that

$$a_k = \frac{1}{2\pi i}(A)\int_{T_1}(\tau - a)^{-k-1}F_1(\tau)\,d\tau$$

$$= \frac{1}{2\pi i}(A)\int_{T_1}(\tau - a)^{-k-1}F_1(\tau)\,d\tau + \frac{1}{2\pi i}\int_{T_1}(\tau - a)^{-k-1}F_2(\tau)\,d\tau$$

$$= \frac{1}{2\pi i}(A)\int_{T_1}(\tau - a)^{-k-1}\left[F_1(\tau) + F_2(\tau)\right]d\tau$$

$$= \frac{1}{2\pi i}(A)\int_{T_1}(\tau - a)^{-k-1}F(\tau)\,d\tau,\ k \in Z_+,$$

$$a_k = \frac{1}{2\pi i}\int_{T_1}(\tau - a)^{-k-1}F_2(\tau)\,d\tau$$

$$= \frac{1}{2\pi i}\int_{T_1}(\tau - a)^{-k-1}F_2(\tau)\,d\tau + \frac{1}{2\pi i}(A)\int_{T_1}(\tau - a)^{-k-1}F_1(\tau)\,d\tau$$

$$= \frac{1}{2\pi i}(A)\int_{T_1}(\tau - a)^{-k-1}\left[F_2(\tau) + F_1(\tau)\right]d\tau$$

$$= \frac{1}{2\pi i}(A)\int_{T_1}(\tau - a)^{-k-1}F(\tau)\,d\tau,\ k \in Z\backslash Z_+,$$

$$a_k = \frac{1}{2\pi i}(A)\int_{T_2}(\tau - a)^{-k-1}F_2(\tau)\,d\tau$$

$$= \frac{1}{2\pi i}(A)\int_{T_2}(\tau - a)^{-k-1}F_2(\tau)\,d\tau + \frac{1}{2\pi i}\int_{T_2}(\tau - a)^{-k-1}F_1(\tau)\,d\tau$$

$$= \frac{1}{2\pi i}(A)\int_{T_2}(\tau - a)^{-k-1}\left[F_2(\tau) + F_1(\tau)\right]d\tau$$

$$= \frac{1}{2\pi i}(A)\int_{T_2}(\tau - a)^{-k-1}F(\tau)\,d\tau,\ k \in Z\backslash Z_+.$$

$$a_k = \frac{1}{2\pi i}\int_{T_2}(\tau - a)^{-k-1}F_1(\tau)\,d\tau$$

$$= \frac{1}{2\pi i}\int_{T_2}(\tau - a)^{-k-1}F_1(\tau)\,d\tau + \frac{1}{2\pi i}(A)\int_{T_2}(\tau - a)^{-k-1}F_2(\tau)\,d\tau$$

$$= \frac{1}{2\pi i}(A)\int_{T_2}(\tau - a)^{-k-1}\left[F_1(\tau) + F_2(\tau)\right]d\tau$$

$$= \frac{1}{2\pi i}(A)\int_{T_2}(\tau - a)^{-k-1}F(\tau)\,d\tau,\ k \in Z_+.$$

Theorem 2.5 is proved.

3. Laurent coefficients of Cauchy type integrals of finite complex measures

For a measurable real function f on the interval $[a\,,\,b]$ we write

$$(f > \lambda) = \{x \in [a\,,\,b]\,:\, f\,(x) > \lambda\}\,, (f < \lambda) = \{x \in [a\,,\,b]\,:\, f\,(x) < \lambda\}\,,$$
$$(f \geq \lambda) = \{x \in [a\,,\,b]\,:\, f\,(x) \geq \lambda\}\,, (f \leq \lambda) = \{x \in [a\,,\,b]\,:\, f\,(x) \leq \lambda\}\,.$$

Definition 3.1. We denote by $M\,([a\,,\,b]\,;\,C)$ the class of measurable complex-valued functions f on the interval $[a\,,\,b]$ for which a finite limit $\lim\limits_{\lambda\to+\infty} \lambda\, m\,(|f| > \lambda)$ exist.

In [4] the author showed that the Q-integral and the Q'-integral coincide on the function class $M\,([a\,,\,b]\,;\,C)$, that is, if $f \in M\,([a\,,\,b]\,;\,C)$, then for the existence of the integral $(Q)\int_a^b f\,(x)\,dx$ it is necessary and sufficient that the integral $(Q')\int_a^b f\,(x)\,dx$ exist, and in that case these integrals are equal.

Let be a measurable real 2π - periodic function such that for every interval $[\alpha\,,\,\beta] \subset R$ there exists a finite limit

$$\lim_{\lambda\to+\infty} \lambda\, m\,\{x \in [\alpha\,,\,\beta]\,:\, |f\,(x)| > \lambda\}\,.$$

We write (see [2])

$$P\,(f;q;x;t) = \frac{\pi}{2} ctg \frac{\pi q^{n-1}}{2} f\,(t)$$

for $t \in \left(x - \pi q^{n-1}, x - \pi q^{n}\right) \bigcup \left(x + \pi q^{n}, x + \pi q^{n-1}\right)\,,\ n \in Z_{+},\ 0 < q < 1,$

$$P_1\,(f;x) = \lim_{q\to1-}\ \lim_{\eta\to+\infty}\ \eta\, m\,\{\,t \in (x, x+\pi)\,:\, |P\,(f;q;x;t)| > \eta\}\,, \tag{3.1}$$

$$P_2\,(f;x) = \lim_{q\to1-}\ \lim_{\eta\to+\infty}\ \eta\, m\,\{\,t \in (x-\pi, x)\,:\, |P\,(f;q;x;t)| > \eta\}\,, \tag{3.2}$$

$$r_{\lambda,f}\,(x) = \begin{cases} sgn\,(P_2\,(f;x) - P_1\,(f;x)) & for\ f\,(x) > \lambda, \\ 0 & for\ |f\,(x)| \leq \lambda, \\ sgn\,(P_1\,(f;x) - P_2\,(f;x)) & for\ f\,(x) < -\lambda, \end{cases}$$

under the assumption that the finite limits on the right-hand sides of the equations (3.1) and (3.2) exist for almost all $x \in [0,\,2\pi)$. For every measurable complex function $f = Ref + iImf$ defined on the interval $[0,\,2\pi)$ we write $r_{\lambda,f}\,(x) = r_{\lambda,Ref}\,(x) + i r_{\lambda,Imf}\,(x)$ under the assumption that $r_{\lambda,Ref}\,(x)$ and $r_{\lambda,Imf}\,(x)$ exist for almost all $x \in [0,\,2\pi)$.

Definition 3.2. Let $SM\,([0, 2\pi]\,;\,C)$ denote the class of complex measurable functions f on the interval $[0,\,2\pi)$ for which $r_{\lambda,f}\,(x)$ exists for almost all $x \in [0,\,2\pi)$, the limit

$$\lim_{\varepsilon\to0+}\ \lim_{\lambda\to+\infty} \lambda \int_{\alpha-\varepsilon}^{\beta-\varepsilon} r_{\lambda,f}\,(t)\,dt$$

exist for every interval $[\alpha\,,\,\beta) \subset [0,\,2\pi)$, and the function $f - \tilde{\nu}$ satisfies condition (2.5), where $\tilde{\nu}$ stands for the function conjugate to the measure ν defined by the rule

$$\nu\,([\alpha\,,\,\beta)) = \frac{\pi}{2} \lim_{\varepsilon\to0+}\ \lim_{\lambda\to+\infty} \lambda \int_{\alpha-\varepsilon}^{\beta-\varepsilon} r_{\lambda,f}\,(t)\,dt$$

for every interval $[\alpha\,,\,\beta) \subset [0,\,2\pi)$.

We note that for every function $f=\tilde{\nu}+g$, where ν is a finite Borel measure on the interval $[0,2\pi)$ and the measurable function g satisfies condition (2.1), the value $r_{\lambda,f}(x)$ exists for almost all $x\in[0,\,2\pi)$, and, for every interval $[\alpha\,,\,\beta)\subset[0,\,2\pi)$, one has the equation (see [2])

$$\nu_s([\alpha\,,\,\beta))=\frac{\pi}{2}\lim_{\varepsilon\to0+}\lim_{\lambda\to+\infty}\lambda\int_{\alpha-\varepsilon}^{\beta-\varepsilon}r_{\lambda,f}(t)\,dt,\tag{3.3}$$

where ν_s stands for the singular part of ν.

Equality (3.3) shows that the class of functions $SM([0,2\pi]\,;\,C)$ coincides with the class of functions admitting a representation of the form $f=\tilde{\nu}_s+g$, where ν_s is a finite singular complex Borel measure on the interval $[0,2\pi)$ and the measurable complex function g on $[0,2\pi)$ satisfies condition (2.1).

In [4] the author shows that the Q-integral and Q'-integral has the additivity property in $SM([0,2\pi]\,;\,C)$, that is, if the functions $f_1,\,f_2\in SM([0,2\pi]\,;\,C)$ are Q-integrable (Q'-integrable) on the interval $[0\,,\,2\pi]$, then their sum f_1+f_2 also belongs to $SM([0,2\pi]\,;\,C)$, the sum is Q-integrable (Q'-integrable) on this interval and Q-integral (Q'-integral) from the sum equals the sum of Q-integrals (Q'-integrals).

Definition 3.3. If a function f belongs to the class $SM([0,2\pi]\,;\,C)$ and is Q'-integrable on the interval $[0,\,2\pi]$, then is said to be N-integrable on $[0,\,2\pi]$. The sum

$$(Q')\int_0^{2\pi}f(x)\,dx+\frac{\pi}{2i}\lim_{\lambda\to+\infty}\lambda\int_0^{2\pi}r_{\lambda,f}(x)\,dx$$

is referred to as the N^+-integral, and the difference

$$(Q')\int_0^{2\pi}f(x)\,dx-\frac{\pi}{2i}\lim_{\lambda\to+\infty}\lambda\int_0^{2\pi}r_{\lambda,f}(x)\,dx$$

as the N^--integral of on $[0,\,2\pi]$; these integrals are denoted by $(N^+)\int_0^{2\pi}f(x)\,dx$ and $(N^-)\int_0^{2\pi}f(x)\,dx$, respectively.

The equation (3.3) shows that, if the singular measure ν_s in the decomposition $f=\tilde{\nu}_s+g\in SM([0,2\pi]\,;\,C)$ is known, then N^+ - and N^--integrals are evaluated by the formulae

$$\begin{aligned}(N^+)\int_0^{2\pi}f(x)\,dx&=(Q')\int_0^{2\pi}f(x)\,dx-i\int_0^{2\pi}d\nu_s(x)\\&=(A)\int_0^{2\pi}g(x)\,dx-i\int_0^{2\pi}d\nu_s(x),\\(N^-)\int_0^{2\pi}f(x)\,dx&=(Q')\int_0^{2\pi}f(x)\,dx+i\int_0^{2\pi}d\nu_s(x)\\&=(A)\int_0^{2\pi}g(x)\,dx+i\int_0^{2\pi}d\nu_s(x).\end{aligned}$$

For a complex measurable function f on the interval $[a,\,b]$ we denote by f^* a 2π-periodic function defined for $x\in[0,\,2\pi)$ by the equation

$$f^*(x)=\frac{b-a}{2\pi}f\left(a+\frac{b-a}{2\pi}x\right).$$

Definition 3.4. If a function f^* is N-integrable on the interval $[0, 2\pi]$, then the function f is said to be N-integrable on the interval $[a, b]$, and the N^+- and N^--integrals of f on $[a, b]$ are defined by the formulae

$$\left(N^+\right)\int_a^b f(x)dx = \left(N^+\right)\int_0^{2\pi} f^*(x)dx, \left(N^-\right)\int_a^b f(x)dx = \left(N^-\right)\int_0^{2\pi} f^*(x)dx.$$

Example 3.1. If the Lebesgue integrable function φ on $[a, b]$ is Holder continuous at a point $x_0 \in (a, b)$, that is, there are numbers $\delta > 0$, $C > 0$, $\alpha \in (0, 1\,]$ such that $|\varphi(x) - \varphi(x_0)| \leq C\,|x - x_0|^\alpha$ for every $x \in (x_0 - \delta; x_0 + \delta)$, then

$$\left(N^+\right)\int_a^b \frac{\varphi(x)}{x - x_0}dx = v.p.\int_a^b \frac{\varphi(x)}{x - x_0}dx - \pi i\varphi(x_0) = \lim_{\varepsilon\to 0+}\int_a^b \frac{\varphi(x)}{x - x_0 + i\varepsilon}dx,$$

$$\left(N^-\right)\int_a^b \frac{\varphi(x)}{x - x_0}dx = v.p.\int_a^b \frac{\varphi(x)}{x - x_0}dx + \pi i\varphi(x_0) = \lim_{\varepsilon\to 0+}\int_a^b \frac{\varphi(x)}{x - x_0 - i\varepsilon}dx.$$

Note that (see [4]) N^+- and N^--integrals have the additivity property and for these integrals the change of variables formulas hold.

Let a measurable function f be given on a simple closed Lyapunov contour Γ and let $\xi = \xi(s)$, $s \in [0, l]$ be the parametric equation of Γ, where s is the arc length of the part of the contour from the point $\xi_0 = \xi(0)$ to the point $\xi_s = \xi(s)$. We write $f^*(t) = f(\xi(t)) \cdot \xi'(t)$, $t \in [0, l]$.

Definition 3.5. If a function f^* is N-integrable on the interval $[0, l]$, then the function f is said to be N-integrable on the contour Γ, and N^+- and N^--integrals of f on Γ are defined by the formulas

$$\left(N^+\right)\int_\Gamma f(\tau)\,d\tau = \left(N^+\right)\int_0^l f^*(t)\,dt,$$

$$\left(N^-\right)\int_\Gamma f(\tau)\,d\tau = \left(N^-\right)\int_0^l f^*(t)\,dt.$$

From the change of variables formulas for N^+- and N^--integrals it follows that if a contour Γ is given by a parametric equation $\xi = \xi(t)$, $t \in [a, b]$, where $\xi = \xi(t)$ is differentiable on $[a, b]$, the derivative $\xi'(t)$ is Holder continuous and satisfies the conditions: $|\xi'(t)| \geq D_0 > 0$, $t \in [a\,, b]$ and $\xi'(a) = \xi'(b)$, then the N-integrability of the function f on Γ is equivalent to N-integrability of the function $f(\xi(t))\cdot\xi'(t)$ on $[a, b]$, and the N^+ - and N^--integrals of these functions are equal.

We need the following theorems proved by the author in [4] and [7].

Theorem 3.1. [7]. *Let ν_0 be a finite complex Borel measure on the circle T_0 and $F^+(z) = \frac{1}{2\pi i}\int_{T_0}\frac{d\nu_0(\tau)}{\tau - z}$, $z \in D^+$ is Cauchy type integral of the measure ν_0. Then for the Taylor coefficients a_k, $k \in Z_+$, of $F^+(z)$ by expanding Taylor series $F^+(z) = \sum_{k=0}^\infty a'_k z^k$ holds the equality*

$$a'_k = \frac{1}{2\pi i}\left(N^+\right)\int_{T_0}\tau^{-k-1}F^+(\tau)\,d\tau,\ k \in Z_+,$$

where $F^+(\tau)$ are the non-tangential boundary values of $F^+(z)$ as $z \to \tau \in T_0$.

Theorem 3.2. [4]. *Let ν_0 be a finite complex Borel measure on the circle T_0 and $F^+(z)$, $z \in D^+$ be the Cauchy type integral of the measure ν_0. Then*

$$\left(N^+\right) \int_{T_0} \tau^k F^+(\tau)\, d\tau = 0,\ k \in Z_+,$$

where $F^+(\tau)$ are the non-tangential boundary values of $F^+(z)$ as $z \to \tau \in T_0$.

Theorem 3.3. [7]. *Let ν_0 be a finite complex Borel measure on the circle T_0 and $F^-(z) = \frac{1}{2\pi i}\int_{T_0} \frac{d\nu_0(\tau)}{\tau - z}$, $z \in D^-$ is Cauchy type integral of the measure ν. Then for the Taylor coefficients b_k, $k \in N$, of $F^-(z)$ by expanding Taylor series $F^-(z) = \sum_{k=1}^{\infty} \frac{b'_k}{z^k}$ holds the equality*

$$b'_k = \frac{1}{2\pi i}\left(N^-\right) \int_{T_0} \tau^{-k-1} F^-(\tau)\, d\tau,\ k \in N,$$

where $F^-(\tau)$ are the non-tangential boundary values of $F^-(z)$ as $z \to \tau \in T_0$.

Theorem 3.4. [4]. *Let ν be a finite complex Borel measure on the circle T_0 and $F^-(z)$, $z \in D^-$ is Cauchy type integral of the measure ν. Then for the Taylor coefficients b_k, $k \in N$, of $F^-(z)$ by expanding Taylor series* (1.1) *holds the equality*

$$\left(N^-\right) \int_{T_0} \tau^{-k} F^-(\tau)\, d\tau = 0,\ k \in N,$$

where $F^-(\tau)$ are the non-tangential boundary values of $F^-(z)$ as $z \to \tau \in T_0$.

Theorem 3.5. *Let ν be a finite complex Borel measure on the set $T = T_1 \bigcup T_2$ and $F(z) = \frac{1}{2\pi i}\int_T \frac{d\nu(\tau)}{\tau - z} = \frac{1}{2\pi i}\int_{T_1} \frac{d\nu_1(\tau)}{\tau - z} + \frac{1}{2\pi i}\int_{T_2} \frac{d\nu_2(\tau)}{\tau - z}$, $z \in G$ is Cauchy type integrals of the measure ν on the annulus G. Then for the Laurent coefficients a_k, $k \in Z$, of $F(z)$ by expanding Laurent series* (1.1) *holds the equalities*

$$\begin{aligned} a_k &= \frac{1}{2\pi i}\left(N^+\right) \int_{T_1} (\tau - a)^{-k-1} F(\tau)\, d\tau \\ &= \frac{1}{2\pi i}\left(N^-\right) \int_{T_2} (\tau - a)^{-k-1} F(\tau)\, d\tau, k \in Z, \end{aligned} \tag{3.4}$$

where $F(\tau)$ are the non-tangential boundary values of $F(z)$ as $z \to \tau \in T$.

Proof. We denote by

$$F_1(z) = \frac{1}{2\pi i}\int_{T_1} \frac{d\nu_1(\tau)}{\tau - z}, \{z \in C : |z - a| < R\},$$

$$F_2(z) = \frac{1}{2\pi i}\int_{T_2} \frac{d\nu_2(\tau)}{\tau - z}, \{z \in C : |z - a| > r\}.$$

It follows from the theorems 3.1, 3.2, 3.3 and 3.4 that

$$a_k = \frac{1}{2\pi i}\left(N^+\right) \int_{T_1} (\tau - a)^{-k-1} F_1(\tau)\, d\tau,\ k \in Z_+, \tag{3.5}$$

$$\left(N^+\right) \int_{T_1} (\tau - a)^{-k-1} F_1(\tau)\, d\tau = 0,\ k \in Z \backslash Z_+, \tag{3.6}$$

$$a_k = \frac{1}{2\pi i}\left(N^-\right) \int_{T_2} (\tau - a)^{-k-1} F_2(\tau)\, d\tau,\ k \in Z \backslash Z_+. \tag{3.7}$$

$$\left(N^{-}\right)\int_{T_2}(\tau-a)^{-k-1}F_2(\tau)\,d\tau=0,\ k\in Z_{+}. \tag{3.8}$$

Therefore, it follows from the equations (2.7)-(2.10), (3.5)-(3.8) and from the additivity of N^{+}-, N^{-}-integrals (see [4]) that

$$a_k=\frac{1}{2\pi i}\left(N^{+}\right)\int_{T_1}(\tau-a)^{-k-1}F_1(\tau)\,d\tau$$

$$=\frac{1}{2\pi i}\left(N^{+}\right)\int_{T_1}(\tau-a)^{-k-1}F_1(\tau)\,d\tau+\frac{1}{2\pi i}\int_{T_1}(\tau-a)^{-k-1}F_2(\tau)\,d\tau$$

$$=\frac{1}{2\pi i}\left(N^{+}\right)\int_{T_1}(\tau-a)^{-k-1}\left[F_1(\tau)+F_2(\tau)\right]d\tau$$

$$=\frac{1}{2\pi i}\left(N^{+}\right)\int_{T_1}(\tau-a)^{-k-1}F(\tau)\,d\tau,\ k\in Z_{+},$$

$$a_k=\frac{1}{2\pi i}\int_{T_1}(\tau-a)^{-k-1}F_2(\tau)\,d\tau$$

$$=\frac{1}{2\pi i}\int_{T_1}(\tau-a)^{-k-1}F_2(\tau)\,d\tau+\frac{1}{2\pi i}\left(N^{+}\right)\int_{T_1}(\tau-a)^{-k-1}F_1(\tau)\,d\tau$$

$$=\frac{1}{2\pi i}\left(N^{+}\right)\int_{T_1}(\tau-a)^{-k-1}\left[F_2(\tau)+F_1(\tau)\right]d\tau$$

$$=\frac{1}{2\pi i}\left(N^{+}\right)\int_{T_1}(\tau-a)^{-k-1}F(\tau)\,d\tau,\ k\in Z\backslash Z_{+},$$

$$a_k=\frac{1}{2\pi i}\left(N^{-}\right)\int_{T_2}(\tau-a)^{-k-1}F_2(\tau)\,d\tau$$

$$=\frac{1}{2\pi i}\left(N^{-}\right)\int_{T_2}(\tau-a)^{-k-1}F_2(\tau)\,d\tau+\frac{1}{2\pi i}\int_{T_2}(\tau-a)^{-k-1}F_1(\tau)\,d\tau$$

$$=\frac{1}{2\pi i}\left(N^{-}\right)\int_{T_2}(\tau-a)^{-k-1}\left[F_2(\tau)+F_1(\tau)\right]d\tau$$

$$=\frac{1}{2\pi i}\left(N^{-}\right)\int_{T_2}(\tau-a)^{-k-1}F(\tau)\,d\tau,\ k\in Z\backslash Z_{+}.$$

$$a_k=\frac{1}{2\pi i}\int_{T_2}(\tau-a)^{-k-1}F_1(\tau)\,d\tau$$

$$=\frac{1}{2\pi i}\int_{T_2}(\tau-a)^{-k-1}F_1(\tau)\,d\tau+\frac{1}{2\pi i}\left(N^{-}\right)\int_{T_2}(\tau-a)^{-k-1}F_2(\tau)\,d\tau$$

$$=\frac{1}{2\pi i}\left(N^{-}\right)\int_{T_2}(\tau-a)^{-k-1}\left[F_1(\tau)+F_2(\tau)\right]d\tau$$

$$=\frac{1}{2\pi i}\left(N^{-}\right)\int_{T_2}(\tau-a)^{-k-1}F(\tau)\,d\tau,\ k\in Z_{+}.$$

Theorem 3.5 is proved.

References

[1] A.B.Aleksandrov, A-integrability of the boundary values of harmonic functions, *Mathematical Notes,* **30:1** (1981), 515–523.

[2] R.A.Aliev, Representability of analytic functions in terms of their boundary values, *Mathematical Notes*, **73:1-2** (2003), 8-20.

[3] R.A.Aliev, Existence of angular boundary values and Cauchy-Green formula, *Journal of Mathematical Physics Analysis Geometry*,**7:1** (2011), 3–18.

[4] R.A.Aliev, $N^{\pm}$–integrals and boundary values of Cauchy-type integrals of finite measures, *Sbornik: Mathematics,* **205:7** (2014), 913-935.

[5] R.A.Aliev, On the properties of Q- and Q'-inteqrals of the function measurable on the real axis, *Proceedings of the Institute of Mathematics and Mechanics, NAS of Azerbaijan*, **41:1** (2015), 56-62.

[6] R.A.Aliev, On properties of Hilbert transform of finite complex measures, *Complex analysis and operator theory*, **10:1** (2016), 171–185.

[7] R.A.Aliev, On Taylor coefficients of Cauchy type integrals of finite complex measures, *Complex variables and elliptic equations*, **60:12** (2015), 1727–1738.

[8] R.A.Aliev, Riesz's equality for the Hilbert transform of the finite complex measures, *Azerbaijan Journal of Mathematics*, **6:1** (2016), 126–135.

[9] M.P. Efimova, On the properties of the Q-integral, *Math. Notes*, **90:3-4** (2011), 322-332.

[10] M.P. Efimova, Sufficient condition for a change of variable in generalized Q-integration, *Izv. Saratov Univ. (N.S.), Ser. Math. Mech. Inform.*, **13:1(2)** (2013), 43-46.

[11] M.P. Efimova, The sufficient condition for integrability of a generalized Q-integral and points of integrability, Moscow University Mathematics Bulletin, **70:4** (2015), 181-184.

[12] P. Koosis, *Introduction to H^p spaces*, 2nd ed., Cambridge Univ. Press, 1998.

[13] W. Rudin, *Real and complex analysis*, Mc Graw-Hill, New York, 1987, 416 pp.

[14] T.S.Salimov, The A-integral and boundary values of analytic functions, *Mathematics of the USSR-Sbornik*, **64:1** (1989), 23–40.

[15] T.S.Salimov, On E.Titchmarsh's theorem on the conjugate function, *Proc. A. Razmadze Math. Inst*,**102** (1993), 99–114. (Russian)

[16] V.Smirnoff, Sur les valeurs limites des fonctions regulieres a l'interieur d'un cercle, *J. Soc. Phys.-Math. Leningrade*, **2:2** (1929), 22-37.

[17] V.A.Skvortsov, A-integrable martingale sequences and Walsh series, *Izvestia: Mathematics*, **65:3** (2001), 607–616.

[18] E.C.Titchmarsh, On conjugate functions, *Proc. of the London Math. Soc.*, **9** (1929), 49–80.

[19] P.L.Ul'yanov, Application of A-integration to a class of trigonometric series, *Mathematics of the USSR-Sbornik* , **35(77)** (1954), 469-490. (in Russian)

[20] P.L.Ul'yanov, On the A-Cauchy integral. I, *Russian Mathematical Surveys*, **11:5(71)** (1956), 223–229. (in Russian)

[21] P.L.Ul'yanov, Integrals of Cauchy type, *Twelve Papers on Approximations and Integrals*, Amer. Math. Soc., Trans. **2/44**, 1965, 129–150.

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