Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 43, Number 1, 2017, Pages 3–67

A SURVEY ON MULTINORMED VON NEUMANN ALGEBRAS

ANAR DOSI

Abstract. The present paper is devoted to multinormed von Neumann algebras. We pick up basic facts on von Neumann algebras to introduce their locally convex versions. A key construction in this direction is a central topology of a von Neumann algebra. Every multinormed von Neumann algebra can be realized as a local von Neumann algebra on a certain domain in a Hilbert space. It admits the predual (unique up to an isometry), which is an ℓ_1-normed space. As the main result we describe multinormed L^∞-algebras which are locally convex analogs of abelian von Neumann algebras.

1. Introduction

The operator algebra $\mathcal{B}(H)$ of all bounded linear operators on a Hilbert space H possesses many interesting locally convex topologies apart from the operator norm topology. The most important ones among them are the strong operator topology (SOT) and the weak operator topology (WOT), the latter is weaker than SOT. Since both admit the same supply of continuous functionals, the strong closure of a convex subset in $\mathcal{B}(H)$ is reduced to its weak closure. Von Neumann algebras occur as weakly (or strongly) closed, unital $*$-subalgebras of $\mathcal{B}(H)$. One of the key properties of von Neumann algebras is a double commutant theorem $\mathcal{M} = \mathcal{M}''$ proven by von Neumann (1929), where $\mathcal{M} \subseteq \mathcal{B}(H)$ is a von Neumann algebra. Another key property is to admit Banach preduals, that is, for a von Neumann algebra $\mathcal{M}$ there corresponds a uniquely defined Banach space $\mathcal{M}_*$ such that its normed dual $(\mathcal{M}_*)^*$ is reduced to $\mathcal{M}$ [26, 1.13.3]. In particular, $\mathcal{M}$ possesses w^*-topology which coincides with WOT on every bounded subset of $\mathcal{M}$.

Multinormed (or locally convex) von Neumann algebras occur as inverse limits of von Neumann algebras whose connecting maps are w^*-continuous $*$-homomorphisms. Their operator realizations are reduced to operator algebras of unbounded operators. The theory of $*$-algebras of unbounded operators in a Hilbert space (or briefly, O*-algebras) has a great importance in many problems of the representation theory of Lie groups, quantum field theory and statistical physics. A survey of basic results on O*-algebras can be found in the monograph [29] by Schmüdgen. Recall that an O*-algebra is defined as a unital $*$-algebra

2000 *Mathematics Subject Classification.* Primary 46K10; Secondary 47L25, 47L60.
Key words and phrases. Multinormed von Neumann algebra, unbounded operators, central topology, ℓ_1-normed space.

$\mathcal{A}$ of linear transformations on a common dense subspace $\mathcal{D}$ of a Hilbert space H which leaves invariant $\mathcal{D}$. In this case, every $x \in \mathcal{A}$ being an unbounded operator on H admits an unbounded dual $x^{\bigstar}$ which leaves invariant $\mathcal{D}$, therefore $x^{*} = x^{\bigstar}|\mathcal{D} \in \mathcal{A}$ and the correspondence $x \mapsto x^{*}$ turns out to be a natural involution on $\mathcal{A}$. Thus an O*-algebra $\mathcal{A}$ on a dense subspace $\mathcal{D}$ is the $*$-subalgebra of the enveloping O*-algebra $L^{*}(\mathcal{D}) = \left\{x \in L(\mathcal{D}) : \mathcal{D} \subseteq \operatorname{dom}\left(x^{\bigstar}\right), x^{\bigstar}(\mathcal{D}) \subseteq \mathcal{D}\right\}$ on $\mathcal{D}$, where $L(\mathcal{D})$ is the algebra of all linear transformations on the space $\mathcal{D}$. The algebra $L^{*}(\mathcal{D})$ can be equipped with various polynormed (or locally convex) topologies (see [29, 3.3]) generalizing SOT and WOT from $\mathcal{B}(H)$. For example, the seminorms $p_{\zeta,\eta}(x) = |\langle x\zeta, \eta\rangle|$, $\zeta, \eta \in \mathcal{D}$, and $p_{\zeta}(x) = \|x\zeta\|$, $\zeta \in \mathcal{D}$, define the weak and strong operator topologies in $L^{*}(\mathcal{D})$, respectively. But the uniform operator topology has some diversity in this context (see [29, Ch. 4], [20, Ch. I]), which depends on the considered O*-algebra on $\mathcal{D}$. A $*$-representation of a unital $*$-algebra is defined as a $*$-homomorphism into $L^{*}(\mathcal{D})$ for a certain domain $\mathcal{D}$. The concepts of a generalized vector, standard systems and modular systems in this framework allow to develop the Tomita-Takesaki theory of O*-algebras with its physical applications [20]. Further developments of these ideas, namely considering partial O*-algebras and their representations, may be found in [1].

The O*-algebras appeared in the realization problem of an abstract quantum (or local operator) space as well. The representation theorem [4] for quantum spaces asserts that each quantum space V can be realized (up to a topological matrix isomorphism) as a concrete quantum space on a quantum domain $\mathcal{E}$. This result generalizes the representation theorem for multinormed (or locally) C^*-algebras proved in [19] by Inoue. Confirm that a *quantum domain in a Hilbert space* H is defined as an upward filtered family $\mathcal{E} = \{H_{\alpha} : \alpha \in \Lambda\}$ of its closed subspaces whose *union space* $\mathcal{D} = \cup\mathcal{E}$ is dense in H. Equivalently, a quantum domain is an upward filtered family of projections $\mathcal{E} = \{p_{\alpha} : \alpha \in \Lambda\}$ in $\mathcal{B}(H)$ such that $\vee_{\alpha} p_{\alpha} = 1_{H}$, where $\vee_{\alpha} p_{\alpha}$ is the least upper bound in $\mathcal{B}(H)$ of the projection net. We define [4], [5], [6] the algebra of all noncommutative continuous functions on $\mathcal{D}$ to be the O*-subalgebra $C_{\mathcal{E}}^{*}(\mathcal{D}) = \{x \in L(\mathcal{D}) : p_{\alpha}x \subseteq xp_{\alpha}, xp_{\alpha} \in \mathcal{B}(H), \alpha \in \Lambda\} \subseteq L^{*}(\mathcal{D})$. The $*$-algebra $C_{\mathcal{E}}^{*}(\mathcal{D})$ can be equipped with the canonical polynormed topology defined by the family $\|x\|_{\alpha} = \|xp_{\alpha}\|$, $x \in C_{\mathcal{E}}^{*}(\mathcal{D})$, $\alpha \in \Lambda$ of C^*-seminorms, which results in a unital multinormed C^*-algebra structure on it. This topology represents the operator norm topology in the locally convex space context. The representation theorem asserts that if $\mathcal{A}$ is a unital multinormed C^*-algebra with its upward filtered family $\{\|\cdot\|_{\alpha}\}$ of C^*-seminorms then there is a quantum domain $\mathcal{E} = \{p_{\alpha}\}$ and $*$-homomorphism $\varphi : \mathcal{A} \to C_{\mathcal{E}}^{*}(\mathcal{D})$ such that $\|\varphi(a)\|_{\alpha} = \|a\|_{\alpha}$, $a \in \mathcal{A}$ for all α (see [19], [4]). A similar result for multinormed von Neumann algebras was obtained in [14] by Fragoulopoulou. The representation theorem from [14] asserts that each multinormed von Neumann algebra is topologically $*$-isomorphic to a strongly closed $*$-subalgebra in $C_{\mathcal{E}}^{*}(\mathcal{D})$. This type of realization of a multinormed von Neumann algebra would lead in to the locally convex version of von Neumann algebras to be defined as strongly closed $*$-subalgebras in $C_{\mathcal{E}}^{*}(\mathcal{D})$ (see [21], [22]). But this type motivation needs to be done with more analysis than a direct comparison with respect to the normed case. For instance, we don't know whether the algebra $C_{\mathcal{E}}^{*}(\mathcal{D})$ plays indeed the role of $\mathcal{B}(H)$ in the locally convex space theory. Recent investigations in the duality theory for quantum spaces indicate that is not indeed the case. The bicommutant property fails to be true for strongly closed unital $*$-subalgebras in $C^*_{\mathcal{E}}(\mathcal{D})$. This property in its general setting has been investigated for unbounded operators in [2]. As shown in [9] the algebra $C^*_{\mathcal{E}}(\mathcal{D})$ has a well duality behavior if $\mathcal{E}$ admits a gradation, that is, each $p_\alpha = \sum_{\kappa\in\alpha} e_\kappa$ is a finite sum of an orthogonal family (e_κ) of projections such that $\sum_\kappa e_\kappa = 1$. In this case a quantum space version of the operator space $\mathcal{T}(H)$ of all trace class operators on a Hilbert space H can be constructed using the inductive limit of operator spaces. Namely, $\mathcal{T}_{\mathcal{E}}(\mathcal{D}) = \operatorname{op}\bigoplus_\kappa \mathcal{T}(e_\kappa(H))$ is the quantum direct sum of the operator subspaces $\mathcal{T}(e_\kappa(H)) \subseteq \mathcal{T}(H)$. If $\mathcal{T}_{\mathcal{E}}(\mathcal{D})'_\beta$ is the strong quantum dual (see [12]) of $\mathcal{T}_{\mathcal{E}}(\mathcal{D})$ then $\mathcal{T}_{\mathcal{E}}(\mathcal{D})'_\beta = C^*_{\mathcal{E}}(\mathcal{D})$ up to a (canonical) topological matrix isomorphism. We couldn't expect similar property for strongly closed $*$-subalgebras of $C^*_{\mathcal{E}}(\mathcal{D})$ in the general case. Actually, we don't know a possible link between the multinormed von Neumann algebras, $*$-subalgebras in $C^*_{\mathcal{E}}(\mathcal{D})$ which admit preduals, and so called locally von Neumann algebras.

The aim of the present manuscript is to introduce a reader to the latest problems on multinormed von Neumann algebras providing a necessary background from operator algebras. As a basic material we use G. K. Pederson's book [24] on modern analysis, M. Takesaki's book [30] on operator algebras, and the results from [10], [11]. We clarify many details from [30] to satisfy ordinary demands of graduate students. The material provided can also be used as a lecture notes on locally convex operator algebras. The main approach to the multinormed von Neumann algebras is based on the multinormed completions of a von Neumann algebra. If $\mathcal{M}$ is a von Neumann algebra with its central projections $\mathcal{E} = (e_\iota)$, $\vee_\iota e_\iota = 1$ then $\mathcal{M}$ possesses a multinormed (or locally multiplicative convex) topology with its defining family of C^*-seminorms $\|x\|_\iota = \|xe_\iota\|$, $x \in \mathcal{M}$ called *a central topology in* $\mathcal{M}$, whose completion $\mathcal{M}_{\mathcal{E}}$ is a multinormed von Neumann algebra. It turns out that the multinormed von Neumann algebras are precisely the central completions of von Neumann algebras. Every multinormed von Neumann algebra A admits a unique (up the isometry) predual X (in the bornological sense), which is an ℓ_1-normed space (see Definition 4.1). As in the normed case, multinormed von Neumann algebras admit operator $*$-representations. A locally convex version of $\mathcal{B}(H)$ is reduced to the $*$-algebra $C^*_{\mathcal{E}}(\mathcal{D})$ of all noncommutative continuous functions over *a commutative domain* $\mathcal{E} = \{p_\alpha : \alpha \in \Lambda\}$, that is, $\mathcal{E}$ is a mutually commuting family of projections in $\mathcal{B}(H)$. Note that $C^*_{\mathcal{E}}(\mathcal{D})$ is a multinormed von Neumann algebra. Actually, it is a central completion of the commutant $\mathcal{E}'$ in $\mathcal{B}(H)$. In particular, it admits the unique predual, which is $\mathcal{T}_{\mathcal{E}}(\mathcal{D})$ whenever $\mathcal{E}$ admits a gradation. The representation theorem for multinormed C^*-algebras mentioned above is generalized in the following way. If A is a multinormed C^*-algebra with its upward filtered family of C^*-seminorms $\{\|\cdot\|_\alpha\}$, then there is a (commutative) domain $\mathcal{E} = \{e_\alpha : \alpha \in \Lambda\}$ in a Hilbert space H and $*$-embedding $\varphi : \mathcal{A} \hookrightarrow C^*_{\mathcal{E}}(\mathcal{D})$ such that $\|\varphi(a)\|_\alpha = \|a\|_\alpha$, $a \in \mathcal{A}$, $\alpha \in \Lambda$, where $\mathcal{D}$ is the union space of $\mathcal{E}$. A strongly closed $*$-subalgebra $\mathcal{A} \subseteq C^*_{\mathcal{E}}(\mathcal{D})$ is said to be *a local von Neumann algebra on* $\mathcal{D}$ if $\mathcal{A}p_\alpha \subseteq \mathcal{A}$ for all α. In this case, $\mathcal{A}p_\alpha$ is a von Neumann algebra and $\mathcal{A}$ turns out to be an inductive limit of these von Neumann algebras (see Proposition 5.3). For instance, the center $Z(C^*_{\mathcal{E}}(\mathcal{D}))$

of $C_{\mathcal{E}}^{*}(\mathcal{D})$ is a local von Neumann algebra on $\mathcal{D}$, which is the strong closure of the unital associative subalgebra generated by $\mathcal{E}$. Analog of Sakai theorem (see [26, 1.16.7] for the normed case) sounds in the following way. If $\mathcal{A}$ is a multinormed von Neumann algebra with its upward filtered family $\{\|\cdot\|_{\alpha} : \alpha \in \Lambda\}$ of C^*-seminorms, then there exists a domain $\mathcal{E} = \{p_{\alpha} : \alpha \in \Lambda\}$ in a Hilbert space H and $*$-isomorphism $\mathcal{A} \to C_{\mathcal{E}}^{*}(\mathcal{D})$ onto a local von Neumann subalgebra on $\mathcal{D}$ such that $\|a\|_{\alpha} = \|ap_{\alpha}\|$, $a \in \mathcal{A}$ for all α. Thus the multinormed von Neumann algebras are precisely $*$-subalgebras in $C_{\mathcal{E}}^{*}(\mathcal{D})$ which admit preduals, and they are in turn the local von Neumann algebras. Finally, we classify abelian multinormed von Neumann algebras. Namely, let $\mathcal{T}$ be a locally compact Hausdorff topological space equipped with a positive Radon integral $\int : C_c(\mathcal{T}) \to \mathbb{C}$, and let μ be the related Radon measure on $\mathcal{T}$. A domain in $L^{\infty}(\mathcal{T})$ corresponds to a family $\mathcal{E} = (E_{\iota})_{\iota \in \Xi}$ of measurable subsets in $\mathcal{T}$ whose union is (almost) covering $\mathcal{T}$, and $\|x\|_{\iota,\infty} = \|x \cdot e_{\iota}\|_{\infty}$, $x \in L^{\infty}(\mathcal{T})$, $e_{\iota} = [E_{\iota}]$ are the related C^*-seminorms on $L^{\infty}(\mathcal{T})$, whose completion $L^{\infty}(\mathcal{T})_{\mathcal{E}}$ is a multinormed W^*-algebra. We say that $\mathcal{E}$ is *a measurable covering of* $\mathcal{T}$ if for every nonempty compact subset $C \subseteq \mathcal{T}$ we have $C \cap E_{\iota} = \varnothing$ for all ι but at most countable many of them. We prove (see below Theorem 8.1) that $L^{\infty}(\mathcal{T})_{\mathcal{E}}$ is represented by means of measurable, E-locally, essentially bounded functions f, that is, esssup $|f|E_{\iota}| < \infty$ for all ι, and the bounded part of $L^{\infty}(\mathcal{T})_{\mathcal{E}}$ is reduced to $L^{\infty}(\mathcal{T})$. Moreover, the bornological predual $L^{1}(\mathcal{T})_{\mathcal{E}}$ of $L^{\infty}(\mathcal{T})_{\mathcal{E}}$ consists of those $g \in L^{1}(\mathcal{T})$ such that $\|g\|_1 = \int_{E_{\alpha}} |g|$ for some $\alpha \in \Lambda$. For applications of local operator algebras from $C_{\mathcal{E}}^{*}(\mathcal{D})$ to the quantum moment problems we refer the reader to [6].

2. Preliminaries

All vector space are assumed to be complex. The norm dual of a normed space X is denoted by X^*. The closed unit ball of X is denoted by $\operatorname{ball} X$. The closed subspace of X generated by a subset $S \subseteq X$ is denoted by $\langle S \rangle$. The canonical duality between X and X^* is given by the pairing $\langle x, x^* \rangle = x^*(x)$ for all $x \in X$ and $x^* \in X^*$. For a separating subspaces $Y \subseteq X$ and $Y' \subseteq X^*$ the related weak topology in X and weak* topology in X^* are denoted by $\sigma(X, Y')$ and $\sigma(X^*, Y)$, respectively. The Banach algebra of all bounded linear operators on a Banach space X is denoted by $\mathcal{B}(X)$, which is a C^*-algebra for a Hilbert space $X = H$. For an operator $x \in \mathcal{B}(H)$ we put $|x| = (x^*x)^{1/2}$ for the absolute value of x, and $x = u|x|$ for its polar decomposition. The countable Hilbert space power of H is denoted by $\ell^2(H)$ called the amplification of H. Note that $\ell^2(H)$ consists of those sequences $\zeta = (\zeta_n)_n$ taken from H such that $\|\zeta\|^2 = \sum_{n=1}^{\infty} \|\zeta_n\|^2 < \infty$. The space $\ell^2(H)$ can be treated as the Hilbert space tensor product $H \otimes_{\sigma} \ell^2$ (see below Subsection 2.4). The commutant of a subset $\mathcal{M} \subseteq \mathcal{B}(H)$ is denoted by $\mathcal{M}'$, that is, $x \in \mathcal{M}'$ iff $xy = yx$ for all $y \in \mathcal{M}$. Confirm that $\mathcal{M}'$ is a unital C^*-algebra. The positive elements of a C^*-algebra A is denoted by A_+ whereas A_h denotes the set of all self-adjoint (or hermitian) elements of A. By a projection e in $\mathcal{B}(H)$ we mean a self-adjoint idempotent $e^* = e = e^2$. The projection $1 - e$ is denoted by $e^{\perp}$. Everywhere below $\mathcal{T}$ denotes a locally compact (Hausdorff) space. The C^*-algebra equipped with the uniform norm of all complex continuous functions on $\mathcal{T}$ vanishing at infinity is denoted by $C_0(\mathcal{T})$ whereas $C(\mathcal{T})$ denotes the $*$-algebra of all continuous functions on $\mathcal{T}$. Note that $C_0(\mathcal{T}) = C(\mathcal{T})$ whenever $\mathcal{T}$ is a compact space. The C^*-algebra $C_0(\mathcal{T})$ contains the $*$-algebra $C_c(\mathcal{T})$ of all complex continuous functions f on $\mathcal{T}$ with their compact supports $\operatorname{supp} f$ as a dense $*$-subalgebra. The characteristic function of a subset $S \subseteq \mathcal{T}$ is denoted by $[S]$.

2.1. Remarks on the weak* topology and weak operator topology. Let X be a normed space, $Y \subseteq X$ a subspace and let $Y^\perp = \{x^* \in X^* : \langle Y, x^* \rangle = \{0\}\}$ be the polar of Y in the dual space X^*. A subspace $Y \subseteq X$ is said to be separating if its polar $Y^\perp$ in X^* is vanishing. The weak* topology (briefly, w^*-topology) $\sigma(X^*, Y)$ on X^* is given by means of the seminorms $p_y(x^*) = |\langle y, x^* \rangle|$, $x^* \in X^*$, $y \in Y$. Thus a neighborhood filter base of the origin consists of sets $\{\max\{p_{y_i} : 1 \le i \le n\} \le \varepsilon\}$, where $\{y_i\} \subseteq Y$ is a finite subset and $\varepsilon > 0$.

Lemma 2.1. *Let X be a normed space, $B \subseteq X^*$ a bounded subset, and let $Y \subseteq X$ be a dense subspace. Then $\sigma(X^*, X)|B = \sigma(X^*, Y)|B$.*

Proof. Take a net $(x_\lambda^*)_\lambda \subseteq B$ with $x^* = \sigma(X^*, Y)\text{-}\lim_\lambda x_\lambda^* \in B$, and $x \in X$. For every $\varepsilon > 0$ one can find $y \in Y$ such that $\|x - y\| \le \varepsilon / (4 \sup \|B\|)$. But $\lim_\lambda p_y(x^* - x_\lambda^*) = 0$, therefore $p_y(x^* - x_\lambda^*) \le \varepsilon/2$ for all $\lambda \ge \lambda_0$, where $p_y(z^*) = |\langle y, z^* \rangle|$, $z^* \in X^*$. Then

$$\begin{aligned} p_x(x^* - x_\lambda^*) &= |\langle x, x^* - x_\lambda^* \rangle| \le |\langle x - y, x^* - x_\lambda^* \rangle| \\ &+ p_y(x^* - x_\lambda^*) \le \|x - y\| \, \|x^* - x_\lambda^*\| + p_y(x^* - x_\lambda^*) \\ &\le \|x - y\| \, 2 \sup \|B\| + p_y(x^* - x_\lambda^*) \le \varepsilon, \end{aligned}$$

that is, $x^* = \sigma(X^*, X)\text{-}\lim_\lambda x_\lambda^*$ either. Finally, take a neighborhood filter base $\mathcal{O}_X(x^*)$ of the fixed point $x^* \in B$ in B with respect to $\sigma(X^*, X)|B$-topology. Similarly, $\mathcal{O}_Y(x^*)$ is a neighborhood filter base with respect to $\sigma(X^*, Y)|B$-topology. Then $x^* = \sigma(X^*, Y)\text{-}\lim \mathcal{O}_Y(x^*) = \sigma(X^*, X)\text{-}\lim \mathcal{O}_Y(x^*)$, that is, $\mathcal{O}_Y(x^*)$ dominates $\mathcal{O}_X(x^*)$. Since $\mathcal{O}_X(x^*) \supseteq \mathcal{O}_Y(x^*)$, we derive that $\mathcal{O}_X(x^*) = \mathcal{O}_Y(x^*)$. Hence $\sigma(X^*, X)|B = \sigma(X^*, Y)|B$. □

Similarly, we define the polar $Z^\perp$ of a subspace $Z \subseteq X^*$ to be $\{x \in X : \langle x, Z \rangle = \{0\}\}$. It is well known (see [24, 2.3.5]) that $Y^- = Y^{\perp\perp}$, and $Z^{-w^*} = Z^{\perp\perp}$ due to the known (see [24, 2.4.10]) Bipolar Theorem. It is immediate that $\langle \cup_\iota Y_\iota \rangle^\perp = \cap_\iota Y_\iota^\perp$ and $\langle \cup_\iota Z_\iota \rangle^\perp = \cap_\iota Z_\iota^\perp$ for families $\{Y_\iota\}$ and $\{Z_\iota\}$ of subspaces in X and X^*, respectively. If $\{Z_\iota\}$ is a family of w^*-closed subspaces in X^* then

$$(\cap_\iota Z_\iota)^\perp = \left\langle \cup_\iota Z_\iota^\perp \right\rangle. \tag{2.1}$$

Indeed, since $\cap_\iota Z_\iota \subseteq Z_\iota$ for every ι, it follows that $\cup_\iota Z_\iota^\perp \subseteq (\cap_\iota Z_\iota)^\perp$, which in turn implies that $\left\langle \cup_\iota Z_\iota^\perp \right\rangle \subseteq (\cap_\iota Z_\iota)^\perp$. Using Bipolar Theorem, we derive that $\cap_\iota Z_\iota = (\cap_\iota Z_\iota)^{-w^*} = (\cap_\iota Z_\iota)^{\perp\perp} \subseteq \left\langle \cup_\iota Z_\iota^\perp \right\rangle^\perp = \cap_\iota Z_\iota^{\perp\perp} = \cap_\iota Z_\iota^{-w^*} = \cap_\iota Z_\iota$, that is, $\cap_\iota Z_\iota = \left\langle \cup_\iota Z_\iota^\perp \right\rangle^\perp$. It follows that $(\cap_\iota Z_\iota)^\perp = \left\langle \cup_\iota Z_\iota^\perp \right\rangle^{\perp\perp} = \left\langle \cup_\iota Z_\iota^\perp \right\rangle$.

Similar formula takes place for the closed subspaces $\{Y_\iota\}$ of X. Namely,

$$(\cap_\iota Y_\iota)^\perp = \left\langle \cup_\iota Y_\iota^\perp \right\rangle^{-w^*}. \tag{2.2}$$

Indeed, since $\left(\left\langle \cup_\iota Y_\iota^\perp \right\rangle^{-w^*}\right)^\perp = \left\langle \cup_\iota Y_\iota^\perp \right\rangle^\perp = \cap_\iota Y_\iota^{\perp\perp} = \cap_\iota Y_\iota$, we have $\left\langle \cup_\iota Y_\iota^\perp \right\rangle^{-w^*} = \left(\left\langle \cup_\iota Y_\iota^\perp \right\rangle^{-w^*}\right)^{\perp\perp} = \left(\cap_\iota Y_\iota\right)^\perp$.

Now consider the WOT-topology in $\mathcal{B}(H)$, which is given by means of the seminorms $p_{\zeta,\eta}(x) = |(x\zeta,\eta)|$, $x \in \mathcal{B}(H)$, $\zeta,\eta \in H$.

Lemma 2.2. *Let B be a bounded subset of $\mathcal{B}(H)$, and $S \subseteq H$ whose span is dense in H. Then WOT on B can be defined by means of the seminorms $p_{\zeta,\eta}$, $\zeta,\eta \in S$.*

Proof. Let τ be the topology in $\mathcal{B}(H)$ given by the family $\{p_{\zeta,\eta} : \zeta,\eta \in S\}$, which is obviously Hausdorff and weaker than WOT. Note that τ is also given by the family $\{p_{\zeta,\eta} : \zeta,\eta \in Y\}$ of seminorms, where Y is the linear span of S. Prove that WOT$|B = \tau|B$. As in Lemma 2.1, take a net $(x_\lambda)_\lambda \subseteq B$ with $x = \tau\text{-}\lim_\lambda x_\lambda \in B$, and $\zeta,\eta \in \operatorname{ball} H$. For every $1 \geq \varepsilon > 0$ one can find $\zeta_1,\eta_1 \in Y$ such that $\|\zeta - \zeta_1\|, \|\eta - \eta_1\| \leq \varepsilon / (12 \sup \|B\|)$ (we can assume that $\sup\|B\| \geq 1$). Since $\lim_\lambda p_{\zeta_1,\eta_1}(x - x_\lambda) = 0$, we have $p_{\zeta_1,\eta_1}(x - x_\lambda) \leq \varepsilon/2$ for all $\lambda \geq \lambda_0$. Then

$$\begin{aligned}
p_{\zeta,\eta}(x - x_\lambda) &= |((x - x_\lambda)\zeta,\eta)| \leq |((x - x_\lambda)(\zeta - \zeta_1),\eta)| \\
&\quad + |((x - x_\lambda)\zeta_1,\eta - \eta_1)| + |((x - x_\lambda)\zeta_1,\eta_1)| \\
&\leq \|x - x_\lambda\| \|\zeta - \zeta_1\| \|\eta\| + \|x - x_\lambda\| \|\zeta_1\| \|\eta - \eta_1\| + p_{\zeta_1,\eta_1}(x - x_\lambda) \\
&\leq (\|\zeta - \zeta_1\| \|\eta\| + \|\zeta - \zeta_1\| \|\eta - \eta_1\| + \|\zeta\| \|\eta - \eta_1\|) 2\sup\|B\| + p_{\zeta_1,\eta_1}(x - x_\lambda) \\
&\leq (\|\zeta - \zeta_1\| + \|\zeta - \zeta_1\| \|\eta - \eta_1\| + \|\eta - \eta_1\|) 2\sup\|B\| + p_{\zeta_1,\eta_1}(x - x_\lambda) \leq \varepsilon
\end{aligned}$$

that is, $x =$WOT-$\lim_\lambda x_\lambda$ either. Finally, consider WOT-neighborhood filter base $\mathcal{O}_{\mathrm{WOT}}(x)$ of the fixed point x in B, and let $\mathcal{O}_\tau(x)$ be the neighborhood filter base with respect to $\tau|B$-topology. Then $x = \tau\text{-}\lim \mathcal{O}_\tau(x) =$WOT-$\lim \mathcal{O}_\tau(x)$, that is, $\mathcal{O}_\tau(x) = \mathcal{O}_{\mathrm{WOT}}(x)$. Hence WOT$|B = \tau|B$. □

2.2. The extreme points. Recall that an extreme point of a convex subset $\mathcal{C} \subseteq X$ of a vector space X is a point of $\mathcal{C}$ that can not be expressed as a nontrivial convex combination of elements from $\mathcal{C}$. The set of all extreme points of $\mathcal{C}$ is called the extremal boundary of $\mathcal{C}$ denoted by $\partial\mathcal{C}$. The well known (see [24, 2.5.4]) Krein-Milman theorem asserts that for each convex, compact subset $\mathcal{C}$ of a vector space X equipped with the weak topology $\sigma(X,Y)$ of a dual pair (X,Y), the convex hull of $\partial\mathcal{C}$ is dense in $\mathcal{C}$. It is not hard to see that the extremal boundary of $\operatorname{ball} C_0(\mathcal{T})$ consists of unitary functions, that is,

$$\partial \operatorname{ball} C_0(\mathcal{T}) = \{f \in C_0(\mathcal{T}) : |f| = 1\}. \tag{2.3}$$

Indeed, take $f \in \operatorname{ball} C_0(\mathcal{T})$. If $|f(t_0)| < 1$ for some $t_0 \in \mathcal{T}$ then $\alpha = \sup|f(U)| < 1$ for a compact neighborhood U of t_0, and we can choose $h \in C_0(\mathcal{T})$ such that $0 \leq h \leq 1$, $h(t_0) = 1$ and $\operatorname{supp} h \subseteq U$ (see [24, 1.7.5]). Put $g = (1-\alpha)h$. Then $f \pm g \in \operatorname{ball} C_0(\mathcal{T})$ and $f = \frac{1}{2}((f+g) + (f-g))$, that is, $f \notin \partial \operatorname{ball} C_0(\mathcal{T})$. Conversely, if f is unitary and $f = \frac{1}{2}(g+h)$ for some $g,h \in \operatorname{ball} C_0(\mathcal{T})$, then $f(t)$ is an extreme point of the unit circle in $\mathbb{C}$, therefore $g(t) = h(t)$, that is, $f \in \partial \operatorname{ball} C_0(\mathcal{T})$. In particular, presence of an extreme point in $\operatorname{ball} C_0(\mathcal{T})$ involves compactness of $\mathcal{T}$ automatically, that is, $C_0(\mathcal{T})$ is unital. Further, $\partial \operatorname{ball} C_0(\mathcal{T})_+$ consists of projections in $C_0(\mathcal{T})$, that is,

$$\partial \operatorname{ball} C_0(\mathcal{T})_+ = \{[S] : S \subseteq \mathcal{T} \text{ is clopen}\}. \tag{2.4}$$

Indeed, if $[S] = \frac{1}{2}(g+h)$ for some $g, h \in \operatorname{ball} C_0(\mathcal{T})_+$, then $g(t) = h(t) = 1$, $t \in S$ and $0 \leq g(t) = h(t) = 0$ for all $t \in \mathcal{T} - S$, that is, $g = h = [S]$ and $[S] \in \partial \operatorname{ball} C_0(\mathcal{T})_+$. Conversely, take $f \in \partial \operatorname{ball} C_0(\mathcal{T})_+$. If $0 < f(t_0) < 1$ for some $t_0 \in \mathcal{T}$ then as above choose the function $h \in \operatorname{ball} C_0(\mathcal{T})_+$ and small $\varepsilon > 0$ such that $f(1 \pm \varepsilon h) \in \operatorname{ball} C_0(\mathcal{T})_+$. But $f = \frac{1}{2}(f(1+\varepsilon h) + f(1-\varepsilon h))$, which is impossible. Thus f can only take the values 0 or 1, that is, $f = [S]$ for a clopen subset $S \subseteq \mathcal{T}$.

Now let us consider the general case of an arbitrary C^*-algebra A.

Lemma 2.3. *If x^*x is not a projection for some $x \in \operatorname{ball} A$ then $x \notin \partial \operatorname{ball} A$.*

Proof. Note that $|x|$ is not a projection as well, and the C^*-subalgebra B in the unitization A_1 of A generated by 1 and $|x|$ is abelian. Therefore it is an isomorphic copy of $C_0(\mathcal{T})$, and $|x| \in \operatorname{ball} B_+$. Using (2.4), we conclude that $|x| \notin \partial \operatorname{ball} B_+$, that is, there is $a \in \operatorname{ball} B_+$ such that $\||x|(1 \pm a)\| \leq 1$ and $a|x| \neq 0$ (that is, $|x| = f$ and $a = \varepsilon h$ in the notations above related to (2.4)). Then $\|x(1 \pm a)\|^2 = \|(1 \pm a)x^*x(1 \pm a)\| = \left\|(1 \pm a)|x|^2(1 \pm a)\right\| = \||x|(1 \pm a)\|^2 \leq 1$ and $\|xa\|^2 = \|ax^*xa\| = \left\|a|x|^2 a\right\| = \||x|a\|^2 \neq 0$, that is, $x(1 \pm a) \neq x$. Since $x = \frac{1}{2}(x(1+a) + x(1-a))$, it follows that $x \notin \partial \operatorname{ball} A$. □

Lemma 2.4. *If $x \in \partial \operatorname{ball} A$ then $p = x^*x$ and $q = xx^*$ are projections and $(1-q)A(1-p) = \{0\}$.*

Proof. The fact that p is a projection follows from Lemma 2.3. Moreover,

$$\|x(1-p)\|^2 = \|(x(1-p))^* x(1-p)\|$$
$$= \|(1-p)x^*x(1-p)\| = \|(1-p)p(1-p)\| = 0,$$

that is, $x = xp$, which in turn implies that $q = xx^* = xpx^* \in A_h$ and $q^2 = xpx^*xpx^* = xp^3x^* = xpx^* = q$. Thus p and q are projections. As above

$$\|x^*(1-q)\|^2 = \|(1-q)xx^*(1-q)\| = \|(1-q)q(1-q)\| = 0,$$

that is, $x^* = x^*q = (qx)^*$ or $x = qx$. Further, take $a \in (1-q)(\operatorname{ball} A)(1-p) \subseteq \operatorname{ball} A$, that is, $(1-q)a = a(1-p) = a$. Since $x^*a = x^*qa = x^*q(1-q)a = 0$, $a^*x = a^*(1-q)x = a^*(1-q)qx = 0$ and $a^*a = (1-p)a^*a(1-p)$, it follows that

$$\|x \pm a\|^2 = \|(x \pm a)^*(x \pm a)\| = \|x^*x \pm x^*a \pm a^*x + a^*a\|$$
$$= \|p + (1-p)a^*a(1-p)\| = \max\{\|p\|, \|(1-p)a^*a(1-p)\|\} = 1.$$

Since $x = \frac{1}{2}((x+a) + (x-a))$, it follows that $x = x + a$ or $a = 0$. Thus $(1-q)(\operatorname{ball} A)(1-p) = \{0\}$, which implies that $(1-q)A(1-p) = \{0\}$. □

Proposition 2.1. *Let A be a C^*-algebra. Then A is unital iff $\partial \operatorname{ball} A \neq \varnothing$.*

Proof. First take $x \in \partial \operatorname{ball} A$, and let $\{e_\iota\} \subseteq \operatorname{ball} A$ be an approximate unit for A (see [23, 3.1.1]). Using Lemma 2.4, we obtain that $(1-q)\{e_\iota\}(1-p) = \{0\}$ and

$$q + p - qp = \lim_\iota (qe_\iota + e_\iota p - qe_\iota p) = \lim_\iota (e_\iota - (1-q)e_\iota(1-p)) = \lim_\iota e_\iota,$$

that is, $q + p - qp$ is a unit for A. Conversely, suppose that A has the unit 1, and $1 = \frac{1}{2}(a+b)$ for some $a, b \in \operatorname{ball} A$. Then $1 = \frac{1}{2}(a^* + b^*)$ and $1 = \frac{1}{2}(a_h + b_h)$

with self-adjoint elements $a_h = \frac{1}{2}(a + a^*)$ and $b_h = \frac{1}{2}(b + b^*)$. Since $b_h = 2 - a_h$, it follows that a_h and b_h commute, and the unital C^*-subalgebra B generated by $\{a_h, b_h\}$ is abelian. In particular, for a certain compact space K we have $B = C(K)$ up to an isometric $*$-isomorphism. Using (2.3), we obtain that $\partial\,\mathrm{ball}\,C(K)$ consists of unitaries. In particular, $1 = \frac{1}{2}(a_h + b_h) \in \partial\,\mathrm{ball}\,C(K)$ implies that $a_h = b_h = 1$. But the relation $1 = \frac{1}{2}(a + a^*)$ implies that a and a^* commute, that is, a is normal. Using again (2.3), we conclude that $a = a^* = 1$. Similarly, we get $b = b^* = 1$. Whence $1 \in \partial\,\mathrm{ball}\,A$. □

2.3. The Banach algebra $\mathcal{B}^1(H)$. Let H be a Hilbert space with an orthonormal basis $(\epsilon_i)_{i\in I}$. The $*$-ideal (two-sided) of all finite-rank operators on H is denoted by $\mathcal{B}_f(H)$ whose closure in $\mathcal{B}(H)$ is reduced to the ideal $\mathcal{B}_0(H)$ of compact operators, which is $*$-ideal automatically. Confirm that $\mathcal{B}_f(H)$ is spanned by one-rank operators $\zeta \odot \eta \in \mathcal{B}(H)$, $(\zeta \odot \eta)(\theta) = (\theta, \eta)\zeta$, $\zeta, \eta, \theta \in H$. Recall that the trace of a positive operator $x \in \mathcal{B}(H)_+$ is defined as $\mathrm{tr}(x) = \sum_{i\in I}(x\epsilon_i, \epsilon_i)$. If $\mathrm{tr}(|x|^p) < \infty$ for some $p > 0$ then $T \in \mathcal{B}_0(H)$ automatically. Confirm that $\zeta\odot\zeta \geq 0$ and $\mathrm{tr}(\zeta \odot \zeta) = \sum_{i\in I}((\zeta \odot \zeta)\epsilon_i, \epsilon_i) = \sum_{i\in I}(\epsilon_i, \zeta)(\zeta, \epsilon_i) = \sum_{i\in I}|(\zeta, \epsilon_i)|^2 = \|\zeta\|^2 = (\zeta, \zeta)$. The trace class operators $\mathcal{B}^1(H)$ is defined as span of those $x \in \mathcal{B}_0(H)_+$ with $\mathrm{tr}(x) < \infty$. Similarly, $\mathcal{B}^2(H) = \{x \in \mathcal{B}_0(H) : \mathrm{tr}(x^*x) < \infty\}$ is the class of all Hilbert-Schmidt operators. Since $x = \sum_{k=0}^3 i^k x_k$ with $x_k \geq 0$, for every x in $\mathcal{B}^1(H)$, it follows that $\mathrm{tr}(x) = \sum_{k=0}^3 i^k\,\mathrm{tr}(x_k)$ extends tr to a linear functional on $\mathcal{B}^1(H)$. In particular, $4\,\mathrm{tr}(\zeta \odot \eta) = \sum_{k=0}^3 i^k\,\mathrm{tr}\left(\left(\zeta + i^k\eta\right) \odot \left(\zeta + i^k\eta\right)\right) = \sum_{k=0}^3 i^k\left(\zeta + i^k\eta, \zeta + i^k\eta\right) = 4(\zeta, \eta)$, that is, $\mathrm{tr}(\zeta \odot \eta) = (\zeta, \eta)$. Note also that

$$|\zeta \odot \eta| = ((\zeta \odot \eta)^*(\zeta \odot \eta))^{1/2} = ((\eta \odot \zeta)(\zeta \odot \eta))^{1/2} = ((\zeta, \zeta)\eta \odot \eta)^{1/2}$$

$$= \|\zeta\|\,\|\eta\|\left(\|\eta\|^{-1}\eta \odot \|\eta\|^{-1}\eta\right)^{1/2} = \|\zeta\|\,\|\eta\|\left(\|\eta\|^{-1}\eta \odot \|\eta\|^{-1}\eta\right),$$

which in turn implies that $\mathrm{tr}(|\zeta \odot \eta|) = \|\zeta\|\,\|\eta\|$. It is well known (see [24, 3.4.8, 3.4.12]) $\mathcal{B}^1(H)$ and $\mathcal{B}^2(H)$ are $*$-ideals of $\mathcal{B}(H)$ and $\mathcal{B}_f(H) \subseteq \mathcal{B}^1(H) \subseteq \mathcal{B}^2(H) \subseteq \mathcal{B}_0(H)$. Moreover, $\mathcal{B}^1(H) = \{x \in \mathcal{B}(H) : \mathrm{tr}(|x|) < \infty\}$ is a Banach algebra under the norm $\|x\|_1 = \mathrm{tr}(|x|)$ (see [24, 3.4.12]), whereas $\mathcal{B}^2(H)$ is a Hilbert space under the inner product $(x, y) = \mathrm{tr}(y^*x)$, $x, y \in \mathcal{B}^2(H)$. Further, the bilinear form $\langle x, y\rangle = \mathrm{tr}(xy)$ implements the duality between the pair of Banach spaces $\mathcal{B}_0(H)$ and $\mathcal{B}^1(H)$, and the pair $\mathcal{B}^1(H)$ and $\mathcal{B}(H)$ as well. Thus $\mathcal{B}_0(H)^* = \mathcal{B}^1(H)$ and $\mathcal{B}^1(H)^* = \mathcal{B}(H)$ up to isometric isomorphisms of Banach spaces. In particular, $\mathcal{B}(H)$ possesses the w^*-topology $\sigma\left(\mathcal{B}(H), \mathcal{B}^1(H)\right)$ defined by means of the family $p_x(y) = |\langle x, y\rangle|$, $y \in \mathcal{B}(H)$, $x \in \mathcal{B}^1(H)$ of seminorms. For $\zeta, \eta \in \ell^2(H)$ we define the linear functional $\omega_{\zeta,\eta} : \mathcal{B}(H) \to \mathbb{C}$, $\langle y, \omega_{\zeta,\eta}\rangle = \sum_{n=1}^\infty (y\zeta_n, \eta_n)$. Note that $\sum_n |(y\zeta_n, \eta_n)| = \|y\| \sum_n \|\zeta_n\|\,\|\eta_n\| \leq \|y\|\,\|\zeta\|\,\|\eta\| < \infty$ for every $y \in \mathcal{B}(H)$. For a subspace $\mathcal{M} \subseteq \mathcal{B}(H)$ we define its polar $\mathcal{M}^\perp$ to be $\left\{x \in \mathcal{B}^1(H) : \langle x, \mathcal{M}\rangle = \{0\}\right\}$ in $\mathcal{B}^1(H)$.

Proposition 2.2. *Let $\mathcal{M}$ be a w^*-closed subspace in $\mathcal{B}(H)$. Then $\mathcal{M}_* = \mathcal{B}^1(H)/\mathcal{M}^\perp$ is a Banach space predual of $\mathcal{M}$, that is, $(\mathcal{M}_*)^* = \mathcal{M}$ up to an isometric isomorphism. Moreover, $\sigma(\mathcal{M}, \mathcal{M}_*)$-continuous linear functionals on $\mathcal{M}$ consists of those $\omega_{\zeta,\eta} : \mathcal{M} \to \mathbb{C}$ with $\zeta, \eta \in \ell^2(H)$. Thus the*

w^*-topology of a w^*-closed subspace $\mathcal{M}$ in $\mathcal{B}(H)$ is defined by means of the seminorms $p_{\zeta,\eta}(y) = |\langle y, \omega_{\zeta,\eta}\rangle|$, $y \in \mathcal{M}$, where $\zeta, \eta \in \ell^2(H)$, therefore $\sigma(\mathcal{M}, \mathcal{M}_*) = \sigma\left(\mathcal{B}(H), \mathcal{B}^1(H)\right)|\mathcal{M}$.

Proof. Based on the Bipolar Theorem (see [24, 2.4.11]), we conclude that $\mathcal{M} = \left(\mathcal{M}^\perp\right)^\perp$. Then the adjoint mapping π^* to the quotient mapping $\pi : \mathcal{B}^1(H) \to \mathcal{M}_*$ implements an isometric isomorphism of $(\mathcal{M}_*)^*$ onto $\left(\mathcal{M}^\perp\right)^\perp$, that is, $(\mathcal{M}_*)^* = \mathcal{M}$ up to an isometric isomorphism (see [24, 2.4.13]). The duality between $\mathcal{M}_*$ and $\mathcal{M}$ is given by the pairing $\langle x^\sim, y\rangle = \operatorname{tr}(xy)$, where $x \in \mathcal{B}^1(H)$ with its class $x^\sim \in \mathcal{B}^1(H)/\mathcal{M}^\perp$ and $y \in \mathcal{M}$. In particular, every $\sigma(\mathcal{M}, \mathcal{M}_*)$-continuous linear functional on $\mathcal{M}$ is given by $x^\sim \in \mathcal{M}_*$ as $\langle y, \varphi_x\rangle = \langle x^\sim, y\rangle$, $y \in \mathcal{M}$. Fix $x^\sim \in \mathcal{M}_*$ with the polar decomposition $x = u|x|$ of $x \in \mathcal{B}^1(H)$. Note that $|x| = u^*x \in \mathcal{B}^1(H) \subseteq \mathcal{B}_0(H)$, which in turn implies that $|x|$ is diagonalizable. Thus for a suitable orthonormal basis $(\epsilon_i)_{i\in I}$ we have $|x| = \sum_i \lambda_i \epsilon_i \odot \epsilon_i$. Then $x = \sum_i \lambda_i^{1/2} u\epsilon_i \odot \lambda_i^{1/2}\epsilon_i = \sum_i \zeta_i \odot \eta_i$ and $\sum \|\zeta_i\|^2 = \sum \lambda_i (u^*u\epsilon_i, \epsilon_i) = \sum \lambda_i = \operatorname{tr}(|x|) = \|x\|_1$ (confirm that u^*u is the projection onto $|x|(H)^-$ and $|x|\epsilon_i = \lambda_i\epsilon_i$), $\sum_i \|\eta_i\|^2 = \sum \lambda_i = \|x\|_1$. It follows that

$$\langle y, \varphi_x\rangle = \operatorname{tr}(xy) = \operatorname{tr}(yx) = \operatorname{tr}\left(\sum_i y\zeta_i \odot \eta_i\right) = \sum_i \operatorname{tr}(y\zeta_i \odot \eta_i) = \sum_i (y\zeta_i, \eta_i)$$

for all $y \in \mathcal{M}$. But

$$\sum_i |(y\zeta_i, \eta_i)| = \|y\| \sum_i \|\zeta_i\| \|\eta_i\| \leq \|y\| \left(\sum_i \|\zeta_i\|^2\right)^{1/2} \left(\sum_i \|\eta_i\|^2\right)^{1/2} = \|y\| \|x\|_1$$

for every $x \in x^\sim$ (see [24, 3.4.10]). In particular, $\sum_i |(y\zeta_i, \eta_i)| \leq \|y\| \|x^\sim\| < \infty$, where $\|x^\sim\| = \inf\{\|x\|_1 : x \in x^\sim\}$. Moreover, $(y\zeta_i, \eta_i) = 0$ for all i but countable many of them, and $\langle y, \varphi_x\rangle = \sum_{n=1}^\infty (y\zeta_{i_n}, \eta_{i_n}) = \omega_{\zeta,\eta}(y)$, where $\zeta = (\zeta_{i_n})_n$, $\eta = (\eta_{i_n})_n \in \ell^2(H)$.

Conversely, take $\zeta, \eta \in \ell^2(H)$ and put $x = \sum_n \zeta_n \odot \eta_n$. Note that

$$\sum_n \|\zeta_n \odot \eta_n\|_1 = \sum_n \operatorname{tr}(|\zeta_n \odot \eta_n|) = \sum_n \|\zeta_n\| \|\eta_n\| \operatorname{tr}\left(\|\eta_n\|^{-1} \eta_n \odot \|\eta_n\|^{-1} \eta_n\right)$$

$$= \sum_n \|\zeta_n\| \|\eta_n\| \left(\|\eta_n\|^{-1} \eta_n, \|\eta_n\|^{-1} \eta_n\right) = \sum_n \|\zeta_n\| \|\eta_n\| \leq \|\zeta\| \|\eta\| < \infty.$$

Since $\mathcal{B}^1(H)$ is a Banach algebra, it follows that $x \in \mathcal{B}^1(H)$. Furthermore,

$$\omega_{\zeta,\eta}(y) = \sum_{n=1}^\infty (y\zeta_n, \eta_n) = \sum_{n=1}^\infty \operatorname{tr}(y\zeta_n \odot \eta_n) = \operatorname{tr}\left(\sum_{n=1}^\infty (\zeta_n \odot \eta_n) y\right)$$
$$= \operatorname{tr}(xy) = \operatorname{tr}(x^\sim y) = \langle y, \varphi_x\rangle$$

for all $y \in \mathcal{M}$, that is, $\omega_{\zeta,\eta} : \mathcal{M} \to \mathbb{C}$ is a $\sigma(\mathcal{M}, \mathcal{M}_*)$-continuous linear functional. □

Confirm that if $\zeta, \eta \in H$ the related family $\{p_{\zeta,\eta}\}$ of seminorms defines the weak operator topology (briefly WOT) on $\mathcal{M}$, that is, WOT is weaker than $\sigma(\mathcal{M}, \mathcal{M}_*)$.

The related WOT-continuous functionals on $\mathcal{M}$ consists of those $\omega_{\zeta,\eta}$ with finite support elements ζ,η from $\ell^2(H)$ (see [24, 4.6.4]). The identity mapping $(\mathrm{ball}\,\mathcal{M},\sigma(\mathcal{M},\mathcal{M}_*))\rightarrow(\mathrm{ball}\,\mathcal{M},\mathrm{WOT})$ is continuous, and $\mathrm{ball}\,\mathcal{M}=\mathrm{ball}\,(\mathcal{M}_*)^*$ is a w^*-compact set by Alaoglu's Theorem [24, 2.5.2], that is, $(\mathrm{ball}\,\mathcal{M},\sigma(\mathcal{M},\mathcal{M}_*))$ is a compact topological space, which in turn implies that the identity mapping is a homeomorphism (see [24, 1.6.8]). Hence WOT$=\sigma(\mathcal{M},\mathcal{M}_*)$ one every bounded subset of $\mathcal{M}$. Based on Lemma 2.2, we conclude that the w^*-topology of $\mathrm{ball}\,\mathcal{B}(H)$ can be given by means of the family $\{p_{\zeta,\eta}:\zeta,\eta\in S\}$ for a subset $S\subseteq H$ whose span is dense in H. Finally, note that the family $p_\zeta(y)=\|y\zeta\|$, $y\in\mathcal{M}$, $\zeta\in H$ defines the strong operator topology (briefly SOT). Since $p_{\zeta,\eta}(y)=|(y\zeta,\eta)|\leq\|y\zeta\|\,\|\eta\|=p_\zeta(y)\,\|\eta\|$, $y\in\mathcal{M}$ it follows that WOT is weaker than SOT. Moreover, both have the same supply of continuous linear functionals (in the case of $\mathcal{M}=\mathcal{B}(H)$ it is reduced to $\mathcal{B}_f(H)$), therefore the WOT-closure of a convex subset of $\mathcal{M}$ coincides with its SOT-closure.

Remark 2.1. Let $\mathcal{S}\subseteq\mathcal{B}(H)$ be a non-empty subset. Then the commutant $\mathcal{S}'$ is a unital, WOT-closed, subspace of $\mathcal{B}(H)$. Indeed, take a net $(x_\lambda)_{\lambda\in\Lambda}\subseteq\mathcal{S}'$ with $x=$SOT-$\lim_\lambda x_\lambda\in\mathcal{B}(H)$. For every $y\in\mathcal{S}$ and $\zeta\in H$ we have

$$p_\zeta(xy-yx)=\|xy\zeta-yx\zeta\|\leq\|xy\zeta-x_\lambda y\zeta\|$$

$$+\|x_\lambda y\zeta-yx\zeta\|=p_{y\zeta}(x-x_\lambda)+\|yx_\lambda\zeta-yx\zeta\|$$

$$=p_{y\zeta}(x-x_\lambda)+\|y\|\,\|x_\lambda\zeta-x\zeta\|=p_{y\zeta}(x-x_\lambda)+\|y\|\,p_\zeta(x_\lambda-x)\rightarrow 0$$

for large λ, that is, $p_\zeta(xy-yx)=0$ for all ζ. Hence $xy=yx$ for all $y\in\mathcal{S}$, that is, $x\in\mathcal{S}'$. Thus $\mathcal{S}'$ is a unital SOT-closed, subspace. Being a convex subset, it is WOT-closed automatically. Since WOT is weaker than w^*-topology $\sigma\left(\mathcal{B}(H),\mathcal{B}^1(H)\right)$, we conclude that $\mathcal{S}'$ is a unital, w^*-closed, subspace. If S is a $*$-subspace (that is, $S^*\subseteq S$) then $\mathcal{S}'$ is a unital w^*-closed, $*$-subalgebra.

Now let $e\in\mathcal{B}(H)$ be a projection with $S=e(H)$, and consider the reduction mapping $\mathcal{B}(H)\rightarrow e\mathcal{B}(H)e=\mathcal{B}(S)$, $x\mapsto exe=x_S$, which is a $*$-linear mapping. For all $\zeta,\eta\in\ell^2(S)$ we have $p_{\zeta,\eta}(x_S)=|\langle x_S,\omega_{\zeta,\eta}\rangle|=|\sum\langle x\zeta_n,\eta_n\rangle|=p_{\zeta,\eta}(x)$, which means that the restriction is w^*-continuous as well (see Proposition 2.2). For a subset $\mathcal{R}\subseteq\mathcal{B}(H)$ we use the notation $\mathcal{R}_e$ to denote the range $\{x_S:x\in\mathcal{R}\}$.

Corollary 2.1. *Let $\mathcal{M}$ be a w^*-closed, $*$-subspace in $\mathcal{B}(H)$. Then $\mathcal{M}_e$ is a w^*-closed, $*$-subspace of $\mathcal{B}(S)$. If $e\in\mathcal{M}$ then $\mathcal{M}'_e\subseteq(\mathcal{M}_e)'$.*

Proof. Since $\mathrm{ball}\,\mathcal{B}(H)$ is w^*-compact, so is $\mathcal{M}\cap\mathrm{ball}\,\mathcal{B}(H)$. Since the reduction is a w^*-continuous mapping, it follows that $\mathcal{M}_e\cap\mathrm{ball}\,\mathcal{B}(S)=\mathrm{ball}\,\mathcal{M}_e=e(\mathrm{ball}\,\mathcal{M})e$ is w^*-compact in $\mathcal{B}(S)$. So are all $\mathcal{M}_e\cap n\,\mathrm{ball}\,\mathcal{B}(S)$, $n\in\mathbb{N}$. Using Krein-Smulian Theorem (see [24, 2.5.9]), we conclude that $\mathcal{M}_e$ is a w^*-closed, $*$-subspace in $\mathcal{B}(S)$. Further, suppose $\mathcal{M}$ contains the projection e. Take $y\in\mathcal{M}'$ and $x\in\mathcal{M}$. Then $y_Sx_S=(yexe)_S=(exey)_S=x_Sy_S$, that is, $y_S\in(\mathcal{M}_e)'$. □

Remark 2.2. In the general case, the mapping $\mathcal{B}(H)\rightarrow\mathcal{B}(H)$, $x\mapsto exf$ is w^*-continuous for every couple of projections $e,f\in\mathcal{B}(H)$, for $\langle exf,\omega_{\zeta,\eta}\rangle=\sum_n\langle xf\zeta_n,e\eta_n\rangle=\omega_{f\zeta,e\eta}(x)$, $x\in\mathcal{B}(H)$, where $f\zeta=(f\zeta_n)_n$, $e\eta=(e\eta_n)_n\in\ell^2(H)$. Using very similar arguments from Corollary 2.1, we conclude that $e\mathcal{M}f$ is w^*-closed for every w^*-closed subspace $\mathcal{M}\subseteq\mathcal{B}(H)$.

2.4. The cross-norms on the algebraic tensor product. Let X and Y be Banach spaces. A norm $\|\cdot\|$ on the algebraic tensor product $X \otimes Y$ is said to be a cross-norm if $\|x \otimes y\| = \|x\| \|y\|$ for all $x \in X$ and $y \in Y$. We have a canonical duality between $X \otimes Y$ and $X^* \otimes Y^*$ given by $\langle x \otimes y, x^* \otimes y^* \rangle = \langle x, x^* \rangle \langle y, y^* \rangle$. The injective cross-norm $\|\cdot\|_\lambda$ on $X \otimes Y$ is defined by the rule $\|z\|_\lambda = \sup \{ |\langle z, x^* \otimes y^* \rangle| : x^* \in \operatorname{ball} X^*, y^* \in \operatorname{ball} Y^* \}$. Similarly, we have the projective cross-norm $\|z\|_\pi = \inf \{ \sum_{i=1}^n \|x_i\| \|y_i\| : z = \sum_{i=1}^n x_i \otimes y_i \}$. Obviously, $\|\cdot\|_\alpha \leq \|\cdot\|_\pi$ (or simply $\alpha \leq \pi$) for a cross-norm $\|\cdot\|_\alpha$ on $X \otimes Y$. Moreover, every cross-norm α defines a seminorm α^* on $X^* \otimes Y^*$ by the rule $\|f\|_{\alpha^*} = \sup \{ |\langle z, f \rangle| : \|z\|_\alpha \leq 1 \}$.

Proposition 2.3. *Let α be a cross-norm on $X \otimes Y$. Then $\lambda \leq \alpha \leq \pi$ iff α^* is a cross-norm on $X^* \otimes Y^*$.*

Proof. If α^* is a cross-norm on $X^* \otimes Y^*$ then

$$\begin{aligned} \|z\|_\lambda &\leq \sup \{ \|z\|_\alpha \|x^* \otimes y^*\|_{\alpha^*} : x^* \in \operatorname{ball} X^*, y^* \in \operatorname{ball} Y^* \} \\ &= \sup \{ \|z\|_\alpha \|x^*\| \|y^*\| : x^* \in \operatorname{ball} X^*, y^* \in \operatorname{ball} Y^* \} \leq \|z\|_\alpha \end{aligned}$$

for all $z \in X \otimes Y$, that is, $\lambda \leq \alpha \leq \pi$. Conversely, suppose $\|\cdot\|_\alpha$ is a cross-norm on $X \otimes Y$ such that $\lambda \leq \alpha \leq \pi$. Then

$$\begin{aligned} \|x^*\| \|y^*\| &= \sup \{ |\langle \operatorname{ball} X, x^* \rangle| \, |\langle \operatorname{ball} Y, y^* \rangle| \} \\ &= \sup \{ |\langle x \otimes y, x^* \otimes y^* \rangle| : x \in \operatorname{ball} X, y \in \operatorname{ball} Y \} \\ &\leq \sup \{ |\langle x \otimes y, x^* \otimes y^* \rangle| : \|x \otimes y\|_\alpha \leq 1 \} \leq \sup \{ |\langle z, x^* \otimes y^* \rangle| : \|z\|_\alpha \leq 1 \} \\ &= \|x^* \otimes y^*\|_{\alpha^*} \leq \|x^* \otimes y^*\|_{\lambda^*} . \end{aligned}$$

Note that $|\langle z, x^* \otimes y^* \rangle| \leq \|z\|_\lambda \|x^*\| \|y^*\|$ by its very definition of the injective cross-norm. Then $\|x^* \otimes y^*\|_{\lambda^*} = \sup \{ |\langle z, x^* \otimes y^* \rangle| : \|z\|_\lambda \leq 1 \} \leq \|x^*\| \|y^*\|$, which in turn implies that $\|x^* \otimes y^*\|_{\alpha^*} = \|x^*\| \|y^*\|$, that is, α^* is a cross-norm on $X^* \otimes Y^*$. $\square$

The norm completion of $X \otimes Y$ with respect to a cross-norm α on is denoted by $X \otimes_\alpha Y$. Take $z^* \in (X \otimes_\alpha Y)^*$. Then $|\langle x \otimes y, z^* \rangle| \leq \|x \otimes y\|_\alpha \|z^*\|_{\alpha^*} = \|x\| \|y\| \|z^*\|_{\alpha^*}$. Define $\Phi(z^*) : Y \to X^*$, $\langle x, \Phi(z^*) y \rangle = \langle x \otimes y, z^* \rangle$. Then

$$\begin{aligned} \|\Phi(z^*)\| &= \sup \|\Phi(z^*) \operatorname{ball} Y\| = \sup |\langle \operatorname{ball} X, \Phi(z^*) \operatorname{ball} Y \rangle| \\ &= \sup |\langle (\operatorname{ball} X) \otimes (\operatorname{ball} Y), z^* \rangle| \leq \|z^*\|_{\alpha^*} , \end{aligned}$$

that is, $\Phi : (X \otimes_\alpha Y)^* \to \mathcal{B}(Y, X^*)$ is a well defined linear contraction. A slight modification of Φ generates a linear contraction $(X \otimes_\alpha Y)^* \to \mathcal{B}(X, Y^*)$ as well.

Proposition 2.4. *The mapping $\Phi : (X \otimes_\pi Y)^* \to \mathcal{B}(Y, X^*)$ implements an isometry of $(X \otimes_\pi Y)^*$ onto $\mathcal{B}(Y, X^*)$.*

Proof. As we have just seen Φ is a contraction for every cross-norm α. Prove that it is an isometry for the projective cross-norm π. Take $z^* \in (X \otimes_\pi Y)^*$ and $z = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$. Then

$$|\langle z, z^* \rangle| \leq \sum_i |\langle x_i \otimes y_i, z^* \rangle| = \sum_i |\langle x_i, \Phi(z^*) y_i \rangle| \leq \sum_i \|x_i\| \|y_i\| \|\Phi(z^*)\| ,$$

which in turn implies that $|\langle z, z^* \rangle| \leq \|z\|_\pi \|\Phi(z^*)\|$. That is, $\|z^*\|_{\pi^*} = \sup |\langle \operatorname{ball}(X \otimes_\pi Y), z^* \rangle| = \sup |\langle \operatorname{ball}(X \otimes Y), z^* \rangle| \leq \|\Phi(z^*)\|$. Finally, take $T \in \mathcal{B}(Y, X^*)$. Define $z^* : X \otimes Y \to \mathbb{C}$ to be $z^*(x \otimes y) = \langle x, Ty \rangle$. Since $|\langle z, z^* \rangle| \leq \sum_i |\langle x_i \otimes y_i, z^* \rangle| = \sum_i |\langle x_i, Ty_i \rangle| \leq \sum_i \|x_i\| \|y_i\| \|T\|$, we derive that $z^* \in (X \otimes_\pi Y)^*$ and $\|z^*\| \leq \|T\|$. But $T = \Phi(z^*)$. Whence Φ is an isometry of $(X \otimes_\pi Y)^*$ onto $\mathcal{B}(Y, X^*)$. □

Thus $(X \otimes_\pi Y)^* = \mathcal{B}(Y, X^*) = \mathcal{B}(X, Y^*)$ up to isometric isomorphisms from Proposition 2.4.

Proposition 2.5. *Let X and Y be Banach spaces. If $z \in X \otimes_\pi Y$ then $\|z\| = \inf \sum_{n=1}^\infty \|x_n\| \|y_n\|$, where the greatest lower bound is taken over all absolutely convergent expansions $z = \sum_{n=1}^\infty x_n \otimes y_n$.*

Proof. Take $z \in X \otimes_\pi Y$ and choose a sequence $(u_n)_n \subseteq X \otimes Y$ with $\|z - u_n\| \leq 3^{-1}2^{-n}(\|z\| + \varepsilon/2)$. Put $z_n = u_n - u_{n-1}$ with $u_{-1} = 0$. Then $z = \sum_{n=1}^\infty z_n$ and

$$\sum_{n=1}^\infty \|z_n\| \leq \sum_{n=1}^\infty \|z - u_n\| + \|z - u_{n-1}\|$$

$$\leq \sum_{n=1}^\infty \frac{1}{3}\left(\frac{1}{2^n} + \frac{1}{2^{n-1}}\right)(\|z\| + \varepsilon/2) = \|z\| + \varepsilon/2.$$

But $\|z_n\| + \varepsilon 2^{-n-1} \geq \sum_{k=1}^{m_n} \|x_{nk}\| \|y_{nk}\|$ for some representation $z_n = \sum_{k=1}^{m_n} x_{nk} \otimes y_{nk}$. Hence $z = \sum_{n=1}^\infty \sum_{k=1}^{m_n} x_{nk} \otimes y_{nk}$ with $\sum_{n=1}^\infty \sum_{k=1}^{m_n} \|x_{nk}\| \|y_{nk}\| \leq \sum_{n=1}^\infty \|z_n\| + \varepsilon 2^{-n-1} \leq \|z\| + \varepsilon$. □

Corollary 2.2. *Let $T_i : X_i \to Y_i$, $i = 1, 2$ be quotient mappings of Banach spaces. Then so is $T_1 \otimes T_2 : X_1 \otimes_\pi X_2 \to Y_1 \otimes_\pi Y_2$, $(T_1 \otimes T_2)(x_1 \otimes x_2) = T_1 x_1 \otimes T_2 x_2$.*

Proof. Take $y \in Y_1 \otimes_\pi Y_2$. For $\varepsilon > 0$ we have $\|y\| + \varepsilon > \sum_{n=1}^\infty \|y_{1,n}\| \|y_{2,n}\|$ for some expansion $y = \sum_{n=1}^\infty y_{1,n} \otimes y_{2,n}$ by Proposition 2.5. Choose positive small $\varepsilon_{i,n}$, $i = 1, 2$ such that $\sum_{n=1}^\infty (\|y_{1,n}\| + \varepsilon_{1,n})(\|y_{2,n}\| + \varepsilon_{2,n}) \leq \|y\| + \varepsilon$. Pick $x_{i,n} \in X_i$ so that $T_i(x_{i,n}) = y_{i,n}$, $\|x_{i,n}\| \leq \|y_{i,n}\| + \varepsilon_{i,n}$ for $i = 1, 2$. Put $x = \sum_{n=1}^\infty x_{1,n} \otimes x_{2,n} \in X_1 \otimes_\pi X_2$. Then $(T_1 \otimes T_2) x = \sum_{n=1}^\infty y_{1,n} \otimes y_{2,n} = y$ and

$$\|x\| \leq \sum_{n=1}^\infty \|x_{1,n}\| \|x_{2,n}\| \leq \sum_{n=1}^\infty (\|y_{1,n}\| + \varepsilon_{1,n})(\|y_{2,n}\| + \varepsilon_{2,n}) \leq \|y\| + \varepsilon.$$

Taking into account that $\|T_1 \otimes T_2\| \leq 1$, we conclude that $\|y\| = \inf\{\|x\| : (T_1 \otimes T_2) x = y\}$, that is, $T_1 \otimes T_2$ is a quotient mapping. □

Now let H_1 and H_2 be Hilbert spaces. For elements $\zeta = \sum_{i=1}^n \zeta_{1,i} \otimes \zeta_{2,i}$, $\eta = \sum_{j=1}^m \eta_{1,j} \otimes \eta_{2,j}$ of $H_1 \otimes H_2$ we define $(\zeta, \eta) = \sum_{i,j} (\zeta_{1,i}, \eta_{1,j})(\zeta_{2,i}, \eta_{2,j})$. Using an orthonormal basis for the closed subspace $\langle \zeta_{2,i}, \eta_{2,j} : i, j \rangle$ one can easily prove that $(\cdot, \cdot)$ is a well defined sesquilinear form on $H_1 \otimes H_2$. Note that if $\zeta = \sum_{i=1}^n \zeta_i \otimes \epsilon_i$ with an orthonormal set $\{\epsilon_i\}$ in H_2 then $(\zeta, \zeta) = \sum_i (\zeta_i, \zeta_i) = \sum_{i=1}^n \|\zeta_i\|^2 \geq 0$, that is, $(\cdot, \cdot)$ is an inner product on $H_1 \otimes H_2$. Since $\|\zeta_1 \otimes \zeta_2\|^2 = (\zeta_1 \otimes \zeta_2, \zeta_1 \otimes \zeta_2) = (\zeta_1, \zeta_1)(\zeta_2, \zeta_2) = \|\zeta_1\|^2 \|\zeta_2\|^2$, we deal with the cross-norm σ on $H_1 \otimes H_2$ whose completion $H_1 \otimes_\sigma H_2$ is a Hilbert space called the Hilbert space tensor product. Take an orthonormal basis $(\epsilon_i)_{i \in I}$ (or just I) for H_2. As we have confirmed above every element $\zeta \in H_1 \otimes H_2$ admits a unique expansion $\zeta = \sum_{i \in F} \zeta_i \otimes \epsilon_i$ for a finite subset $F \subseteq I$. Moreover, $\|\zeta\|^2 = (\zeta, \zeta) =$

$\sum_{i,j} (\zeta_i, \zeta_j) (\epsilon_i, \epsilon_j) = \sum_{i\in F} \|\zeta_i\|^2$. It follows that the linear mapping

$$U : \bigoplus_I H_1 \to H_1 \otimes_\sigma H_2, \quad U\left((\zeta_i)_{i\in I}\right) = \sum_{i\in I} \zeta_i \otimes \epsilon_i$$

is an isometry of the Hilbert sum $\bigoplus_I H_1$ onto $H_1 \otimes_\sigma H_2$. In particular, there are isometries $U_i : H_1 \to H_1 \otimes_\sigma H_2$, $U_i(\zeta) = \zeta \otimes \epsilon_i$, $i \in I$ such that $U_i(H_1) \perp U_j(H_1)$ for $i \neq j$, and $H_1 \otimes_\sigma H_2 = \bigoplus_{i\in I} U_i(H_1)$. Note also that $U_i^* : \bigoplus_I H_1 \to H_1$ are the canonical projections $(\eta_j)_{j\in I} \mapsto \eta_i$, for

$$(U_i\zeta, \eta) = \left(U_i\zeta, \sum_{j\in I} \eta_j \otimes \epsilon_j\right) = \left(\zeta \otimes \epsilon_i, \sum_{j\in I} \eta_j \otimes \epsilon_j\right) = (\zeta, \eta_i) = (\zeta, U_i^*\eta)$$

for all $\zeta \in H_1$ and $\eta \in \bigoplus_I H_1$. Note also that $U_i U_j^* \left(\sum_k \zeta_k \otimes \epsilon_k\right) = \zeta_j \otimes \epsilon_i$.

Corollary 2.3. *For the Hilbert spaces H_1 and H_2 we have $\lambda \leq \sigma \leq \pi$.*

Proof. Take $f = \overline{\eta_1} \otimes \overline{\eta_2} \in H_1^* \otimes H_2^*$, where $\eta_i \in H_i$ and $\langle \zeta_i, \overline{\eta_i}\rangle = (\zeta_i, \eta_i)$ for all $\zeta_i \in H_i$. Take $\zeta \in \operatorname{ball} H_1 \otimes_\sigma H_2$ with $\zeta = \sum_i \zeta_i \otimes \epsilon_i$. Then $\|\zeta\|^2 = \sum_i \|\zeta_i\|^2 \leq 1$ and $\langle \zeta, f\rangle = \sum_i \langle \zeta_i \otimes \epsilon_i, f\rangle = \sum_i \langle \zeta_i, \eta_1\rangle \langle \epsilon_i, \eta_2\rangle$. It follows that

$$\begin{aligned}|\langle \zeta, f\rangle| &\leq \sum_i |\langle \zeta_i, \eta_1\rangle| \, |\langle \eta_2, \epsilon_i\rangle| \leq \|\eta_1\| \sum_i \|\zeta_i\| \, |\langle \eta_2, \epsilon_i\rangle| \\ &\leq \|\eta_1\| \left(\sum_i \|\zeta_i\|^2\right)^{1/2} \left(\sum_i |\langle \eta_2, \epsilon_i\rangle|^2\right)^{1/2} \\ &= \|\eta_1\| \, \|\zeta\| \, \|\eta_2\| \leq \|\overline{\eta_1}\| \, \|\overline{\eta_2}\|,\end{aligned}$$

which in turn implies that $\|f\|_{\sigma^*} = \sup |\langle \operatorname{ball} H_1 \otimes_\sigma H_2, f\rangle| \leq \|\overline{\eta_1}\| \, \|\overline{\eta_2}\|$. Conversely,

$$\begin{aligned}\|\overline{\eta_1}\| \, \|\overline{\eta_2}\| &= \sup\{|(\zeta_1, \eta_1)(\zeta_2, \eta_2)| : \zeta_i \in \operatorname{ball} H_i\} \\ &= \sup\{|(\zeta_1 \otimes \zeta_2, \eta_1 \otimes \eta_2)| : \zeta_1 \otimes \zeta_2 \in \operatorname{ball} H_1 \otimes_\sigma H_2\} \\ &\leq \sup\{|\langle \zeta, f\rangle| : \zeta \in \operatorname{ball} H_1 \otimes_\sigma H_2\} = \|f\|_{\sigma^*},\end{aligned}$$

that is, $\|\cdot\|_{\sigma^*}$ is a cross-norm on $H_1^* \otimes H_2^*$. Using Proposition 2.3, we conclude that $\lambda \leq \sigma \leq \pi$. ☐

Now let $x_1 \in \mathcal{B}(H_1)$. The operator $x_1 \otimes 1$ on $H_1 \otimes H_2$ is bounded, for

$$\begin{aligned}\|(x_1 \otimes 1)\zeta\|^2 &= \left\|(x_1 \otimes 1)\sum_{i\in F} \zeta_i \otimes \epsilon_i\right\|^2 = \left\|\sum_{i\in F} x_1\zeta_i \otimes \epsilon_i\right\|^2 = \sum_{i\in F} \|x_1\zeta_i\|^2 \\ &\leq \sum_{i\in F} \|x_1\|^2 \|\zeta_i\|^2 = \|x_1\|^2 \|\zeta\|^2.\end{aligned}$$

Thus $x_1 \otimes 1$ admits an extension up to a bounded operator $x_1 \otimes 1 \in \mathcal{B}(H_1 \otimes_\sigma H_2)$ and $\|x_1 \otimes 1\| \leq \|x_1\|$. But

$$\begin{aligned}\|x_1\| &= \sup \|x_1(\operatorname{ball} H_1)\| \\ &= \sup \|x_1(\operatorname{ball} H_1) \otimes \epsilon_i\| = \sup \|(x_1 \otimes 1)((\operatorname{ball} H_1) \otimes \epsilon_i)\| \\ &\leq \|x_1 \otimes 1\| \sup \|(\operatorname{ball} H_1) \otimes \epsilon_i\| \leq \|x_1 \otimes 1\|,\end{aligned}$$

that is, $\|x_1 \otimes 1\| = \|x_1\|$. Similarly, using an orthonormal basis $(\vartheta_v)_{v\in J}$ for H_1 we derive that $H_1 \otimes_\sigma H_2 = \bigoplus_J H_2$ up to an isometric isomorphism and $1 \otimes x_2 \in$

$\mathcal{B}(H_1 \otimes_\sigma H_2)$ with $\|1 \otimes x_2\| = \|x_2\|$. In particular, $x_1 \otimes x_2 = (x_1 \otimes 1)(1 \otimes x_2) \in \mathcal{B}(H_1 \otimes_\sigma H_2)$ and $\|x_1 \otimes x_2\| \leq \|x_1\| \|x_2\|$. Moreover,

$$\begin{aligned}(\|x_1\| - \varepsilon)(\|x_2\| - \varepsilon) &\leq \|x_1\zeta_1\| \|x_2\zeta_2\| = ((x_1\zeta_1, x_1\zeta_1)(x_2\zeta_2, x_2\zeta_2))^{1/2} \\ &= (x_1\zeta_1 \otimes x_2\zeta_2, x_1\zeta_1 \otimes x_2\zeta_2)^{1/2} = \|x_1\zeta_1 \otimes x_2\zeta_2\| \\ &\leq \|x_1 \otimes x_2\| \|\zeta_1 \otimes \zeta_2\| \leq \|x_1 \otimes x_2\|\end{aligned}$$

for some unit vectors $\zeta_1 \in H_1$ and $\zeta_2 \in H_2$, that is, $\|x_1 \otimes x_2\| = \|x_1\| \|x_2\|$. Note also that

$$\begin{aligned}((x_1 \otimes x_2)(\zeta_1 \otimes \zeta_2), \eta_1 \otimes \eta_2) &= (x_1\zeta_1, \eta_1)(x_2\zeta_2, \eta_2) = (\zeta_1, x_1^*\eta_1)(\zeta_2, x_2^*\eta_2) \\ &= (\zeta_1 \otimes \zeta_2, (x_1^* \otimes x_2^*)(\eta_1 \otimes \eta_2))\end{aligned}$$

for all $\zeta_i \in H_1$ and $\eta_i \in H_2$. It follows that $(x_1 \otimes x_2)^* = x_1^* \otimes x_2^*$.

Corollary 2.4. *Let $e_i \in \mathcal{B}(H_i)$, $i = 1, 2$ be projections. Then $e_1 \otimes e_2 \in \mathcal{B}(H_1 \otimes_\sigma H_2)$ is a projection and $(e_1 \otimes e_2)(H_1 \otimes_\sigma H_2) = e_1H_1 \otimes_\sigma e_2H_2$.*

Proof. First note that $(e_1 \otimes e_2)^* = e_1^* \otimes e_2^* = e_1 \otimes e_2$ and $(e_1 \otimes e_2)^2 = $ $= (e_1 \otimes 1)(1 \otimes e_2)(e_1 \otimes 1)(1 \otimes e_2) = (e_1 \otimes 1)^2 (1 \otimes e_2)^2 = \left(e_1^2 \otimes 1\right)\left(1 \otimes e_2^2\right) = $ $e_1 \otimes e_2$, that is, $e_1 \otimes e_2$ is a projection. Take an orthonormal basis $(\epsilon_i)_{i \in J}$ for $e_2(H_2)$, and extend it to an orthonormal basis $(\epsilon_i)_{i \in I}$ for H_2, where $J \subseteq I$. Then

$$\begin{aligned}(e_1 \otimes e_2)(H_1 \otimes_\sigma H_2) &= (e_1 \otimes e_2)\left(\bigoplus_{i \in I} U_i(H_1)\right) = (e_1 \otimes e_2)\left(\bigoplus_{i \in I} H_1 \otimes \epsilon_i\right) \\ &= \bigoplus_{i \in J} e_1H_1 \otimes \epsilon_i = e_1H_1 \otimes_\sigma e_2H_2,\end{aligned}$$

that is, $e_1 \otimes e_2$ is the projection onto $e_1H_1 \otimes_\sigma e_2H_2$. □

2.5. Bounded matrices. Now let $\mathcal{M}_I(\mathcal{B}(H_1))$ be the $*$-algebra of all $I \times I$-matrices $x = [x_{ij}]_{i,j \in I}$ over the algebra $\mathcal{B}(H_1)$, where I is an orthonormal basis for H_2. Recall that $[x_{ij}]_{i,j \in I}^* = [x_{ji}]_{i,j \in I}^*$. Every $x \in \mathcal{M}_I(\mathcal{B}(H_1))$ is acting on $H_1 \otimes_\sigma H_2 = \bigoplus_I H_1$ as $x(\zeta_i)_{i \in I} = \left(\sum_j x_{ij}\zeta_j\right)_{i \in I}$. Put $M_I(\mathcal{B}(H_1)) = \{x \in \mathcal{M}_I(\mathcal{B}(H_1)) : x \in \mathcal{B}(H_1 \otimes_\sigma H_2)\}$ to be the $*$-subalgebra of all bounded matrices.

Lemma 2.5. *There is a canonical identification $\mathcal{B}(H_1 \otimes_\sigma H_2) = M_I(\mathcal{B}(H_1))$ of $*$-algebras.*

Proof. Take $x \in \mathcal{B}(H_1 \otimes_\sigma H_2)$ and define $x_{ij} = U_i^* x U_j \in \mathcal{B}(H_1)$ for all $i, j \in I$. For every $\zeta = \sum_{j \in I} \zeta_j \otimes \epsilon_j \in H_1 \otimes_\sigma H_2$ we have

$$\begin{aligned}x\zeta &= \sum_k x(\zeta_j \otimes \epsilon_j) = \sum_j xU_j(\zeta_j) \\ &= \sum_j \sum_i U_i^* x U_j(\zeta_j) \otimes \epsilon_i = \sum_i \left(\sum_j U_i^* x U_j(\zeta_j)\right) \otimes \epsilon_i \\ &= \sum_i \left(\sum_j x_{ij}\zeta_j\right) \otimes \epsilon_i = [x_{ij}]_{i,j \in I}\, \zeta,\end{aligned}$$

that is, x is identified with the matrix $[x_{ij}]_{i,j\in I}$. For $\zeta,\eta\in H_1\otimes_\sigma H_2$ we have

$$(x\zeta,\eta)=\sum_i\left(\sum_j x_{ij}\zeta_j,\eta_i\right)=\sum_{i,j}\left(\zeta_j,x_{ij}^*\eta_i\right)$$
$$=\sum_j\left(\zeta_j,\sum_i x_{ij}^*\eta_i\right)=\sum_i\left(\zeta_i,\sum_j x_{ji}^*\eta_j\right)$$
$$=\left(\zeta,[x_{ji}^*]_{i,j\in I}\,\eta\right),$$

which means that $x^*=\left[x_{ji}^*\right]_{i,j\in I}$. It is a routine calculation to verify that the matrix of xy is the multiplication of related matrices. Thus the mapping $x\mapsto[x_{ij}]_{i,j\in I}$ implements the $*$-isomorphism of $\mathcal{B}(H_1\otimes_\sigma H_2)$ onto $M_I(\mathcal{B}(H_1))$. □

Note that the matrix of $U_iU_j^*\left(\sum_k\zeta_k\otimes\epsilon_k\right)=\zeta_j\otimes\epsilon_i$ is the elementary matrix $e_{ij}=[\delta_{ij}1]$. Put $\mathcal{E}=\{e_{ij}:i,j\in I\}$ to be the set of all elementary matrices, and

$$\mathcal{B}(H_1)\otimes\mathbb{C}=\{x_1\otimes 1\in\mathcal{B}(H_1\otimes_\sigma H_2):x_1\in\mathcal{B}(H_1)\},$$

which is a $*$-subalgebra in $\mathcal{B}(H_1\otimes_\sigma H_2)$.

Corollary 2.5. *The matrix of $x_1\otimes 1$ is diagonal with the same diagonal entry x_1. Moreover, $\mathcal{B}(H_1)\otimes\mathbb{C}=\mathcal{E}'$.*

Proof. First note that $U_i^*(x_1\otimes 1)U_j\zeta_j=U_i^*(x_1\zeta_j\otimes\epsilon_j)=\delta_{ij}x_1\zeta_j$ for all $\zeta_j\in H_1$, that is, $(x_1\otimes 1)_{ij}=U_i^*(x_1\otimes 1)U_j=\delta_{ij}x_1$ by Lemma 2.5. Thus $x_1\otimes 1=[\delta_{ij}x_1]_{i,j\in I}$. Further, based on Lemma 2.5, we derive that $e_{ij}x$ has only nontrivial ith row $[x_{jk}]_{k\in I}$ whereas xe_{ij} has only nontrivial jth column $[x_{ik}]_{k\in I}$, where $x\in M_I(\mathcal{B}(H_1))$. In particular, $(e_{ij}x)_{ij}=x_{jj}$ and $(xe_{ij})_{ij}=x_{ii}$. Therefore $x\in\mathcal{E}'$ iff $x_{jk}=0$ for all j,k, $j\neq k$, and $x_{jj}=x_{ii}$ for all i,j, that is, $x=[\delta_{ij}x_1]$ with $x_1=x_{ii}$. Whence $\mathcal{B}(H_1)\otimes\mathbb{C}=\mathcal{E}'$. □

Now consider the faithful C^*-representation $\pi:\mathcal{B}(H_1)\to\mathcal{B}(H_1\otimes_\sigma H_2)$, $\pi(x_1)=x_1\otimes 1$ called the diagonal representation. For a subset $\mathcal{S}\subseteq\mathcal{B}(H_1)$ we obtain the set $\pi(\mathcal{S})$ of diagonal matrices such that $\pi(\mathcal{S})\subseteq\mathcal{B}(H_1)\otimes\mathbb{C}=\mathcal{E}'$ by Corollary 2.5. Therefore $\mathcal{E}\subseteq\mathcal{E}''\subseteq\pi(\mathcal{S})'$. The set of all bounded matrices over $\mathcal{S}'$ is denoted by $M_I(\mathcal{S}')$, which is a $*$-subalgebra of $M_I(\mathcal{B}(H_1))$ (see Lemma 2.5).

Corollary 2.6. *For a subset $\mathcal{S}\subseteq\mathcal{B}(H_1)$ we have $\pi(\mathcal{S})'=M_I(\mathcal{S}')$. In particular, $\pi(\mathcal{S})''=\pi(\mathcal{S}'')$.*

Proof. Note that $x\in\pi(\mathcal{S})'$ iff

$$[x_{ij}y_1]_{i,j\in I}=[x_{ij}]_{i,j\in I}[\delta_{ij}y_1]_{i,j\in I}=[\delta_{ij}y_1]_{i,j\in I}[x_{ij}]_{i,j\in I}=[y_1x_{ij}]_{i,j\in I}$$

for all $y_1\in\mathcal{S}$. The latter is possible iff $x_{ij}y_1=y_1x_{ij}$ for all i,j, that is, $x\in M_I(\mathcal{S}')$. Consequently, $\pi(\mathcal{S})'=M_I(\mathcal{S}')$. Further, if $x_1\in\mathcal{S}''$ then

$$[x_1,\mathcal{S}']=\{0\}\Rightarrow[\pi(x_1),\pi(\mathcal{S}')]$$
$$=\{0\}\Rightarrow[\pi(x_1),M_I(\mathcal{S}')]=\{0\}\Rightarrow[\pi(x_1),\pi(\mathcal{S})']=\{0\}$$
$$\Rightarrow\pi(x_1)\in\pi(\mathcal{S})'',$$

that is, $\pi(\mathcal{S}'') \subseteq \pi(\mathcal{S})''$. Conversely, take $x \in \pi(\mathcal{S})''$. Since $\mathcal{E} \subseteq \pi(\mathcal{S})'$, it follows that $x \in \mathcal{E}'$. Using Corollary 2.5, we conclude that $x = \pi(x_1)$ for a certain $x_1 \in \mathcal{B}(H_1)$. Then

$$\left[\pi(x_1), \pi(\mathcal{S})'\right] = \{0\} \Rightarrow \left[\pi(x_1), M_I(\mathcal{S}')\right]$$

$$= \{0\} \Rightarrow \left[\pi(x_1), \pi(\mathcal{S}')\right] = \{0\} \Rightarrow \left[x_1, \mathcal{S}'\right] = \{0\} \Rightarrow x_1 \in \mathcal{S}'',$$

that is, $x \in \pi(\mathcal{S}'')$. Whence $\pi(\mathcal{S})'' = \pi(\mathcal{S}'')$. □

3. Background of von Neumann algebras

In this section we provide the survey with basic facts on von Neumann algebras.

3.1. Nondegenerate $*$-subalgebras. Let H be a Hilbert space and let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a $*$-subalgebra. Note that $\mathcal{M} \subseteq \mathcal{M}''$, and in the equality case we say that $\mathcal{M}$ is a von Neumann algebra, which is a unital C^*-algebra automatically. A von Neumann algebra $\mathcal{M}$ with the trivial center $\mathcal{Z}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$ is called a factor. So is the C^*-algebra $\mathcal{B}(H)$. Since $\mathcal{M}' = \mathcal{M}'''$ for every $*$-subalgebra $\mathcal{M}$, it follows that $\mathcal{M}'$ is a von Neumann algebra.

Lemma 3.1. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a $*$-subalgebra, $e \in \mathcal{B}(H)$ a projection such that $\mathcal{M}e(H) \subseteq e(H)$. Then $e \in \mathcal{M}'$. If $\mathcal{M}'e(H) \subseteq e(H)$ for a projection e and von Neumann algebra $\mathcal{M}$, then $e \in \mathcal{M}$. In particular, the projection e_ζ onto $\langle \mathcal{M}'\zeta \rangle$ belongs to $\mathcal{M}$, whereas the projection e'_ζ onto $\langle \mathcal{M}\zeta \rangle$ belongs to $\mathcal{M}'$, where $\zeta \in H$.*

Proof. By assumption, $xe = exe$ for every $x \in \mathcal{M}$. Since $x^* \in \mathcal{M}$ for $x \in \mathcal{M}$, we derive that $xe = exe = (exe)^{**} = (ex^*e)^* = (x^*e)^* = ex$, that is, $e \in \mathcal{M}'$. Now let $\mathcal{M}$ be a von Neumann algebra and e a projection with $\mathcal{M}'e(H) \subseteq e(H)$. Then $e \in \mathcal{M}'' = \mathcal{M}$. Finally, $\mathcal{M}\langle \mathcal{M}\zeta \rangle \subseteq \langle \mathcal{M}\zeta \rangle$ and $\mathcal{M}'\langle \mathcal{M}'\zeta \rangle \subseteq \langle \mathcal{M}'\zeta \rangle$ for $\zeta \in H$, therefore $e'_\zeta \in \mathcal{M}'$ and $e_\zeta \in \mathcal{M}$. □

Lemma 3.2. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a $*$-subalgebra such that $\langle \mathcal{M}H \rangle = H$, that is, $\mathcal{M}$ is a nondegenerate $*$-subalgebra. Then $\langle \mathcal{M}\zeta \rangle = \langle \mathcal{M}''\zeta \rangle$ for every $\zeta \in H$.*

Proof. Let e be a projection onto the $\mathcal{M}$-invariant closed subspace $\langle \mathcal{M}\zeta \rangle$. By Lemma 3.1, $e \in \mathcal{M}'$. Note that

$$\left(e^\perp\zeta, H\right) = \left(e^\perp\zeta, \langle \mathcal{M}H \rangle\right) = \left(e^\perp\zeta, \mathcal{M}H\right)^- = \left(\mathcal{M}^* e^\perp\zeta, H\right)^- = \left(\mathcal{M}e^\perp\zeta, H\right)^-$$

$$= \left(e^\perp\mathcal{M}\zeta, H\right)^- = \left(e^\perp e\mathcal{M}\zeta, H\right)^- = \{0\},$$

that is, $e^\perp\zeta = 0$, which in turn implies that $\zeta = e\zeta \in \langle \mathcal{M}\zeta \rangle$. Further, since $e \in \mathcal{M}'$, it follows that $ye = ey$ for all $y \in \mathcal{M}''$, thereby $\mathcal{M}''\langle \mathcal{M}\zeta \rangle \subseteq \langle \mathcal{M}\zeta \rangle$. In particular, $\mathcal{M}''\zeta \subseteq \langle \mathcal{M}\zeta \rangle$, which in turn implies that $\langle \mathcal{M}''\zeta \rangle \subseteq \langle \mathcal{M}\zeta \rangle \subseteq \langle \mathcal{M}''\zeta \rangle$, that is, $\langle \mathcal{M}\zeta \rangle = \langle \mathcal{M}''\zeta \rangle$. □

Now consider the amplification $\ell^2(H) = H \otimes_\sigma \ell^2$ and the diagonal $*$-representation $\pi : \mathcal{B}(H) \to \mathcal{B}\left(H \otimes_\sigma \ell^2\right)$, $\pi(x) = x \otimes 1$ from Subsection 2.5.

Corollary 3.1. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a $*$-subalgebra such that $\langle \mathcal{M}H \rangle = H$. Then $\langle \pi(\mathcal{M})\zeta \rangle = \langle \pi(\mathcal{M})''\zeta \rangle$ for every $\zeta \in \ell^2(H)$.*

Proof. For $\eta \in H$ we have $\eta = \lim_n \sum_{m=1}^{k_n} x_{nm}\zeta_{nm}$ for some $\{x_{nm}\} \subseteq \mathcal{M}$ and $\{\zeta_{nm}\} \subseteq H$. Then $\eta\otimes\epsilon_i = \lim_n \sum_{m=1}^{k_n} x_{nm}\zeta_{nm}\otimes\epsilon_i = \lim_n \sum_{m=1}^{k_m} \pi(x_{nm})(\zeta_{nm}\otimes\epsilon_i) \in \langle \pi(\mathcal{M})\ell^2(H)\rangle$, where $(\epsilon_i)_i$ is an orthonormal basis for ℓ^2. Thus $H\otimes\epsilon_i \subseteq \langle \pi(\mathcal{M})\ell^2(H)\rangle$ for all i, which in turn implies that $\langle \pi(\mathcal{M})\ell^2(H)\rangle = \ell^2(H)$, that is, $\pi(\mathcal{M})$ is a nondegenerate $*$-subalgebra. Using Lemma 3.2, we conclude that $\langle \pi(\mathcal{M})\zeta\rangle = \langle \pi(\mathcal{M})''\zeta\rangle$ for every $\zeta \in \ell^2(H)$. □

3.2. The bicommutant theorem of von Neumann. Now we formulate the well known result of von Neumann on the bicommutant of a $*$-subalgebra.

Proposition 3.1. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a $*$-subalgebra. If $\mathcal{M}$ is nondegenerate then $\mathcal{M}'' = \mathcal{M}^{-w^*} = \mathcal{M}^{-WOT} = \mathcal{M}^{-SOT}$.*

Proof. Assume that $\langle \mathcal{M}H\rangle = H$, and take $x \in \mathcal{M}''$. Using Corollary 2.6, we derive that $\pi(x) \in \pi(\mathcal{M})''$. For every $\zeta \in \ell^2(H)$, we obtain that $\pi(x)\zeta \in \pi(\mathcal{M})''\zeta \subseteq \langle \pi(\mathcal{M})\zeta\rangle$ thanks to Corollary 3.1. Take $\varepsilon > 0$ and nonzero $\zeta, \eta \in \ell^2(H)$. Then $\|\pi(x)\zeta - \pi(y)\zeta\| \leq \varepsilon\|\eta\|^{-1}$ for a certain $y \in \mathcal{M}$. It follows that

$$p_{\zeta,\eta}(x-y) = \left|\sum_n ((x-y)\zeta_n, \eta_n)\right| \leq \sum_n |((x-y)\zeta_n, \eta_n)| \leq \sum_n \|(x-y)\zeta_n\|\|\eta_n\|$$

$$\leq \left(\sum_n \|(x-y)\zeta_n\|^2\right)^{1/2}\left(\sum_n \|\eta_n\|^2\right)^{1/2} = \|\pi(x-y)\zeta\|\|\eta\| \leq \varepsilon,$$

that is, $x \in \mathcal{M}^{-w^*}$. As we have confirmed above in Remark 2.1, $\mathcal{M}''$ is WOT-closed (in particular, w^*-closed). Therefore

$$\mathcal{M}'' = \mathcal{M}^{-w^*} \subseteq \mathcal{M}^{-\mathrm{WOT}} = \mathcal{M}^{-\mathrm{SOT}} \subseteq \mathcal{M}'',$$

that is, the assertion follows. □

Corollary 3.2. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a w^*-closed, $*$-subalgebra. Then $\mathcal{M}$ is unital whose unit e is the projection onto $\langle \mathcal{M}H\rangle$, and $\mathcal{M}'' = \mathbb{C}1 + \mathcal{M}$. In particular, every unital, w^*-closed, $*$-subalgebra is a von Neumann algebra.*

Proof. Since $\operatorname{ball}\mathcal{B}(H)$ is w^*-compact, so is $\mathcal{M}\cap\operatorname{ball}\mathcal{B}(H)$. By Krein-Milman Theorem, the extremal boundary $\partial\operatorname{ball}\mathcal{M}$ is non-empty. Using Proposition 2.1, we conclude that $\mathcal{M}$ is unital. Put $S = e(H)$ for the unit e of $\mathcal{M}$. Note that $S \subseteq \mathcal{M}H \subseteq \langle \mathcal{M}H\rangle = \langle e\mathcal{M}H\rangle \subseteq \langle e(H)\rangle = S$, that is, $S = \langle \mathcal{M}H\rangle$. Since $xe = ex$ for all $x \in \mathcal{M}$, it follows that $e \in \mathcal{Z}(\mathcal{M})$ and both $\mathcal{M}$ and $\mathcal{M}'$ leave invariant the subspace S. Consider the reduction $\mathcal{B}(H) \to e\mathcal{B}(H)e = \mathcal{B}(S)$. Using Corollary 2.1, we conclude that $\mathcal{M}_e$ is w^*-closed, $*$-subalgebra in $\mathcal{B}(S)$ (in particular, it is norm closed). But $\mathcal{M}_e$ contains the unit of $\mathcal{B}(S)$, that is, it is nondegenerate and $\mathcal{M}_e = (\mathcal{M}_e)''$ thanks to Proposition 3.1. Since e is a unit for $\mathcal{M}$, we conclude that $\mathcal{M}_{e^\perp} = 0$, and

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_e & 0 \\ 0 & 0_{S^\perp} \end{bmatrix} \Rightarrow \begin{bmatrix} (\mathcal{M}_e)' & 0 \\ 0 & \mathcal{B}(S^\perp) \end{bmatrix}$$

$$\subseteq \mathcal{M}' \Rightarrow \mathcal{M}'' \subseteq \begin{bmatrix} (\mathcal{M}_e)'' & * \\ * & \mathbb{C}1_{S^\perp} \end{bmatrix} = \begin{bmatrix} \mathcal{M}_e & * \\ * & \mathbb{C}1_{S^\perp} \end{bmatrix}.$$

Finally, take $a \in \mathcal{M}''$. Since $e \in \mathcal{M}'$, it follows that $a = a_S \oplus a_{S^\perp} = a_S \oplus \lambda 1_{S^\perp}$ for some $\lambda \in \mathbb{C}$, and $(a - \lambda 1)_{S^\perp} = 0$, that is, $(a - \lambda 1)_S = b_S \in \mathcal{M}_e$ for some $b \in \mathcal{M}$. But $b_{S^\perp} = 0$ as well, therefore $a = \lambda 1 + b$. Whence $\mathcal{M}'' = \mathbb{C}1 + \mathcal{M}$. □

Corollary 3.3. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra, $e \in \mathcal{M}$ a projection. Then $\mathcal{M}'_e = (\mathcal{M}_e)'$.*

Proof. The inclusion $\mathcal{M}'_e \subseteq (\mathcal{M}_e)'$ was proven in Corollary 2.1. Put $S = e(H)$, and consider the closed subspace $N = \langle \mathcal{M}S \rangle$, which is $\mathcal{M}$-invariant obviously. Actually it is $\mathcal{M}'$-invariant either. Indeed, $\mathcal{M}'N \subseteq \langle \mathcal{M}'\mathcal{M}e(H) \rangle \subseteq \langle \mathcal{M}e\mathcal{M}'(H) \rangle \subseteq \langle \mathcal{M}S \rangle = N$. Using Lemma 3.1, we conclude that the projection p onto N is a central element of $\mathcal{M}$. Take a unitary $u \in (\mathcal{M}_e)'$. We extend it to an element $u' \in \mathcal{M}'$ in the following way. First put $u_0 : \mathcal{M}S \to \mathcal{M}S$, $u_0 \sum_{k=1}^n x_k \zeta_k = \sum_{k=1}^n x_k u \zeta_k$ for $\{x_k\} \subseteq \mathcal{M}$ and $\{\zeta_k\} \subseteq S$. Note that

$$\left\| \sum_{k=1}^n x_k u \zeta_k \right\|^2 = \sum_{k,m} (x_m^* x_k u \zeta_k, u \zeta_m)$$

$$= \sum_{k,m} (e x_m^* x_k e u \zeta_k, u \zeta_m) = \sum_{k,m} (u e x_m^* x_k e \zeta_k, u \zeta_m)$$

$$= \sum_{k,m} (e x_m^* x_k e \zeta_k, \zeta_m) = \sum_{k,m} (x_k \zeta_k, x_m \zeta_m) = \left\| \sum_{k=1}^n x_k \zeta_k \right\|^2,$$

that is, u_0 is a well defined isometry, which in turn is extended up to a unitary $u_0 \in \mathcal{B}(N)$. We extend u_0 up to a partial isometry $u' \in \mathcal{B}(H)$ by setting $u'(N^\perp) = \{0\}$. For every $y \in \mathcal{M}$ we have $py = yp$ (since $p \in \mathcal{M} \cap \mathcal{M}'$) and $u_0 pyp \left(\sum_{k=1}^n x_k \zeta_k\right) = u_0 \left(\sum_{k=1}^n y x_k \zeta_k\right) = \sum_{k=1}^n y x_k u \zeta_k = pypu_0 \left(\sum_{k=1}^n x_k \zeta_k\right)$, that is, $u'y = u'py = u_0 py = u_0 pyp = pypu_0 p = pypu' = pyu' = yu'$. Thus $u' \in \mathcal{M}'$ and $u = u'_S \in \mathcal{M}'_e$. But $(\mathcal{M}_e)'$ is a C^*algebra (see Remark 2.1), every its element is a linear combination of its unitary elements (see also [24, 3.2.21]). The rest is clear. □

Corollary 3.4. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra, $e \in \mathcal{Z}(\mathcal{M})$ a central projection. Then $\mathcal{Z}(\mathcal{M})_e = \mathcal{Z}(\mathcal{M}_e)$.*

Proof. As above put $S = e(H)$. Using Corollary 3.3, we obtain that $\mathcal{Z}(\mathcal{M})_e = (\mathcal{M} \cap \mathcal{M}')_e \subseteq \mathcal{M}_e \cap \mathcal{M}'_e = \mathcal{M}_e \cap (\mathcal{M}_e)' = \mathcal{Z}(\mathcal{M}_e)$. Conversely, take $y_S \in \mathcal{Z}(\mathcal{M}_e)$ with $y \in \mathcal{M}$. Since $\mathcal{Z}(\mathcal{M}_e) = \mathcal{M}_e \cap \mathcal{M}'_e$, it follows that $ye = xe$ for some $x \in \mathcal{M}'$. But $e \in \mathcal{Z}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$, therefore $xe \in \mathcal{M}'$ either. Taking into account that $xe = ye \in \mathcal{M}$, we conclude that $xe \in \mathcal{Z}(\mathcal{M})$ and $(xe)_S = y_S$, that is, $y_S \in \mathcal{Z}(\mathcal{M})_e$. □

Below (see Corollary 3.8) the assertion of Corollary 3.4 is proved for all projections e from $\mathcal{M}$.

3.3. w^*-closed ideals. Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra and let $\mathfrak{m} \subseteq \mathcal{M}$ be a w^*-closed left ideal of $\mathcal{M}$. Since $p_{\zeta,\eta}(x^*) = |\langle x^*, \omega_{\zeta,\eta} \rangle| = = \left|\sum_n (x^* \zeta_n, \eta_n)\right| = \left|\sum_n (x \eta_n, \zeta_n)\right| = p_{\eta,\zeta}(x)$, $x \in \mathcal{M}$, it follows that $\mathfrak{m}^*$ is w^*-closed, and $\mathfrak{n} = \mathfrak{m} \cap \mathfrak{m}^*$ is a w^*-closed, $*$-subalgebra. By Corollary 3.2, $\mathfrak{n}$ has a unit e.

Lemma 3.3. *If $\mathfrak{m} \subseteq \mathcal{M}$ is a w^*-closed left ideal then $\mathcal{M}e = \mathfrak{m}$ for the unique projection e. If $\mathfrak{m}$ is a two-sided ideal then $e \in \mathcal{Z}(\mathcal{M})$. In particular, every two-sided ideal of a factor is w^*-dense.*

Proof. Fix the unit e of $\mathfrak{n}$. Note that $\mathcal{M}e \subseteq \mathfrak{m}$, for $\mathfrak{m}$ is a left ideal. Conversely, take $x \in \mathfrak{m}$. Then $x^*x = (x^*x)^* \in \mathfrak{n}$, which in turn implies that $|x| = (x^*x)^{1/2} \in \mathfrak{n}$, for $\mathfrak{n}$ is a C^*-algebra. Using the polar decomposition $x = u|x|$, we derive that $x = u|x|e \in \mathcal{M}e$. Now assume that $\mathfrak{m}$ is a two-sided ideal, and take $x \in \mathfrak{m}$. Then $ex \in \mathfrak{m} = \mathcal{M}e$ and $ex^* \in \mathfrak{m}x^* \subseteq \mathfrak{m} = \mathcal{M}e$, which in turn implies that $ex = exe = (ex^*e)^* = (ex^*)^* = xe$, that is, e is a central projection of $\mathfrak{m}$.

Finally, assume that $\mathfrak{m}$ is a two-sided ideal of a factor $\mathcal{M}$. One can easily verify that the w^*-closure of $\mathfrak{m}$ is a two-sided ideal in $\mathcal{M}$ yet. It follows that $\mathfrak{m}^{-w^*} = \mathcal{M}e$ for a central projection e in $\mathcal{M}$. But $\mathcal{M}$ is a factor, therefore $e = 1$ or $\mathfrak{m}^{-w^*} = \mathcal{M}$. □

Corollary 3.5. *Let H be a Hilbert space. Then $\mathcal{B}_f(H)^{-w^*} = \mathcal{B}_f(H)^{-WOT} = \mathcal{B}_f(H)^{-SOT} = \mathcal{B}(H)$.*

Proof. One needs to appeal to Proposition 3.1 and Lemma 3.3. □

Remark 3.1. Let $x = u|x|$ be the polar decomposition of an element x in a von Neumann algebra. Note that $|x| \in \mathcal{M}$, for $\mathcal{M}$ is a C^*-algebra. Actually, $u \in \mathcal{M}$. Indeed, take a unitary $v \in \mathcal{M}'$. Since $xv = vx$ and $|x|v^* = v^*|x|$, it follows that $x = vxv^* = vu|x|v^* = vuv^*|x|$. But vuv^* is a partial isometry, therefore $vuv^* = u$ or $uv = vu$, that is, $u \in \mathcal{M}'' = \mathcal{M}$. In particular, $x^* = |x|u^* \in \mathcal{M}$.

3.4. Projections in von Neumann algebras. Let $\mathcal{M}$ be a von Neumann algebra on H, and let $\mathfrak{P}(\mathcal{M})$ be the set of all projections in $\mathcal{M}$. For $e,\ f \in \mathfrak{P}(\mathcal{B}(H))$ we write $e \leq f$ whenever $e(H) \subseteq f(H)$. In this case, $ef = fe = e$. It is a well defined order in $\mathfrak{P}(\mathcal{B}(H))$. In particular, $\mathfrak{P}(\mathcal{M})$ is an ordering set. Note that $e \leq f$ iff $1 - e \geq 1 - f$. For a subset $S \subseteq \mathfrak{P}(\mathcal{B}(H))$ we define $e_0,\ e_1 \in \mathfrak{P}(\mathcal{B}(H))$ to be the orthogonal projections onto the closed subspaces $\cap_{e\in S} e(H)$ and $\langle \cup_{e\in S} e(H) \rangle$, respectively. That is, $e_0 = \inf S$ and $e_1 = \sup S$. We use the notations $e_0 = \wedge S$ and $e_1 = \vee S$. Note that

$$e_1 = 1 - \wedge(1 - S), \tag{3.1}$$

where $1 - S = \{1 - e : e \in S\}$. Indeed, for every $e \in S$ we have $1 - e \geq \wedge(1 - S)$, which in turn implies that $e = 1 - (1 - e) \leq 1 - \wedge(1 - S)$, that is, $S \leq 1 - \wedge(1 - S)$. Take an upper bound $f \in \mathfrak{P}(\mathcal{B}(H))$, $S \leq f$. Then

$$1 - S \geq 1 - f \Rightarrow \wedge(1 - S) \geq 1 - f \Rightarrow 1 - \wedge(1 - S) \leq f \Rightarrow 1 - \wedge(1 - S) = \vee S = e_1,$$

that is, the equality (3.1) holds.

Lemma 3.4. *The set $\mathfrak{P}(\mathcal{M})$ is a complete lattice.*

Proof. Take $S \subseteq \mathcal{M}$ with $e_0 = \wedge S$ and $e_1 = \vee S$ from $\mathfrak{P}(\mathcal{B}(H))$. Prove that $e_0, e_1 \in \mathcal{M}$. Note that $\mathcal{M}'e_0(H) = \mathcal{M}' \cap_{e\in S} e(H) \subseteq \cap_{e\in S} \mathcal{M}'e(H) \subseteq \cap_{e\in S} e\mathcal{M}'(H) \subseteq e_0(H)$ and $\mathcal{M}'e_1(H) = \mathcal{M}' \langle \cup_{e\in S} e(H) \rangle \subseteq \langle \cup_{e\in S} \mathcal{M}'e(H) \rangle \subseteq \langle \cup_{e\in S} e\mathcal{M}'(H) \rangle \subseteq e_1(H)$, that is, both $e_0(H)$ and $e_1(H)$ are closed $\mathcal{M}'$-invariant subspaces in H. By Lemma 3.1, $e_0, e_1 \in \mathcal{M}$. □

Recall that e_ζ is the projection onto $\langle \mathcal{M}'\zeta\rangle$, whereas e'_ζ is the projection onto $\langle \mathcal{M}\zeta\rangle$, where $\zeta \in H$. As we have seen above in Lemma 3.1, $e_\zeta \in \mathfrak{P}(\mathcal{M})$ and $e'_\zeta \in \mathfrak{P}(\mathcal{M}')$.

Corollary 3.6. *For every $\zeta \in H$ we have*

$$e_\zeta = \wedge \{e \in \mathfrak{P}(\mathcal{M}) : e(\zeta) = \zeta\},$$
$$e'_\zeta = \wedge \{e' \in \mathfrak{P}(\mathcal{M}') : e'(\zeta) = \zeta\}.$$

Proof. If $e(\zeta) = \zeta$ for some $e \in \mathfrak{P}(\mathcal{M})$ then $e_\zeta(H) = \langle \mathcal{M}'\zeta\rangle = \langle \mathcal{M}'e\zeta\rangle = \langle e\mathcal{M}'\zeta\rangle = e\langle \mathcal{M}'\zeta\rangle \subseteq e(H)$, that is, $e_\zeta \le e$. Taking into account that $e_\zeta(\zeta) = \zeta$, we have $e_\zeta = \wedge\{e \in \mathfrak{P}(\mathcal{M}) : e(\zeta) = \zeta\}$ based on Lemma 3.4. Similar argument is applicable to the second equality. ☐

The projections e and f in $\mathcal{M}$ are supposed to be equivalent $e \sim f$ if $u^*u = e$ and $uu^* = f$ for some $u \in \mathcal{M}$, which is an equivalence relation in $\mathfrak{P}(\mathcal{M})$.

Remark 3.2. If $u^*u = e \in \mathfrak{P}(\mathcal{M})$ for some $u \in \mathcal{M}$ then $f = uu^*$ is a projection in $\mathcal{M}$. Indeed, as in the proof of Lemma 2.4, we have $u = ue$. Further, $f = uu^* = ueu^*$ and $f^2 = ueu^*ueu^* = ue^3u^* = ueu^* = f$. As above, $u^* = u^*f$ or $u = fu$.

Remark 3.3. If $e, f \in \mathfrak{P}(\mathcal{M})$ are unitarily equivalent, that is, $vev^* = f$ for a unitary $v \in \mathcal{M}$, then $e \sim f$. Indeed, $(ve)^*(ve) = ev^*ve = e^2 = e$ and $(ve)(ve)^* = vev^* = f$.

Remark 3.4. If $e \sim f$ with $u^*u = e$ and $uu^* = f$, then u is a partial isometry [24, 3.2.16] with its initial space $e(H)$ and the terminal space $f(H)$. Indeed, $ue(\zeta) = uu^*u(\zeta) = fu(\zeta) \in f(H)$ for every $\zeta \in H$, that is, $u(e(H)) \subseteq f(H)$ and $u(1-e)(H) = \{0\}$ (see Remark 3.2). Similarly, $u^*(f(H)) \subseteq e(H)$ and $u^*(1-f)(H) = \{0\}$ But $u^*ue(\zeta) = e(\zeta)$ and $uu^*f(\eta) = f(\eta)$ for all $\zeta, \eta \in H$, that is, $u^* : f(H) \to e(H)$ is the inverse of $u : e(H) \to f(H)$ which means that u is a partial isometry.

Remark 3.5. Let $\{e_i\}$ and $\{f_i\}$ be orthogonal projections in $\mathcal{M}$ such that $e_i \sim f_i$ for all $i \in I$. Thus $u_i^*u_i = e_i$ and $u_iu_i^* = f_i$ for partial isometries $u_i \in \mathcal{M}$, $u_i = u_ie_i = f_iu_i$ (see Remark 3.4). Put $e = \sum_i e_i$ and $f = \sum_i f_i$, which are projections in $\mathcal{M}$. For a finite subset $\lambda \subseteq I$ and $\zeta \in H$ we have $\left\|\sum_{i\in\lambda} u_i\zeta\right\|^2 = \left\|\sum_{i\in\lambda} f_iu_i\zeta\right\|^2 = \sum_{i\in\lambda} \|f_iu_i\zeta\|^2 = \sum_{i\in\lambda}\|u_ie_i\zeta\|^2 \le \sum_{i\in\lambda}\|e_i\zeta\|^2 = \|e\zeta\|^2$, that is, $u = $SOT-$\sum_i u_i \in \mathcal{M}$. Similarly, $\left\|\sum_{i\in\lambda} u_i^*\zeta\right\|^2 = \left\|\sum_{i\in\lambda} e_iu_i^*\zeta\right\|^2 = \sum_{i\in\lambda}\|u_i^*f_i\zeta\|^2 \le \sum_{i\in\lambda}\|f_i\zeta\|^2 = \|f\zeta\|^2$, and $u^* = $SOT-$\sum_i u_i^* \in \mathcal{M}$. It follows that $u^*u = $SOT-$\sum_i u_i^*u_i = \sum_i e_i = e$ and $uu^* = $SOT-$\sum_i u_iu_i^* = \sum_i f_i = f$, that is, $e \sim f$.

The set $\mathfrak{P}_c(\mathcal{M}) = \mathfrak{P}(\mathcal{M}) \cap \mathcal{M}'$ indicates to the set of all central projections in $\mathcal{M}$. If $e \sim f$ and $z \in \mathfrak{P}_c(\mathcal{M})$ then $ze \sim zf$. For $u^*u = e$ and $uu^* = f$ we have $(zu)^*(zu) = ze$ and $(zu)(zu)^* = zf$. Now fix $e \in \mathfrak{P}(\mathcal{M})$. We put

$$z(e) = \wedge\{z \in \mathfrak{P}_c(\mathcal{M}) : e \le z\},$$

which is a projection from $\mathcal{M}$ thanks to Lemma 3.4 called *the central support of* e.

Lemma 3.5. *Let $e \in \mathfrak{P}(\mathcal{M})$. Then $z(e)$ is the projection onto $\langle \mathcal{M}e(H)\rangle$, and $z(we) = wz(e)$ for every $w \in \mathfrak{P}_c(\mathcal{M})$. Moreover, $z(e) = z(f)$ whenever $e \sim f$.*

Proof. Put p to be the projection onto $\langle \mathcal{M}e(H)\rangle$. Note that $\langle \mathcal{M}e(H)\rangle$ is a closed $\mathcal{M}$-invariant and $\mathcal{M}'$-invariant subspace simultaneously. By Lemma 3.1, $p \in \mathcal{M} \cap \mathcal{M}'$. Since $e \leq p$, it follows that $z(e) \leq p$. Take $z \in \mathfrak{P}_c(\mathcal{M})$ with $e \leq z$. Then

$$e(H) \subseteq z(H) \Rightarrow \mathcal{M}e(H) \subseteq \mathcal{M}z(H) \subseteq z\mathcal{M}(H)$$
$$\subseteq z(H) \Rightarrow \langle \mathcal{M}e(H)\rangle \subseteq z(H) \Rightarrow p \leq z,$$

which in turn implies that $p = z(e)$. If $w \in \mathfrak{P}_c(\mathcal{M})$ then

$$z(we)(H) = \langle \mathcal{M}we(H)\rangle = \langle w\mathcal{M}e(H)\rangle = w\langle \mathcal{M}e(H)\rangle = wz(e)(H).$$

But $wz(e)$ is a projection as well, thereby $z(we) = wz(e)$. Finally, suppose $u^*u = e$ and $uu^* = f$ for some $u \in \mathcal{M}$, where $e, f \in \mathfrak{P}(\mathcal{M})$. Using Remark 3.4, we derive that

$$\begin{aligned} z(e)(H) &= \langle \mathcal{M}e(H)\rangle = \langle \mathcal{M}u^*ue(H)\rangle = \langle \mathcal{M}u^*f(H)\rangle \subseteq \langle \mathcal{M}f(H)\rangle = z(f)(H) \\ &= \langle \mathcal{M}uu^*f(H)\rangle = \langle \mathcal{M}ue(H)\rangle \subseteq \langle \mathcal{M}e(H)\rangle = z(e)(H), \end{aligned}$$

that is, $z(e) = z(f)$. □

Corollary 3.7. *The set $\mathfrak{P}_c(\mathcal{M})$ is a complete sublattice of $\mathfrak{P}(\mathcal{M})$.*

Proof. Based on (3.1), it suffices to prove that $e_0 = \wedge S \in \mathfrak{P}_c(\mathcal{M})$ for $S \subseteq \mathfrak{P}_c(\mathcal{M})$. Using Lemma 3.5, we obtain that

$$\begin{aligned} z(e_0)(H) &= \langle \mathcal{M}e_0(H)\rangle = \langle \mathcal{M}\cap_{e\in S} e(H)\rangle \subseteq \cap_{e\in S}\langle \mathcal{M}e(H)\rangle = \cap_{e\in S} z(e)(H) \\ &= (\wedge z(S))(H) = (\wedge S)(H) = e_0(H) \subseteq z(e_0)(H), \end{aligned}$$

for $z(S) = S$. Thus $z(e_0) = e_0$, which means that $e_0 \in \mathfrak{P}_c(\mathcal{M})$. □

Corollary 3.8. *Let $\mathcal{M}$ be a von Neumann algebra, $e \in \mathfrak{P}(\mathcal{M})$. Then $\mathcal{Z}(\mathcal{M})_e = \mathcal{Z}(\mathcal{M}_e)$.*

Proof. First note that $\mathcal{Z}(\mathcal{M})_e$ is a von Neumann algebra on S by Remark 2.2, where $S = e(H)$. The inclusion $\mathcal{Z}(\mathcal{M})_e \subseteq \mathcal{Z}(\mathcal{M}_e)$ follows from Corollary 3.3 (see to the proof of Corollary 3.4). Consider the central support $z(e)$ of e in $\mathcal{M}$. Note that the canonical mapping $\alpha : \mathcal{M}'_{z(e)} \to \mathcal{M}'_e$, $\alpha(u) = ue$ is an isomorphism. Indeed, if $ue = 0$ for some $u \in \mathcal{M}'$ then $uz(e)(H) = u\langle \mathcal{M}e(H)\rangle \subseteq \langle u\mathcal{M}e(H)\rangle = \langle \mathcal{M}ue(H)\rangle = \{0\}$ thanks to Lemma 3.5, that is, $uz(e) = 0$. Now take $x_S \in \mathcal{Z}(\mathcal{M}_e) = \mathcal{M}_e \cap \mathcal{M}'_e$ (see Corollary 3.3). Then $exe = ye$ for some $y \in \mathcal{M}'$. The latter means that $y_S = x_S \in \mathcal{Z}(\mathcal{M}'_e)$ by Proposition 3.1. Then for every $u \in \mathcal{M}'_{z(e)}$ we have $\alpha([yz(e), u]) = [\alpha(yz(e)), \alpha(u)] = [y_S, ue] = 0$, which in turn implies that $[yz(e), u] = 0$. Hence $yz(e) \in \mathcal{Z}\left(\mathcal{M}'_{z(e)}\right)$. Using Corollary 3.4, we obtain that $yz(e) \in \mathcal{Z}(\mathcal{M}')_{z(e)} = \mathcal{Z}(\mathcal{M})_{z(e)}$, that is, $yz(e) = bz(e)$ for some $b \in \mathcal{Z}(\mathcal{M})$. In particular, $yz(e) \in \mathcal{M} \cap \mathcal{M}' = \mathcal{Z}(\mathcal{M})$ and $(yz(e))_S = yz(e)e = ye = exe = x_S$, that is, $x_S \in \mathcal{Z}(\mathcal{M})_e$. □

Corollary 3.9. *Let $\mathcal{M}$ be a von Neumann algebra, $e \in \mathfrak{P}(\mathcal{M})$ with $z(e) = 1$. Then the restriction mapping $\mathcal{Z}(\mathcal{M}) \to \mathcal{Z}(\mathcal{M})_e$ implements an isomorphism of von Neumann algebras.*

Proof. Based on the argument from the proof of Corollary 3.8, and using Lemma 3.5, we conclude that the restriction mapping $\alpha : \mathcal{Z}(\mathcal{M}) \to \mathcal{Z}(\mathcal{M})_e$ is a $*$-isomorphism of von Neumann algebras. In particular, α is w^*-homeomorphism automatically (see [24, E 4.6.10]). □

3.5. Tensor product of von Neumann algebras. Now let $\mathcal{M}_1 \subseteq \mathcal{B}(H_1)$, $\mathcal{M}_2 \subseteq \mathcal{B}(H_2)$ be von Neumann algebras. Put $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$ to be the von Neumann algebra on $H_1 \otimes_\sigma H_2$ generated by all elementary tensors $x_1 \otimes x_2$ with $x_i \in \mathcal{M}_i$, $i = 1, 2$. If $\mathcal{M}_2 = \mathbb{C}1$ we obtain the von Neumann algebra $\mathcal{M}_1 \otimes \mathbb{C}$ considered above in Subsection 2.5. Respectively, we have the von Neumann algebra $\mathbb{C} \otimes \mathcal{M}_2$.

As above we fix an orthonormal basis $(\epsilon_i)_{i \in I}$ for H_2, and consider the one-rank operators $u_{ij} = \epsilon_i \odot \epsilon_j \in \mathcal{B}_f(H_2)$, $i, j \in I$. As in Subsection 2.5, $H_1 \otimes_\sigma H_2 = \bigoplus_I H_1$ and $\mathcal{B}(H_1 \otimes_\sigma H_2) = M_I(\mathcal{B}(H_1))$ thanks to Lemma 2.5. Moreover, in Subsection 2.4, we introduced canonical embeddings $U_i : H_1 \to H_1 \otimes_\sigma H_2$, $U(\zeta) = \zeta \otimes \epsilon_i$, $i \in I$, so that such that $U_i^* : \bigoplus_I H_1 \to H_1$ are the canonical projections. The elementary operators $e_{ij} = U_i U_j^*$, $e_{ij}\left(\sum_k \zeta_k \otimes \epsilon_k\right) = \zeta_j \otimes \epsilon_i$ played a key role in Corollaries 2.5 and 2.6. Note that

$$e_{ij}(\zeta \otimes \eta) = e_{ij}\left(\sum_k \zeta \otimes (\eta, \epsilon_k)\epsilon_k\right) = e_{ij}\left(\sum_k (\eta, \epsilon_k)\zeta \otimes \epsilon_k\right) = (\eta, \epsilon_j)\zeta \otimes \epsilon_i$$
$$= \zeta \otimes (\epsilon_i \odot \epsilon_j)\eta = (1 \otimes u_{ij})(\zeta \otimes \eta),$$

that is, $e_{ij} = 1 \otimes u_{ij}$ for all $i, j \in I$. Now take $x = [x_{ij}]_{ij} \in M_I(\mathcal{B}(H_1))$. By Lemma 2.5, $x_{ij} = U_i^* x U_j \in \mathcal{B}(H_1)$ for all $i, j \in I$. For $\zeta = \sum_k \zeta_k \otimes \epsilon_k \in H_1 \otimes_\sigma H_2$ we have

$$x(\zeta) = \sum_i \left(\sum_j x_{ij}\zeta_j\right) \otimes \epsilon_i$$

$$= \sum_{i,j} x_{ij}\zeta_j \otimes \epsilon_i = \sum_{i,j} (x_{ij} \otimes 1)(\zeta_j \otimes \epsilon_i) = \sum_{i,j} (x_{ij} \otimes 1) e_{ij}\zeta$$

$$= \sum_{i,j} (x_{ij} \otimes 1)(1 \otimes u_{ij})\zeta = \sum_{i,j} (x_{ij} \otimes u_{ij})\zeta.$$

Hence

$$x = \text{SOT-}\sum_{i,j} x_{ij} \otimes u_{ij} \tag{3.2}$$

for all $x \in \mathcal{B}(H_1 \otimes_\sigma H_2)$. In particular, if $x \in M_I(\mathcal{M}_1)$ then $x_{ij} \in \mathcal{M}_1$ for all $i, j \in I$, which in turn implies that $x \in \mathcal{M}_1 \overline{\otimes} \mathcal{B}(H_2)$ thanks to (3.2) and Proposition 3.1.

Proposition 3.2. *For a von Neumann algebra $\mathcal{M}_1 \subseteq \mathcal{B}(H_1)$ we have the canonical identification $\mathcal{M}_1 \overline{\otimes} \mathcal{B}(H_2) = M_I(\mathcal{M}_1)$ of $*$-algebras.*

Proof. We have just seen above $M_I(\mathcal{M}_1) \subseteq \mathcal{M}_1 \overline{\otimes} \mathcal{B}(H_2)$. To prove the reverse inclusion, consider the linear isometric embedding $\sigma_{ij} : \mathcal{B}(H_1) \to \mathcal{B}(H_1 \otimes_\sigma H_2)$, $\sigma_{ij}(x) = x \otimes u_{ij}$ called (i, j)-entrance. In terms of the matrices, we have $x \otimes u_{ij} =$

$(x\otimes 1)\, e_{ij} = [x\delta_{st}]_{s,t}\,[\delta_{ij}1] = [\delta_{ij}x]$ for $x\in\mathcal{B}(H_1)$. Take $\zeta,\eta\in\ell^2(H_1\otimes_\sigma H_2)$. Note that

$$\langle \sigma_{ij}(x), \omega_{\zeta,\eta}\rangle = \sum_n (\sigma_{ij}(x)\zeta_n, \eta_n) = \sum_n \left((x\otimes u_{ij})\sum_k \zeta_{nk}\otimes\epsilon_k, \eta_{nk}\right)$$
$$= \sum_n \left(\sum_k x\zeta_{nk}\otimes u_{ij}\epsilon_k, \eta_n\right) = \sum_n \left(x\zeta_{nj}\otimes\epsilon_i, \sum_k \eta_{nk}\otimes\epsilon_k\right)$$
$$= \sum_n (x\zeta_{nj}, \eta_{ni}) = \left\langle x, \omega_{\zeta^j,\eta^i}\right\rangle,$$

where $\zeta^j = (\zeta_{nj})_n, \eta^i = (\eta_{ni})_n \in \ell^2(H_1)$. Thus σ_{ij} is a w^*-bicontinuous isometry of $\mathcal{B}(H_1)$ into $\mathcal{B}(H_1\otimes_\sigma H_2)$. Consider also the w^*-continuous mapping $\tau_{ij}: \mathcal{B}(H_1\otimes_\sigma H_2)\to\mathcal{B}(H_1\otimes_\sigma H_2)$, $\tau_{ij}(x) = (1\otimes u_{ii})\, x\,(1\otimes u_{jj})$ (see Remark 2.2) for every couple $i,j\in I$. Take $x=[x_{ij}]\in M_I(\mathcal{B}(H_1)) = \mathcal{B}(H_1\otimes_\sigma H_2)$ (see Lemma 2.5). Note that

$$\sigma_{ij}(x_{ij}) = x_{ij}\otimes u_{ij} = (x_{ij}\otimes 1)(1\otimes u_{ij}) = (x_{ij}\otimes 1)(1\otimes u_{ii})(1\otimes u_{jj})$$
$$= (1\otimes u_{ii})(x_{ij}\otimes 1)(1\otimes u_{jj}) = e_{ii}(x_{ij}\otimes 1)\, e_{jj} = e_{ii}\,[\delta_{st}x_{ij}]_{s,t}\, e_{jj}$$
$$= e_{ii}\,[x_{st}]_{s,t}\, e_{jj} = \tau_{ij}(x),$$

that is, the w^*-continuous mapping $\sigma_{ij}^{-1}\tau_{ij}: \mathcal{B}(H_1\otimes_\sigma H_2)\to\mathcal{B}(H_1)$ is acting as $\left(\sigma_{ij}^{-1}\tau_{ij}\right)(x) = x_{ij}$. In particular,

$$M_I(\mathcal{M}_1) = \{x=[x_{ij}]\in\mathcal{B}(H_1\otimes_\sigma H_2) : x_{ij}\in\mathcal{M}_1, i,j\in I\} =$$
$$= \cap_{i,j}\left(\sigma_{ij}^{-1}\tau_{ij}\right)^{-1}(\mathcal{M}_1)$$

turns out to be a w^*-closed, $*$-subalgebra of $\mathcal{B}(H_1\otimes_\sigma H_2)$.

Now take $x_1\in\mathcal{M}_1$ and $x_2\in\mathcal{B}(H_2)$. Since $\mathcal{B}(H_2) = \mathcal{B}_f(H_2)^{-w^*}$ (see Corollary 3.5), it follows that $x_2 = w^*\text{-}\lim_\lambda y_\lambda$ for a net $(y_\lambda)_\lambda\subseteq\mathcal{B}_f(H_2)$. Take an orthonormal basis $(\vartheta_v)_{v\in J}$ for H_1 and fix $\zeta=\sum_v \vartheta_v\otimes\zeta_v, \eta=\sum_v\vartheta_v\otimes\eta_v\in H_1\otimes_\sigma H_2$. Since $\|\zeta\|^2 = \sum_v\|\zeta_v\|^2<\infty$, we can assume that $\zeta = \sum_n\vartheta_{v_n}\otimes\zeta_n$ with $\|\zeta\|^2 = \sum_n\|\zeta_n\|^2<\infty$. Similarly, $\eta=\sum_n\vartheta_{v_n}\otimes\eta_n$ with $\|\eta\|^2=\sum_n\|\eta_n\|^2<\infty$. Note that

$$\omega_{\zeta,\eta}(1\otimes(x_2-y_\lambda)) = ((1\otimes(x_2-y_\lambda))\zeta,\eta)$$
$$= \left(\sum_n \vartheta_{v_n}\otimes(x_2-y_\lambda)\zeta_n, \sum_n\vartheta_{v_n}\otimes\eta_n\right)$$
$$= \sum_n((x_2-y_\lambda)\zeta_n,\eta_n) = \omega_{\overline{\zeta},\overline{\eta}}(x_2-y_\lambda),$$

where $\overline{\zeta} = (\zeta_n)_n, \overline{\eta} = (\eta_n)_n\in\ell^2(H_2)$. In particular, $p_{\zeta,\eta}(1\otimes(x_2-y_\lambda)) = p_{\overline{\zeta},\overline{\eta}}(x_2-y_\lambda)\to 0$ for large λ, that is, $1\otimes x_2 =$WOT-$\lim_\lambda 1\otimes y_\lambda$. Since

$$p_{\zeta,\eta}(x_1\otimes(x_2-y_\lambda)) = p_{\zeta,\eta}((x_1\otimes 1)(1\otimes(x_2-y_\lambda)))$$
$$= |((1\otimes(x_2-y_\lambda))\zeta, (x_1^*\otimes 1)\eta)|$$
$$= p_{\zeta,(x_1^*\otimes 1)\eta}(1\otimes(x_2-y_\lambda))$$

for all $\zeta,\eta \in H_1 \otimes_\sigma H_2$, it follows that $x_1 \otimes x_2 =$WOT-$\lim_\lambda x_1 \otimes y_\lambda$. But

$$\{x_1 \otimes y_\lambda\} \subseteq \operatorname{span}\{x_1 \otimes u_{ij} : i,j \in I\} \subseteq \mathcal{M}_1 \otimes \mathcal{B}_f(H_2)$$

$$= \operatorname{span}\{\mathcal{M}_1 \otimes u_{ij} : i,j \in I\} \subseteq M_I(\mathcal{M}_1),$$

therefore $x_1 \otimes x_2 \in M_I(\mathcal{M}_1)^{-\mathrm{WOT}} = M_I(\mathcal{M}_1)^{-w^*} = M_I(\mathcal{M}_1)$ by virtue of Proposition 3.1. Whence $\mathcal{M}_1 \overline{\otimes} \mathcal{B}(H_2) \subseteq M_I(\mathcal{M}_1)$. □

Corollary 3.10. *Let $\mathcal{M}_1 \subseteq \mathcal{B}(H_1)$ be a von Neumann algebra and H_2 a Hilbert space. Then $(\mathcal{M}_1 \otimes \mathbb{C})' = \mathcal{M}_1' \overline{\otimes} \mathcal{B}(H_2)$. In particular, $(\mathcal{M}_1 \overline{\otimes} \mathcal{B}(H_2))' = \mathcal{M}_1' \otimes \mathbb{C}$.*

Proof. It suffices to use Corollary 2.6 and Proposition 3.2. Namely, $(\mathcal{M}_1 \otimes \mathbb{C})' = \pi(\mathcal{M}_1)' = M_I(\mathcal{M}_1') = \mathcal{M}_1' \overline{\otimes} \mathcal{B}(H_2)$. Finally, $(\mathcal{M}_1 \overline{\otimes} \mathcal{B}(H_2))' = (\mathcal{M}_1'' \overline{\otimes} \mathcal{B}(H_2))' = (\mathcal{M}_1' \overline{\otimes} \mathbb{C})'' = \mathcal{M}_1' \otimes \mathbb{C}$ thanks to Proposition 3.1. □

Note also that $\mathcal{B}(H_1) \overline{\otimes} \mathcal{B}(H_2) = M_I(\mathcal{B}(H_1)) = \mathcal{B}(H_1 \otimes_\sigma H_2)$ thanks to Lemma 2.5 and Proposition 3.2.

Proposition 3.3. *Let $\mathcal{M}_i \subseteq \mathcal{B}(H_i)$ be von Neumann algebras with projection $e_i \in \mathcal{M}_i$, $i = 1,2$. Then $(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)_{e_1 \otimes e_2} = \mathcal{M}_{1,e_1} \overline{\otimes} \mathcal{M}_{2,e_2}$ and $(\mathcal{M}_1' \overline{\otimes} \mathcal{M}_2')_{e_1 \otimes e_2} = \mathcal{M}_{1,e_1}' \overline{\otimes} \mathcal{M}_{2,e_2}'$.*

Proof. First, as we have seen in Corollary 2.4, $e = e_1 \otimes e_2$ is a projection in $\mathcal{M} = \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2 \subseteq \mathcal{B}(H)$, where $H = H_1 \otimes_\sigma H_2$. Note that (see Remark 2.2)

$$e\mathcal{M}e = e(\mathcal{M}_1 \otimes \mathcal{M}_2)^{-\mathrm{WOT}} e \subseteq (e(\mathcal{M}_1 \otimes \mathcal{M}_2)e)^{-\mathrm{WOT}}$$

$$= ((e_1\mathcal{M}_1e_1) \otimes (e_2\mathcal{M}_2e_2))^{-\mathrm{WOT}}$$

$$= (\mathcal{M}_{1,e_1} \otimes \mathcal{M}_{2,e_2})^{-\mathrm{WOT}} = \mathcal{M}_{1,e_1} \overline{\otimes} \mathcal{M}_{2,e_2}$$

thanks to Proposition 3.1. Since the inclusion $\mathcal{M}_{1,e_1} \otimes \mathcal{M}_{2,e_2} \subseteq e\mathcal{M}e$ is obvious, we conclude that $\mathcal{M}_{1,e_1} \overline{\otimes} \mathcal{M}_{2,e_2} \subseteq e\mathcal{M}e$. Consequently, $\mathcal{M}_{1,e_1} \overline{\otimes} \mathcal{M}_{2,e_2} = e\mathcal{M}e$.

Further, $(\mathcal{M}_1' \otimes \mathcal{M}_2')_e \subseteq \mathcal{M}_{1,e_1}' \otimes \mathcal{M}_{2,e_2}' \subseteq (\mathcal{M}_1' \overline{\otimes} \mathcal{M}_2')_e$. Indeed, if $x = \sum_{i=1}^n x_{1i} \otimes x_{2i} \in \mathcal{M}_1' \otimes \mathcal{M}_2'$ then

$$x_{e(H)} = exe = \sum_{i=1}^n x_{1i}e_1 \otimes x_{2i}e_2$$

$$= \sum_{i=1}^n x_{1i,e_1(H_1)} \otimes x_{2i,e_2(H_2)} \in \mathcal{M}_{1,e_1}' \otimes \mathcal{M}_{2,e_2}'.$$

Moreover, for $y = y_{1,e_1(H_1)} \otimes y_{2,e_2(H_2)} \in \mathcal{M}_{1,e_1}' \otimes \mathcal{M}_{2,e_2}'$, we have $y = (y_1 \otimes y_2)_{e(H)} \in (\mathcal{M}_1' \overline{\otimes} \mathcal{M}_2')_e$. Consider the w^*-continuous mapping $r : \mathcal{M}_1' \overline{\otimes} \mathcal{M}_2' \to \mathcal{B}(e(H))$, $r(x) = x_{e(H)}$ (see Remark 2.2). Then

$$(\mathcal{M}_1' \overline{\otimes} \mathcal{M}_2')_e = r(\mathcal{M}_1' \overline{\otimes} \mathcal{M}_2')$$

$$= r\left((\mathcal{M}_1' \otimes \mathcal{M}_2')^{-w^*}\right) \subseteq r(\mathcal{M}_1' \otimes \mathcal{M}_2')^{-w^*} = \left((\mathcal{M}_1' \otimes \mathcal{M}_2')_e\right)^{-w^*}$$

$$\subseteq (\mathcal{M}_{1,e_1}' \otimes \mathcal{M}_{2,e_2}')^{-w^*} = \mathcal{M}_{1,e_1}' \overline{\otimes} \mathcal{M}_{2,e_2}'.$$

Whence $(\mathcal{M}_1' \overline{\otimes} \mathcal{M}_2')_e = \mathcal{M}_{1,e_1}' \overline{\otimes} \mathcal{M}_{2,e_2}'$. □

3.6. Matrix units. Let $\mathcal{M}$ be a von Neumann algebra on H. A family $\{w_{ij} : i,j \in I\} \subseteq \mathcal{M}$ is said to be a matrix unit if $w_{ij}^* = w_{ji}$, $w_{ij}w_{kl} = w_{il}\delta_{jk}$ and SOT-$\sum_i w_{ii} = 1$. Confirm that $w_{ii}^* = w_{ii} = w_{ii}w_{ii}$, that is, $\{w_{ii} : i \in I\} \subseteq \mathfrak{P}(\mathcal{M})$. Fix $i_0 \in I$, and put $e = w_{i_0i_0}$, $H_e = e(H)$ and $w_i = w_{ii_0}$, $i \in I$. Since $w_i = w_{ii}w_{ii_0}$, it follows that $w_iH_e \subseteq w_{ii}H$. Conversely, $w_{ii} = w_{ii_0}w_{i_0i} = w_{ii_0}w_{ii_0}^* = w_{ii_0}w_{i_0i_0}w_{ii_0}^* = w_iew_i^*$, therefore $w_{ii}H = w_iew_i^*H \subseteq w_iH_e$. Hence $w_iH_e = w_{ii}H$. In particular, $H = \bigoplus_{i\in I} w_iH_e$. For every $\eta \in H$ we have $w_i^*\eta = w_{i_0i}\eta = ew_{i_0i}\eta \in eH = H_e$, that is, $w_i^*H \subseteq H_e$. Thus $w_i^*w_i = w_{i_0i}w_{ii_0} = w_{i_0i_0} = e$ and $w_iw_i^* = w_{ii_0}w_{i_0i} = w_{ii}$, which means that $e \sim w_{ii}$ and w_i is the related partial isometry (see Remark 3.4), which implements an isometry $w_i : H_e \to w_{ii}H$ onto for every $i \in I$. In particular, $w_{ii} \sim w_{jj}$ for all $i,j \in I$.

Now fix the canonical basis $(\epsilon_i)_{i\in I}$ of $\ell^2(I)$. Define

$$U : H_e \otimes_\sigma \ell^2(I) = \bigoplus_I H_e \to H, \quad U\left(\sum_i \zeta_i \otimes \epsilon_i\right) = \sum_i w_i\zeta_i$$

(see Subsection 2.4). If $\zeta \in H_e$ then $w_i\zeta = U(\zeta \otimes \epsilon_i)$, that is, U has the dense range.

Lemma 3.6. *The mapping U is an isometric isomorphism of $H_e \otimes_\sigma \ell^2(I)$ onto H.*

Proof. Take $\zeta = \sum_i \zeta_i \otimes \epsilon_i \in H_e \otimes_\sigma \ell^2(I)$. Note that

$$\begin{aligned}(U\zeta,\eta) &= \left(\sum_i w_i\zeta_i,\eta\right) = \sum_i (w_i\zeta_i,\eta) = \sum_i (\zeta_i, w_i^*\eta) = \sum_i (\zeta_i \otimes \epsilon_i, w_i^*\eta \otimes \epsilon_i) \\ &= \left(\sum_i \zeta_i \otimes \epsilon_i, \sum_i w_i^*\eta \otimes \epsilon_i\right) = (\zeta, U^*\eta),\end{aligned}$$

that is, $U^*\eta = \sum_i w_i^*\eta \otimes \epsilon_i$. In particular,

$$\begin{aligned}U^*U(\zeta) &= U^*\sum_j w_j\zeta_j = \sum_{i,j} w_i^*w_j\zeta_j \otimes \epsilon_i = \sum_{i,j} w_{i_0i}w_{ji_0}\zeta_j \otimes \epsilon_i = \sum_{i,j} \delta_{ij}e\zeta_j \otimes \epsilon_i \\ &= \sum_i e\zeta_i \otimes \epsilon_i = \sum_i \zeta_i \otimes \epsilon_i = \zeta\end{aligned}$$

and $UU^*(\eta) = U(\sum_i w_i^*\eta \otimes \epsilon_i) = \sum_i w_iw_i^*\eta = \sum_i w_{ii_0}w_{i_0i}\eta = \sum_i w_{ii}\eta = \eta$. Whence U is a unitary mapping. □

Proposition 3.4. *Let $\mathcal{M}$ be a von Neumann algebra with a matrix unit $\{w_{ij} : i,j \in I\} \subseteq \mathcal{M}$, and $H_e = e(H)$ with $e = w_{i_0i_0}$ for the fixed $i \in I$. Then $\mathcal{M} \simeq \mathcal{M}_e\overline{\otimes}\mathcal{B}(\ell^2(I))$.*

Proof. First note that $\mathcal{M}_e$ is a von Neumann algebra on H_e, $U : H_e\otimes_\sigma\ell^2(I) \to H$ implements an isometric isomorphism thanks to Lemma 3.6, and $\mathcal{M}_e\overline{\otimes}\mathcal{B}(\ell^2(I)) = M_I(\mathcal{M}_e)$ thanks to Proposition 3.2. Take $x = [x_{ij}]_{i,j} \in M_I(\mathcal{M}_e)$. For $\eta \in H$ we have (see to the proof of Lemma 3.6)

$$UxU^*\eta = Ux\left(\sum_i w_i^*\eta \otimes \epsilon_i\right) = U\left(\sum_i \left(\sum_j x_{ij}w_j^*\eta\right) \otimes \epsilon_i\right)$$

$$= \sum_i w_i \sum_j x_{ij} w_j^* \eta = \sum_{i,j} w_i x_{ij} w_j^* \eta,$$

that is, UxU^* =SOT-$\sum_{i,j} w_i x_{ij} w_j^*$. But $w_i x_{ij} w_j^* = w_i e x_{ij} e w_j^* \in \mathcal{M}$, therefore $UxU^* \in \mathcal{M}$. Conversely, take $x \in \mathcal{M}$ and $\zeta = \sum_i \zeta_i \otimes \epsilon_i \in H_e \otimes_\sigma \ell^2(I)$. Then

$$U^* x U \zeta = U^* \left(\sum_j x w_j \zeta_j \right) = \sum_{i,j} w_i^* x w_j \zeta_j \otimes \epsilon_i$$

$$= \sum_i \left(\sum_j w_i^* x w_j \zeta_j \right) \otimes \epsilon_i = [w_i^* x w_j]_{i,j} \zeta,$$

that is, $U^* x U = [w_i^* x w_j]_{i,j}$. But $w_i^* x w_j = w_{i_0 j} x w_{j i_0} = w_{i_0 i_0} w_{i_0 j} x w_{j i_0} w_{i_0 i_0} = e w_{i_0 j} x w_{j i_0} e = e w_i^* x w_j e \in e\mathcal{M}e = \mathcal{M}_e$ for all i, j. Whence $U^* x U \in M_I(\mathcal{M}_e)$. Using again Lemma 3.6, we conclude that $\mathcal{M} \simeq \mathcal{M}_e \overline{\otimes} \mathcal{B}(\ell^2(I))$. $\square$

Corollary 3.11. *Let $\mathcal{M}$ be a von Neumann algebra on H, $(e_i)_{i\in I}$ an orthogonal family of mutually equivalent projections in $\mathcal{M}$ with SOT-$\sum_i e_i = 1$. Then $\mathcal{M} \simeq \mathcal{M}_{e_i} \overline{\otimes} \mathcal{B}(\ell^2(I))$ for every $i \in I$.*

Proof. Fix $i_0 \in I$. Since $e_{i_0} \sim e_i$, it follows that $u_i^* u_i = e_{i_0}$ and $u_i u_i^* = e_i$ for a partial isometry $u_i \in \mathcal{M}$. Put $w_{ij} = u_i u_j^*$ for $i, j \in I$. Then $\{w_{ij} : i, j \in I\}$ is a matrix unit in $\mathcal{M}$. Indeed, first note that $w_{ij}^* = u_j u_i^* = w_{ji}$. Using Remark 3.2, we have $u_k e_{i_0} = u_k = e_k u_k$ and $e_{i_0} u_k^* = u_k^* = u_k^* e_k$ for all $k \in I$. It follows that

$$w_{ij} w_{kl} = u_i u_j^* u_k u_l^* = u_i u_j^* e_j e_k u_k u_l^* = \delta_{jk} u_i u_j^* u_j u_l^* = \delta_{jk} u_i e_{i_0} u_l^* = \delta_{jk} u_i u_l^* = \delta_{jk} w_{il}.$$

Finally, SOT-$\sum_i w_{ii}$ =SOT-$\sum_i e_i = 1$. Using Proposition 3.4, we have $\mathcal{M} \simeq \mathcal{M}_e \overline{\otimes} \mathcal{B}(\ell^2(I))$ with $e = w_{i_0 i_0} = u_{i_0} u_{i_0}^* = e_{i_0}$. $\square$

3.7. The bimodule structure on preduals. Let A be a C^*-algebra, $\omega \in A^*$ and $a \in A$. Define $a\omega \in A^*$ to be $\langle x, a\omega \rangle = \langle xa, \omega \rangle$, $x \in A$. Similarly, we have $\omega a \in A^*$, $\langle x, \omega a \rangle = \langle ax, \omega \rangle$, $x \in A$. Thus $a\omega = \omega \circ R_a$ and $\omega a = \omega \circ L_a$, where R and L are right and left regular representations of A. Note that $(ab)\omega = \omega \circ R_{ab} = \omega \circ R_b \circ R_a = a(\omega \circ R_b) = a(b\omega)$, $\omega(ab) = \omega \circ L_{ab} = \omega \circ L_a \circ L_b = (\omega \circ L_a) b = (\omega a) b$ and $(a\omega) b = (\omega \circ R_a) \circ L_b = \omega \circ R_a \circ L_b = (\omega \circ L_b) \circ R_a = a(\omega \circ L_b) = a(\omega b)$ for all $a, b \in A$. Moreover, $\|\omega a\| = \sup |\langle a \operatorname{ball} A, \omega \rangle| \leq \|\omega\| \|a\|$ and $\|a\omega\| = \sup |\langle (\operatorname{ball} A) a, \omega \rangle| \leq \|\omega\| \|a\|$, that is, A^* is a Banach A-bimodule.

Now assume that $\mathcal{M}$ is a von Neumann algebra on a Hilbert space H. Take $\omega_{\zeta,\eta} \in \mathcal{M}_* \subseteq (\mathcal{M}_*)^{**} = \mathcal{M}^*$ with $\zeta, \eta \in \ell^2(H)$ (see Proposition 2.2) For every $a \in \mathcal{M}$ we have

$$\langle x, a\omega_{\zeta,\eta} \rangle = \langle xa, \omega_{\zeta,\eta} \rangle = \sum_{n=1}^{\infty} (xa\zeta_n, \eta_n) = \langle x, \omega_{a\zeta,\eta} \rangle \text{ and}$$

$$\langle x, \omega_{\zeta,\eta} a \rangle = \langle ax, \omega_{\zeta,\eta} \rangle = \sum_{n=1}^{\infty} (ax\zeta_n, \eta_n) = \sum_{n=1}^{\infty} (x\zeta_n, a^*\eta_n) = \langle x, \omega_{\zeta,a^*\eta} \rangle$$

for all $x \in \mathcal{M}$, where $a\zeta = (a\zeta_n)_n$, $a^*\eta = (a^*\eta_n)_n \in \ell^2(H)$. Using Proposition 2.2, we conclude that $a\omega_{\zeta,\eta} = \omega_{a\zeta,\eta} \in \mathcal{M}_*$ and $\omega_{\zeta,\eta} a = \omega_{\zeta,a^*\eta} \in \mathcal{M}_*$, that is, $\mathcal{M}_*$ is a submodule of the $\mathcal{M}$-bimodule $\mathcal{M}^*$. In particular, for every $x \in \mathcal{M}$ the subspace $\mathcal{M}_* x$ (resp., $x\mathcal{M}_*$) is a left (resp., right) $\mathcal{M}$-submodule of $\mathcal{M}_*$.

For a subspace $\mathcal{V} \subseteq \mathcal{M}_*$, the (absolute) polar of $\mathcal{V}$ in the dual space $\mathcal{M}$ is denoted by $\mathcal{V}^\perp = \{x \in \mathcal{M} : \langle \mathcal{V}, x \rangle = \{0\}\}$. Since $\mathcal{V}^\perp = \cap \{\ker(\omega) : \omega \in \mathcal{V}\}$, it follows that $\mathcal{V}^\perp$ is a w^*-closed subspace in $\mathcal{M}$ (confirm that $\mathcal{M}_*$ is included into $\mathcal{M}^*$ by means of the relation $\langle x, \omega \rangle = \langle \omega, x \rangle$, $\omega \in \mathcal{M}_*$, $x \in \mathcal{M}$). Similarly, $\mathcal{J}^\perp$ denotes the polar of a subspace $\mathcal{J} \subseteq \mathcal{M}$ in the predual $\mathcal{M}_*$, which is a closed subspace.

Lemma 3.7. *For every $e \in \mathfrak{P}(\mathcal{M})$ we have $\mathcal{M}_* e = ((1-e)\mathcal{M})^\perp$ and $e\mathcal{M}_* = (\mathcal{M}(1-e))^\perp$. In particular, $\mathcal{M}_* e$ and $e\mathcal{M}_*$ are closed submodules of $\mathcal{M}_*$ with their polars $(\mathcal{M}_* e)^\perp = (1-e)\mathcal{M}$ and $(e\mathcal{M}_*)^\perp = \mathcal{M}(1-e)$.*

Proof. First take $\omega \in ((1-e)\mathcal{M})^\perp$. Then $\{0\} = \langle \omega, (1-e)\mathcal{M} \rangle = \langle \omega(1-e), \mathcal{M} \rangle$, which in turn implies that $\omega = \omega e \in \mathcal{M}_* e$, that is, $((1-e)\mathcal{M})^\perp \subseteq \mathcal{M}_* e$. Conversely, for every $\omega \in \mathcal{M}_*$ we have $\langle \omega e, (1-e)\mathcal{M} \rangle = \langle \omega, e(1-e)\mathcal{M} \rangle = \{0\}$, that is, $\omega e \in ((1-e)\mathcal{M})^\perp$. Consequently, $\mathcal{M}_* e = ((1-e)\mathcal{M})^\perp$. Similarly, $e\mathcal{M}_* = (\mathcal{M}(1-e))^\perp$.

Finally, based on Bipolar Theorem and Lemma 3.3, we derive that $(\mathcal{M}_* e)^\perp = ((1-e)\mathcal{M})^{\perp\perp} = ((1-e)\mathcal{M})^{-w^*} = (1-e)\mathcal{M}$. Similarly, $(e\mathcal{M}_*)^\perp = \mathcal{M}(1-e)$. □

Proposition 3.5. *The correspondence $\mathcal{V} \mapsto \mathcal{V}^\perp$ implements a bijection between the closed left (resp., right) $\mathcal{M}$-submodules of $\mathcal{M}_*$ and w^*-closed right (resp., left) ideals of $\mathcal{M}$. In particular, for every closed left (resp., right) $\mathcal{M}$-submodule $\mathcal{V}$ of $\mathcal{M}_*$ there correspondence a unique projection e in $\mathcal{M}$ such that $\mathcal{V} = \mathcal{M}_* e$ (resp., $\mathcal{V} = e\mathcal{M}_*$), and closed $\mathcal{M}$-bimodules of $\mathcal{M}_*$ correspond to central projections of $\mathcal{M}$.*

Proof. Take a closed left $\mathcal{M}$-submodule $\mathcal{V} \subseteq \mathcal{M}_*$. Then $\langle \mathcal{V}, \mathcal{V}^\perp \mathcal{M} \rangle \subseteq \langle \mathcal{M}\mathcal{V}, \mathcal{V}^\perp \rangle \subseteq \langle \mathcal{V}, \mathcal{V}^\perp \rangle = \{0\}$, that is, $\mathcal{V}^\perp \mathcal{M} \subseteq \mathcal{V}^\perp$, which means that $\mathcal{V}^\perp$ is a w^*-closed right ideal of $\mathcal{M}$. Using Lemma 3.3, we derive that $\mathcal{V}^\perp = (1-e)\mathcal{M}$ for a certain projection e in $\mathcal{M}$. Using Lemma 3.7, we obtain that $\mathcal{V} = \mathcal{V}^{\perp\perp} = ((1-e)\mathcal{M})^\perp = \mathcal{M}_* e$. Similarly, if $\mathcal{V}$ is a closed right $\mathcal{M}$-submodule in $\mathcal{M}_*$, then $\mathcal{V} = e\mathcal{M}_*$ for a certain projection e. If $\mathcal{V}$ is a closed $\mathcal{M}$-bimodule then e is a central projection thanks to Lemma 3.3. Conversely, if $\mathcal{J}$ is a w^*-closed right ideal in $\mathcal{M}$ then $\mathcal{J} = (1-e)\mathcal{M}$ for a certain projection e in $\mathcal{M}$, and $\mathcal{J}^\perp = \mathcal{M}_* e$ is a closed left $\mathcal{M}$-submodule in $\mathcal{M}_*$ again thanks to Lemmas 3.3 and 3.7. The rest follows from Bipolar Theorem. □

Corollary 3.12. *Let $\mathcal{A} \subseteq \mathcal{M}$ be a subset whose linear span is w^*-dense in $\mathcal{M}$, and $\mathcal{V} \subseteq \mathcal{M}_*$ a closed subspace such that $\mathcal{A} \cdot \mathcal{V} \subseteq \mathcal{V}$. Then $\mathcal{V}$ is a left $\mathcal{M}$-submodule of $\mathcal{M}_*$.*

Proof. Take the polar $\mathcal{V}^\perp$ of $\mathcal{V}$ in $\mathcal{M}$. Then $\langle \mathcal{V}, \mathcal{V}^\perp \mathcal{A} \rangle \subseteq \langle \mathcal{A}\mathcal{V}, \mathcal{V}^\perp \rangle \subseteq \langle \mathcal{V}, \mathcal{V}^\perp \rangle = \{0\}$, that is, $\mathcal{V}^\perp \mathcal{A} \subseteq \mathcal{V}^\perp$. Put $\mathcal{I} = \{x \in \mathcal{M} : \mathcal{V}^\perp x \subseteq \mathcal{V}^\perp\}$, which is a subspace in $\mathcal{M}$ containing $\mathcal{A}$. If $x = w^*\text{-}\lim_\lambda x_\lambda$ for a net $(x_\lambda)_\lambda$ in $\mathcal{I}$, then $yx = w^*\text{-}\lim_\lambda yx_\lambda \in \mathcal{V}^\perp$ for every $y \in \mathcal{V}^\perp$. It follows that $\mathcal{I}$ is w^*-closed, therefore $\mathcal{I} = \mathcal{M}$, that is, $\mathcal{V}^\perp \mathcal{M} \subseteq \mathcal{V}^\perp$ or $\mathcal{V}^\perp$ is a w^*-closed right ideal of $\mathcal{M}$. Using Proposition 3.5, we derive that $\mathcal{V}$ is a closed left $\mathcal{M}$-submodule of $\mathcal{M}_*$. □

Corollary 3.13. *For $e, f \in \mathfrak{P}(\mathcal{M})$ we have $\mathcal{M}_* e \subseteq \mathcal{M}_* f$ iff $(1-e)\mathcal{M} \supseteq (1-f)\mathcal{M}$ (or $e \leq f$). Moreover, for a subset $S \subseteq \mathfrak{P}(\mathcal{M})$ we have*

$$\mathcal{M}_* \wedge S = \cap_{e\in S}\mathcal{M}_* e, \quad \wedge(1-S)\mathcal{M} = \cap_{e\in S}(1-e)\mathcal{M},$$
$$\mathcal{M}_* \vee S = \langle \cup_{e\in S}\mathcal{M}_* e\rangle, \quad \vee(1-S)\mathcal{M} = \langle \cup_{e\in S}(1-e)\mathcal{M}\rangle^{-w^*}.$$

A similar assertion takes place for the right $\mathcal{M}$-submodules of $\mathcal{M}_*$ and w^*-closed left ideals of $\mathcal{M}$.

Proof. Using Proposition 3.5 and Bipolar Theorem, we derive that $\mathcal{M}_* e \subseteq \mathcal{M}_* f$ iff

$$(1-e)\mathcal{M} = ((1-e)\mathcal{M})^{\perp\perp} = (\mathcal{M}_* e)^{\perp} \supseteq (\mathcal{M}_* f)^{\perp} = ((1-f)\mathcal{M})^{\perp\perp} = (1-f)\mathcal{M}.$$

Further, take $S \subseteq \mathfrak{P}(\mathcal{M})$. Since $\cap_{e\in S}\mathcal{M}_* e$ is a closed left $\mathcal{M}$-submodule in $\mathcal{M}_*$, it follows that $\cap_{e\in S}\mathcal{M}_* e = \mathcal{M}_* p$ for some $p \in \mathfrak{P}(\mathcal{M})$ by Proposition 3.5. But $\mathcal{M}_* p \subseteq \mathcal{M}_* e$, $e \in S$, therefore $p \leq \wedge S$. Note also that $\mathcal{M}_* \wedge S \subseteq \mathcal{M}_* e$, $e \in S$, which in turn implies that $\mathcal{M}_* \wedge S \subseteq \cap_{e\in S}\mathcal{M}_* e = \mathcal{M}_* p$ or $\wedge S \leq p$. Whence $p = \wedge S$. Similar argument is applicable to $\cap_{e\in S}(1-e)\mathcal{M} = \wedge(1-S)\mathcal{M}$.

Further, using Lemma 3.7, (3.1) and (2.1), we obtain that

$$\mathcal{M}_*(\vee S) = ((1-\vee S)\mathcal{M})^{\perp} = (\wedge(1-S)\mathcal{M})^{\perp} = (\cap_{e\in S}(1-e)\mathcal{M})^{\perp}$$
$$= \left\langle \cup_{e\in S}((1-e)\mathcal{M})^{\perp}\right\rangle = \langle \cup_{e\in S}\mathcal{M}_* e\rangle,$$

that is, $\mathcal{M}_* \vee S = \langle \cup_{e\in S}\mathcal{M}_* e\rangle$. In particular, $\langle \cup_{e\in S}\mathcal{M}_* e\rangle$ is a left $\mathcal{M}$-submodule. Based on Lemma 3.7 and (2.2), we have also

$$\langle \cup_{e\in S}(1-e)\mathcal{M}\rangle^{-w^*} = \left\langle \cup_{e\in S}(\mathcal{M}_* e)^{\perp}\right\rangle^{-w^*}$$
$$= (\cap_e \mathcal{M}_* e)^{\perp} = (\mathcal{M}_* \wedge S)^{\perp} = (1-\wedge S)\mathcal{M} = \vee(1-S)\mathcal{M},$$

that is, $\vee(1-S)\mathcal{M} = \langle \cup_{e\in S}(1-e)\mathcal{M}\rangle^{-w^*}$. □

Corollary 3.14. *For every $e \in \mathfrak{P}(\mathcal{M})$ the right ideal $e\mathcal{M}$ is a von Neumann algebra with its predual $(e\mathcal{M})_* = \mathcal{M}_* e$ up to an isometric isomorphism. Moreover, for $e, f \in \mathfrak{P}(\mathcal{M})$, $e \leq f$ the adjoint mapping $\iota^* : (\mathcal{M}_* f)^* \to (\mathcal{M}_* e)^*$ to the inclusion $\iota : \mathcal{M}_* e \hookrightarrow \mathcal{M}_* f$ of the closed left $\mathcal{M}$-modules is reduced to the canonical mapping $f\mathcal{M} \to e\mathcal{M}$, $x \mapsto ex$. A similar assertion takes place for the left ideals and related closed right $\mathcal{M}$-modules.*

Proof. Since $e\mathcal{M}$ is w^*-closed subalgebra, it follows that $e\mathcal{M} = (\mathcal{M}_*(1-e))^{\perp}$ by Lemma 3.7. But the adjoint π^* of the quotient mapping $\pi : \mathcal{M}_* \to \mathcal{M}_*/\mathcal{M}_*(1-e)$ is reduced to the inclusion $(\mathcal{M}_*(1-e))^{\perp} \hookrightarrow \mathcal{M}$ (see [24, 2.4.13]), that is, $(\mathcal{M}_*/\mathcal{M}_*(1-e))^* = e\mathcal{M}$ up to an isometry. But $\mathcal{M}_* = \mathcal{M}_*(1-e) \oplus \mathcal{M}_* e$ and there is a well defined contraction $\chi : \mathcal{M}_* \to \mathcal{M}_* e$, $\chi(\omega) = \omega e$ (as we have seen above $\mathcal{M}_*$ is a Banach $\mathcal{M}$-module). Note that $\pi\chi(\omega) = \pi(\omega e) = \pi(\omega e + \omega(1-e)) = \pi(\omega)$ for all $\omega \in \mathcal{M}_*$, and $\pi|\mathcal{M}_* e$ is invertible with the inverse $\sigma : \mathcal{M}_*/\mathcal{M}_*(1-e) \to \mathcal{M}_* e$. Note that $\sigma(\omega^{\sim}) = \sigma(\pi(\omega+y)) = \sigma(\pi\chi(\omega+y)) = \chi(\omega+y) = (\omega+y)e$ for all $y \in \mathcal{M}_*(1-e)$. It follows that $\|\sigma(\omega^{\sim})\| = \|(\omega+y)e\| \leq \|\omega+y\|\|e\| = \|\omega+y\|$, that is, $\|\sigma(\omega^{\sim})\| \leq \inf\|\omega+\mathcal{M}_*(1-e)\| = \|\omega^{\sim}\|$ or $\|\sigma\| \leq 1$. But $\|\pi|\mathcal{M}_* e\| \leq 1$ as well. Hence $\pi\iota : \mathcal{M}_* e \to \mathcal{M}_*/\mathcal{M}_*(1-e)$ implements an isometry, where ι :

$\mathcal{M}_*e \hookrightarrow \mathcal{M}_*$ is the inclusion. Thus the adjoint of the sequence $\mathcal{M}_*e \overset{\iota}{\hookrightarrow} \mathcal{M}_* \overset{\pi}{\longrightarrow} \mathcal{M}_*/\mathcal{M}_*(1-e)$ is reduced to $e\mathcal{M} \overset{\pi^*}{\hookrightarrow} \mathcal{M} \overset{\iota^*}{\longrightarrow} (\mathcal{M}_*e)^*$ and $\iota^*\pi^*$ is an isometry. Note also that ι^* is the quotient mapping $\mathcal{M} \to \mathcal{M}/(\mathcal{M}_*e)^\perp = \mathcal{M}/(1-e)\mathcal{M}$ (see [24, 2.4.13]). The duality between $\mathcal{M}_*e$ and $e\mathcal{M}$ is given by the pairing $\langle \omega e, ex\rangle = \langle \omega, ex\rangle = \langle \omega e, x\rangle$ for all $\omega \in \mathcal{M}_*$, $x \in \mathcal{M}$.

Further, take $e, f \in \mathfrak{P}(\mathcal{M})$ with $e \leq f$, and consider the inclusion $\iota : \mathcal{M}_*e \hookrightarrow \mathcal{M}_*f$ of closed left $\mathcal{M}$-modules. Then $\iota^* : f\mathcal{M} = (\mathcal{M}_*f)^* \to (\mathcal{M}_*e)^* = e\mathcal{M}$ and $\langle \omega e, \iota^*(fx)\rangle = \langle \iota(\omega e), fx\rangle = \langle \omega e, fx\rangle = \langle \omega e, efx\rangle = \langle \omega e, ex\rangle$ for all $\omega \in \mathcal{M}_*$, $x \in \mathcal{M}$, that is, $\iota^*(fx) = ex$ or ι^* is reduced to $f\mathcal{M} \to e\mathcal{M}$, $x \mapsto ex$. □

3.8. The polar decomposition of a normal functional. Let $\mathcal{M}$ be a von Neumann algebra on H. By *a normal functional from* $\mathcal{M}^*$ we mean a w^*-continuous functional on $\mathcal{M}$, that is, an element of the $\mathcal{M}$-bimodule $\mathcal{M}_*$. Take $\omega \in \mathcal{M}_*$ and consider the closed left $\mathcal{M}$-submodule $\langle \mathcal{M}\omega\rangle$ in $\mathcal{M}$ generated by ω. By Proposition 3.5, $\langle \mathcal{M}\omega\rangle = \mathcal{M}_*e$ for a unique projection $e \in \mathfrak{P}(\mathcal{M})$. Similarly, $\langle \omega\mathcal{M}\rangle = e\mathcal{M}_*$.

Lemma 3.8. *Let $\omega \in \mathcal{M}_*$ with $\langle \mathcal{M}\omega\rangle = \mathcal{M}_*e$. Then e is smallest projection in $\mathcal{M}$ with the property $\omega e = \omega$. Similar result takes place for the $\langle \omega\mathcal{M}\rangle$.*

Proof. First note that $\omega = \varphi e$ for some $\varphi \in \mathcal{M}_*$. Therefore $\omega e = \varphi e^2 = \varphi e = \omega$. Conversely, assume that $\omega f = \omega$ for some $f \in \mathfrak{P}(\mathcal{M})$. Then $\mathcal{M}_*e = \langle \mathcal{M}\omega\rangle = \langle \mathcal{M}\omega f\rangle \subseteq \langle \mathcal{M}_*f\rangle = \mathcal{M}_*f$, which in turn implies that $e \leq f$ by virtue of Corollary 3.13. Whence $e = \wedge\{f \in \mathfrak{P}(\mathcal{M}) : \omega f = \omega\}$. □

The projection e from Lemma 3.8 is called the right support of ω denoted by $s_r(\omega)$. Similarly, we have the left support $s_l(\omega)$. Thus $\langle \mathcal{M}\omega\rangle = \mathcal{M}_*s_r(\omega)$ and $\langle \omega\mathcal{M}\rangle = s_l(\omega)\mathcal{M}_*$. Recall that ω admits its adjoint $\omega^* : \mathcal{M} \to \mathbb{C}$, $\langle x, \omega^*\rangle = \overline{\langle x^*, \omega\rangle}$, $x \in \mathcal{M}$. If $\omega = \omega_{\zeta,\eta}$ for some $\zeta, \eta \in \ell^2(H)$ then $\left\langle x, \omega^*_{\zeta,\eta}\right\rangle = \overline{\langle x^*, \omega_{\zeta,\eta}\rangle} = \sum_n \overline{(x^*\zeta_n, \eta_n)} = \sum_n (x\eta_n, \zeta_n) = \langle x, \omega_{\eta,\zeta}\rangle$ for all $x \in \mathcal{M}$, that is, $\omega^* \in \mathcal{M}_*$.

Corollary 3.15. *If $\omega \in \mathcal{M}_*$ then $s_r(\omega^*) = s_l(\omega)$ and $s_l(\omega^*) = s_r(\omega)$. For a hermitian ω we have $s_r(\omega) = s_l(\omega)$ called the support $s(\omega)$ of ω. In particular, a positive ω from $\mathcal{M}_*$ is faithful on $s(\omega)\mathcal{M}s(\omega)$, namely, $\langle x^*x, \omega\rangle = 0$ implies that $xs(\omega) = 0$.*

Proof. First note that $\langle x, (e\omega)^*\rangle = \overline{\langle x^*, e\omega\rangle} = \overline{\langle x^*e, \omega\rangle} = \overline{\langle (ex)^*, \omega\rangle} = \langle ex, \omega^*\rangle = \langle x, \omega^*e\rangle$ for all $x \in \mathcal{M}$, that is, $(e\omega)^* = \omega^*e$. Similarly, $(\omega e)^* = e\omega^*$. Using Lemma 3.8, we derive that $\omega = \omega^{**} = (\omega^*s_r(\omega^*))^* = (s_r(\omega^*)\omega)^{**} = s_r(\omega^*)\omega$, that is, $s_l(\omega) \leq s_r(\omega^*)$. By symmetry, $\omega = \omega^{**} = (s_l(\omega^*)\omega^*)^* = (\omega s_l(\omega^*))^{**} = \omega s_l(\omega^*)$, that is, $s_r(\omega) \leq s_l(\omega^*)$. It follows that $s_r(\omega^*) \leq s_l(\omega^{**}) = s_l(\omega)$, that is, $s_r(\omega) = s_l(\omega^*)$. Similarly, $s_r(\omega^*) = s_l(\omega)$. If $\omega = \omega^*$ then $s_r(\omega) = s_l(\omega) = s(\omega)$. Thus $s(\omega)\omega = \omega s(\omega) = \omega$.

Finally, assume that $\omega \geq 0$. If $\langle x^*x, \omega\rangle = 0$ then using Cauchy-Schwarz inequality (see [24, E 4.3.12]), we have $|\langle x, \omega a\rangle|^2 = |\langle a^{**}x, \omega\rangle|^2 \leq \langle aa^*x, \omega\rangle \langle x^*x, \omega\rangle = 0$ for all $a \in \mathcal{M}$. It follows that $\langle xs(\omega), \mathcal{M}_*\rangle = \langle x, s(\omega)\mathcal{M}_*\rangle = \langle x, \langle \omega\mathcal{M}\rangle\rangle \subseteq \langle x, \omega\mathcal{M}\rangle^- = \{0\}$, that is, $xs(\omega) = 0$. In particular, if $x \in s(\omega)\mathcal{M}s(\omega)$ and $\langle x^*x, \omega\rangle = 0$, then $x = xs(\omega) = 0$. □

Lemma 3.9. *If $\varphi \in \mathcal{M}_*$ and $e \in \mathfrak{P}(\mathcal{M})$ with $\|e\varphi\| = \|\varphi\|$, then $e\varphi = \varphi$.*

Proof. We can assume that $\|\varphi\| = 1$, and put $e^{\perp} = 1 - e$. Prove that $e^{\perp}\varphi = 0$. In the contrary case, we have $\langle b, e^{\perp}\varphi \rangle = \delta > 0$ for some $b \in \operatorname{ball} \mathcal{M}$. Since $\|e\varphi\| = 1$ and $\mathcal{M} = (\mathcal{M}_*)^*$, we also have $\langle a, e\varphi\rangle = 1$ for some $a \in \operatorname{ball} \mathcal{M}$ thanks to Hahn-Banach theorem. Further,

$$\left\|ae + \delta b e^{\perp}\right\|^2 = \left\|\left(ae + \delta b e^{\perp}\right)\left(ae + \delta b e^{\perp}\right)^*\right\| = \left\|\left(ae + \delta b e^{\perp}\right)\left(ea^* + \delta e^{\perp} b^*\right)\right\|$$

$$= \left\|aea^* + \delta^2 b e^{\perp} b^*\right\| \leq \|aea^*\| + \delta^2 \left\|b e^{\perp} b^*\right\|$$

$$\leq \|a\|^2 + \delta^2 \|b\|^2 \leq 1 + \delta^2,$$

that is, $\left\|ae + \delta b e^{\perp}\right\| \leq \left(1 + \delta^2\right)^{1/2}$. It follows that

$$1 + \delta^2 = \langle a, e\varphi\rangle + \delta \left\langle b, e^{\perp}\varphi \right\rangle = \langle ae, \varphi\rangle + \delta \left\langle b e^{\perp}, \varphi \right\rangle = \left\langle ae + \delta b e^{\perp}, \varphi \right\rangle$$

$$\leq \left\|ae + \delta b e^{\perp}\right\| \|\varphi\| = \left\|ae + \delta b e^{\perp}\right\| \leq \left(1 + \delta^2\right)^{1/2} < 1 + \delta^2,$$

a contradiction. Whence $e^{\perp}\varphi = 0$ or $e\varphi = \varphi$. □

Now take $\varphi \in \mathcal{M}_*$. Since $\mathcal{M} = (\mathcal{M}_*)^*$, it follows that $\langle a, \varphi\rangle = \|\varphi\|$ for some $a \in \operatorname{ball} \mathcal{M}$ (Hahn-Banach theorem). Consider the polar decomposition $a^* = u\,|a^*|$ in $\mathcal{M}$, and put $|\varphi| = u^*\varphi \in \mathcal{M}_*$, $e = uu^* \in \mathfrak{P}(\mathcal{M})$, which is the projection onto $a^*(H)^-$, thereby $ea^* = a^*$ or $a = ae$.

Lemma 3.10. *For $\varphi \in \mathcal{M}_*$ we have $|\varphi| \geq 0$, $|||\varphi||| = \|\varphi\|$ and $e\varphi = \varphi$.*

Proof. Put $b = |a^*|$. Note that $a^* = ub$ and $\|b\|^2 = \left\||a^*|^2\right\| = \|aa^*\| = \|a^*\|^2 = \|a\|^2 \leq 1$, that is, $0 \leq b \leq 1$. Further, $\|\varphi\| = \langle a, \varphi\rangle = \langle bu^*, \varphi\rangle = \langle b, u^*\varphi\rangle = \langle b, |\varphi|\rangle$ and

$$\|\varphi\| = \langle b, |\varphi|\rangle \leq \|b\|\,|||\varphi||| \leq |||\varphi||| = \sup|\langle \operatorname{ball}\mathcal{M}, |\varphi|\rangle| = \sup|\langle \operatorname{ball}\mathcal{M}, u^*\varphi\rangle|$$

$$= \sup|\langle (\operatorname{ball}\mathcal{M})\,u^*, \varphi\rangle| \leq \sup\|(\operatorname{ball}\mathcal{M})\,u^*\|\,\|\varphi\| \leq \|\varphi\|,$$

that is, $|||\varphi||| = \langle b, |\varphi|\rangle = \|\varphi\|$. Choose $\theta \in \mathbb{R}$ so that $e^{i\theta}\langle 1 - b, |\varphi|\rangle \geq 0$, and consider the normal element $b + e^{i\theta}(1 - b)$ whose norm is reduced to its spectral radius $r\left(b + e^{i\theta}(1 - b)\right) \leq \sup\left\{\left|t + e^{i\theta}(1 - t)\right| : 0 \leq t \leq 1\right\} \leq 1$ (see [24, 4.3.11]). Then

$$|||\varphi||| = \langle b, |\varphi|\rangle \leq \langle b, |\varphi|\rangle + e^{i\theta}\langle 1 - b, |\varphi|\rangle = \left\langle b + e^{i\theta}(1 - b), |\varphi| \right\rangle \leq |||\varphi|||,$$

which in turn implies that $\langle 1 - b, |\varphi|\rangle = 0$ or $\langle 1, |\varphi|\rangle = \langle b, |\varphi|\rangle = |||\varphi|||$. Thus $|\varphi|$ is a bounded functional on the C^*-algebra with the property $|||\varphi||| = \langle 1, |\varphi|\rangle$, which means that $|\varphi| \geq 0$ (see [24, E 4.3.13]). Further, $u\,|\varphi| = uu^*\varphi = e\,|\varphi|$ and $\|\varphi\| = \langle a, \varphi\rangle = \langle ae, \varphi\rangle = \langle a, e\varphi\rangle \leq \|a\|\,\|e\varphi\| \leq \|e\varphi\| \leq \|\varphi\|$, that is, $\|\varphi\| = \|e\varphi\|$. Using Lemma 3.9, we conclude that $e\varphi = \varphi$. □

Proposition 3.6. *If $\varphi \in \mathcal{M}_*$ then $v = us(|\varphi|)$ is a partial isometry in $\mathcal{M}$, $v^*v = s(|\varphi|) = s_r(\varphi)$, $vv^* = s_l(\varphi)$ and $\varphi = v\,|\varphi|$.*

Proof. Using Lemma 3.10, we derive that $\varphi = e\varphi = uu^*\varphi = u|\varphi|$ with $|\varphi| \geq 0$. But $u^*u|\varphi| = u^*\varphi = |\varphi|$, which implies that $u^*u \geq s_l(|\varphi|) = s(|\varphi|)$. Put $v = us(|\varphi|)$. Then $v^*v = s(|\varphi|)u^*us(|\varphi|) = s(|\varphi|)^2 = s(|\varphi|)$, and $vv^* = us(|\varphi|)^2 u^* = us(|\varphi|)u^*$ is a projection, for $us(|\varphi|)u^*us(|\varphi|)u^* = us(|\varphi|)^2 u^* = us(|\varphi|)u^*$. Hence v is a partial isometry in $\mathcal{M}$. Moreover, $\varphi = u|\varphi| = us(|\varphi|)|\varphi| = v|\varphi|$. Note also that $\varphi s(|\varphi|) = v|\varphi|s(|\varphi|) = v|\varphi| = \varphi$, and

$$|\varphi| = s(|\varphi|)|\varphi| = v^*v|\varphi| = v^*\varphi = v^*\varphi s_r(\varphi)$$

$$= v^*v|\varphi|s_r(\varphi) = s(|\varphi|)|\varphi|s_r(\varphi) = |\varphi|s_r(\varphi),$$

therefore $s(|\varphi|) \geq s_r(\varphi) \geq s(|\varphi|)$, that is, $s_r(\varphi) = s(|\varphi|)$.

Finally, $vv^*\varphi = vv^*v|\varphi| = vs(|\varphi|)|\varphi| = v|\varphi| = \varphi$, which implies that $vv^* \geq s_l(\varphi)$. Then $(v^*s_l(\varphi)v)^2 = v^*s_l(\varphi)vv^*s_l(\varphi)v = v^*s_l(\varphi)^2 v = v^*s_l(\varphi)v$, that is, $v^*s_l(\varphi)v \in \mathfrak{P}(\mathcal{M})$. But $|\varphi| = s(|\varphi|)|\varphi| = v^*v|\varphi| = v^*\varphi = v^*s_l(\varphi)\varphi = v^*s_l(\varphi)v|\varphi|$, therefore $v^*s_l(\varphi)v \geq s(|\varphi|)$. Taking into account that $vv^* \geq s_l(\varphi)$ and $v^*v = s(|\varphi|)$, we obtain that $s_l(\varphi) = vv^*s_l(\varphi)vv^* = v(v^*s_l(\varphi)v)v^* \geq vs(|\varphi|)v^* = vv^*vv^* = (vv^*)^2$. Since $(vs(|\varphi|)v^*)^2 = vs(|\varphi|)v^*vs(|\varphi|)v^* = vs(|\varphi|)^2 v^*$, it follows that $(vv^*)^2$ is a projection. In particular, vv^* is a projection. Thus $s_l(\varphi) \geq vv^*$ and $vv^*\varphi = \varphi$, which in turn implies that $s_l(\varphi) = vv^*$. □

Corollary 3.16. *Let $\omega_{\zeta,\eta} \in \mathcal{M}_*$ with $\zeta, \eta \in H$, and let $e = s_l(\omega_{\zeta,\eta})$ be the left support of $\omega_{\zeta,\eta}$. Then $(1-e)\zeta \perp \langle\mathcal{M}\eta\rangle$.*

Proof. By Proposition 3.6, $\omega_{\zeta,\eta}$ admits a polar decomposition $\omega_{\zeta,\eta} = v|\omega_{\zeta,\eta}|$ with $v^*v = s(|\omega_{\zeta,\eta}|) = s_r(\omega_{\zeta,\eta})$ and $vv^* = s_l(\omega_{\zeta,\eta}) = e$. For every $x \in \mathcal{M}$ we have $(e\zeta, x\eta) = (x^*e\zeta, \eta) = \langle x^*e, \omega_{\zeta,\eta}\rangle = \langle x^*, e\omega_{\zeta,\eta}\rangle = \langle x^*, \omega_{\zeta,\eta}\rangle = (x^*\zeta, \eta) = (\zeta, x\eta)$. It follows that $(1-e)\zeta \perp \langle\mathcal{M}\eta\rangle$. □

4. Multinormed completions of von Neumann algebras

In this section we introduce multinormed (or locally convex) completions of a von Neumann algebra, which in turn are multinormed W^*-algebras.

4.1. Multinormed C^*-algebras. Let A be an associative $*$-algebra. Recall that a seminorm p on A is said to be a *C^*-seminorm* if $p(a^*a) = p(a)^2$ for all $a \in A$. It turns out (see [15, 7.2]) that each C^*-seminorm preserves the involution $*$ on A, and it is submultiplicative. A complete polynormed (or locally convex) algebra A whose topology is defined by a (separated) family of C^*-seminorms $\mathcal{E} = \{\|\cdot\|_\iota : \iota \in \Xi\}$ is called a *multinormed* (or *locally*) *C^*-algebra.* In particular, it is an Arens-Michael algebra expanded into inverse limit $A_\mathcal{E} = \varprojlim\{A_\iota, \varphi_{\iota\kappa}\}$ (see [17, 5.2.10]) of C^*-algebras A_ι such that all connecting maps $\varphi_{\iota\kappa} : A_\kappa \to A_\iota$ ($\iota \leq \kappa$) are $*$-homomorphisms. The set of all finite subsets of $\mathcal{E}$ is denoted by Λ, that is, $\alpha \in \Lambda$ means that $\alpha \subseteq \Xi$ is a finite subset.

Now let $A_\mathcal{E}$ be a multinormed C^*-algebra with its defining family $\mathcal{E} = \{\|\cdot\|_\iota : \iota \in \Xi\}$ of C^*-seminorms, and let $A = \mathfrak{b}(A_\mathcal{E})$ be the set of all bounded elements b in $A_\mathcal{E}$, that is, $\sup_\iota \|b\|_\iota < \infty$. We say that A is *a bounded part of* $A_\mathcal{E}$, which is a C^*-algebra equipped with the norm $\|\cdot\| = \sup_\iota \|\cdot\|_\iota$ (see [25]). Put $J_\iota = \ker\|\cdot\|_\iota$ and $I_\iota = A \cap J_\iota$. Note that $A_\iota = A_\mathcal{E}/J_\iota$ for all $\iota \in \Xi$.

Lemma 4.1. *The ideal I_ι is closed in A and $\|\cdot\| = \|\cdot\|_\iota$ on A/I_ι. Moreover, $A/I_\iota = A_\mathcal{E}/J_\iota$ up to a $*$-isomorphism.*

Proof. If $b = \lim_n b_n$ in the C^*-algebra A for a certain sequence $(b_n) \subseteq I_\iota$, then $\|b\|_\iota \leq \|b - b_n\|_\iota + \|b_n\|_\iota \leq \|b - b_n\|$, that is, $b \in I_\iota$. Thus I_ι is a closed ideal in A. In particular, it is self-adjoint and $(A/I_\iota, \|\cdot\|)$ is a C^*-algebra too. Further, using [28, 5.4], we conclude that $A_\mathcal{E}/J_\iota$ is a C^*-algebra with the quotient norm $\|\cdot\|_\iota$, and the quotient mapping $\pi_\iota : A_\mathcal{E} \to A_\mathcal{E}/J_\iota$ is a $*$-homomorphism. Since A is dense in $A_\mathcal{E}$, it follows that $\pi_\iota : A \to A_\mathcal{E}/J_\iota$ has the dense range. Therefore $\pi_\iota(A) = A_\mathcal{E}/J_\iota$ (see for instance [23, 3.1.5]). In particular, $A/I_\iota = A_\mathcal{E}/J_\iota$ up to a $*$-isomorphism, where A/I_ι is equipped with the quotient C^*-norm $\|\cdot\|_\iota$. Thus we have the C^*-norms $\|\cdot\|$ and $\|\cdot\|_\iota$ on the $*$-algebra A/I_ι. Hence $\|\cdot\| = \|\cdot\|_\iota$. □

4.2. The central topology of a von Neumann algebra. Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space H. *A domain in* $\mathcal{M}$ is defined as a family $\mathcal{E} = \{e_\iota : \iota \in \Xi\} \subseteq \mathfrak{P}_c(\mathcal{M})$ of its central projections such that $\vee\mathcal{E} = 1$. Sometimes we identify Ξ with the domain $\mathcal{E}$ itself. The set of all finite subsets of Ξ is denoted by Λ, that is, $\alpha \in \Lambda$ means that $\alpha \subseteq \Xi$ is a finite subset. For every $\alpha \in \Lambda$ we have the projection $e_\alpha = \vee_{\iota\in\alpha} e_\iota$ in $\mathcal{M}$, which is a central one thanks to Corollary 3.7. Note that $\vee_\alpha e_\alpha =$SOT-$\lim_\alpha e_\alpha = 1$. For each e_ι we set $\|x\|_\iota = \|xe_\iota\|$, $x \in \mathcal{M}$. Then $\|\cdot\|_\iota$ is a C^*-seminorm on $\mathcal{M}$. Indeed, $\|x^*x\|_\iota = \|x^*xe_\iota\| = \|(xe_\iota)^* xe_\iota\| = \|xe_\iota\|^2 = \|x\|_\iota^2$, $x \in \mathcal{M}$. Furthermore, $\|x\| = \sup_\iota \|x\|_\iota$ for every $x \in \mathcal{M}$.

Lemma 4.2. *The following assertions are equivalent:*

(i) $\mathcal{M}e_\iota \subseteq \mathcal{M}e_\kappa$,

(ii) $e_\iota \leq e_\kappa$,

(iii) $\mathcal{M}_*(1 - e_\kappa) \subseteq \mathcal{M}_*(1 - e_\iota)$,

(iv) $\|\cdot\|_\iota \leq C_{\iota\kappa} \|\cdot\|_\kappa$ *for a certain constant* $C_{\iota\kappa} > 0$.

Moreover, if $e_\alpha = \vee_{\iota\in\alpha} e_\iota$ *for a finite subset* $\{e_\iota : \iota \in \alpha\} \subseteq \mathcal{E}$ *and* $\|x\|_\alpha = \|xe_\alpha\|$, $x \in \mathcal{M}$ *is the relevant C^*-seminorm on* $\mathcal{M}$ *then* $\|\cdot\|_\alpha = \max_{\iota\in\alpha} \|\cdot\|_\iota$.

Proof. The equivalence of (i), (ii) and (iii) follows from Corollary 3.13.

$(ii) \Rightarrow (iv)$ If $e_\iota \leq e_\kappa$ then $\|x\|_\iota^2 = \|x^*x\|_\iota = \|x^*e_\iota x\| \leq \|x^*e_\kappa x\| = \|x\|_\kappa^2$ for all $x \in \mathcal{M}$ (see [24, 3.2.9]).

$(iv) \Rightarrow (i)$ If $\|\cdot\|_\iota \leq C_{\iota\kappa} \|\cdot\|_\kappa$ and $x \in \mathcal{M}$ then $\|x(1 - e_\kappa)\|_\iota \leq C_{\iota\kappa} \|x(1 - e_\kappa)\|_\kappa = 0$, that is, $x(1 - e_\kappa)e_\iota = 0$. Then $xe_\iota = xe_\iota e_\kappa \in \mathcal{M}e_\kappa$, that is, $\mathcal{M}e_\iota \subseteq \mathcal{M}e_\kappa$.

Further, fix $\alpha \in \Lambda$. Based on Corollary 3.13, we have $\mathcal{M}e_\alpha = \langle \cup_{\iota\in\alpha} \mathcal{M}e_\iota \rangle^{-w^*} = \left(\sum_{\iota\in\alpha} \mathcal{M}e_\iota\right)^{-w^*}$. In particular, $\|\cdot\|_\alpha \geq \max_{\iota\in\alpha} \|\cdot\|_\iota$. Conversely, one can assume that $e_\alpha = e_1 \vee e_2 = 1$. If $f_i = 1 - e_i$ then $H = e_1e_2(H) \oplus f_2e_1(H) \oplus f_1e_2(H) \oplus f_1f_2(H)$. Moreover, this orthogonal decomposition reduces each $x \in \mathcal{M}$. If $\zeta \in f_1f_2(H)$ then $\zeta \perp e_1(H)$ and $\zeta \perp e_2(H)$, that is, $\zeta \perp (e_1(H) + e_2(H))$. But $e_1(H) + e_2(H)$ is dense in H, for $e_1 \vee e_2 = 1$. Hence $\zeta = 0$. Thus $f_1f_2(H) = \{0\}$ or $H = e_1e_2(H) \oplus f_2e_1(H) \oplus f_1e_2(H)$. It follows that $\|x\| = \max\{\|xe_1e_2\|, \|xf_2e_1\|, \|xf_1e_2\|\} \leq \max\{\|xe_1\|, \|xe_2\|\}$. Consequently, $\|\cdot\|_\alpha = \max_{\iota\in\alpha} \|\cdot\|_\iota$. □

Corollary 4.1. *For every* $x \in \mathcal{M}$ *we have* $\|x\| = \sup\{\|x\|_\iota : \iota \in \Xi\}$.

Proof. Take $x \in \mathcal{M}$. Obviously, $\sup_\iota \{\|x\|_\iota\} \leq \|x\|$. Conversely, note that $\|x\| - \varepsilon/2 \leq \|x\zeta\|$ for some $\zeta \in \operatorname{ball} H$ and $1 =$SOT-$\lim_\alpha e_\alpha$, therefore $\|x\zeta\| =$

$\lim_\alpha \|e_\alpha x\zeta\| = \lim_\alpha \|xe_\alpha\zeta\|$. It follows that $\|x\| - \varepsilon \leq \|x\zeta\| - \varepsilon/2 \leq \|xe_\alpha\zeta\| \leq \|xe_\alpha\| = \|x\|_\alpha$ for some α. But $\|x\|_\alpha = \max_{\iota\in\alpha} \|x\|_\iota$ by virtue of Lemma 4.2. Thus $\|x\| - \varepsilon \leq \max_{\iota\in\alpha} \|x\|_\iota \leq \sup_\iota \{\|x\|_\iota\}$ for every small $\varepsilon > 0$. Whence $\sup_\iota \{\|x\|_\iota\} = \|x\|$. □

The family $\{\|\cdot\|_\iota\}$ of C^*-seminorms on a von Neumann algebra $\mathcal{M}$ associated with the domain $\mathcal{E}$ defines a new polynormed topology in $\mathcal{M}$ called *a central topology*. Note that each central topology is Hausdorff. If $\|x\|_\iota = 0$ for all ι then for each finite index set α, we have $xe_\alpha = 0$ by Lemma 4.2. Since $\vee_\alpha e_\alpha = 1$, it follows that $x = w^*\text{-}\lim_\alpha xe_\alpha = 0$. Furthermore, based on Lemma 4.2 we assert that a central topology can also be defined by means of an upward filtered family of projections (e_α). Two upward filtered families (e_α) and (f_v) of projections in $\mathcal{M}$ are assumed to be *equivalent* (and we write $(e_\alpha) \approx (f_v)$) if for each e_α there corresponds f_v such that $e_\alpha \leq f_v$ and viceversa.

Corollary 4.2. *Let $\mathcal{M}$ be a von Neumann algebra with upward filtered families (e_α) and (f_v) of projections such that $\vee_\alpha e_\alpha = \vee_v f_v = 1$. They define the same central topology in $\mathcal{M}$ iff $(e_\alpha) \approx (f_v)$.*

Proof. The families (e_α) and (f_v) define the same central topology iff for each e_α there corresponds f_v such that $\|\cdot\|_\alpha \leq C_{\alpha v} \|\cdot\|_v$, and viceversa. It remains to use Lemma 4.2. □

Thus there is a one-to-one correspondence between central topologies on a von Neumann algebra $\mathcal{M}$ and the equivalent classes of upward filtered to 1 central projections in $\mathcal{M}$. The completion $\mathcal{M}_\mathcal{E}$ of $\mathcal{M}$ with respect to a central topology turns out to be a multinormed C^*-algebra. We say that it is *a central completion of* $\mathcal{M}$. Based on Corollary 4.2, we can assume that $\mathcal{E}$ contains all $\{e_\alpha : \alpha \in \Lambda\}$ as well.

Definition 4.1. Let X be a normed space. We say that X is an ℓ^1-normed space if it is a locally quotient $\pi : \bigoplus_{\iota\in\Xi} E_\iota \to X$ of the direct sum of some Banach spaces, in the sense that π is onto and for each α it is a quotient mapping of the finite ℓ^1-sum $\overset{1}{\bigoplus_{\iota\in\alpha}} E_\iota$ onto its range, which is a complemented Banach subspace of X.

If $X_\alpha = \pi\left(\overset{1}{\bigoplus_{\iota\in\alpha}} E_\iota\right)$ then X_α is complete in X and $X = \cup_\alpha X_\alpha$. In particular, X is equipped with the bornology $\{\operatorname{ball} X_\alpha\}$.

Examples: 1. The algebraic sum $X = \sum_\iota E_\iota$ in the ℓ^1-sum $E = \overset{1}{\bigoplus_\iota} E_\iota$ of Banach spaces is an ℓ^1-normed space. In this case $X_\alpha = \overset{1}{\bigoplus_{\iota\in\alpha}} E_\iota$ for all α.

2. Each Banach space is obviously an ℓ^1-normed space.

3. If V is a complete operator space then it is a norm-completion of a certain ℓ^1-normed space X. Indeed, there is a (matrix) quotient mapping $\pi : \overset{1}{\bigoplus_\iota} T_{n_\iota} \to V$, where T_{n_ι} is the n_ι-squared trace class matrix algebra (see [3], [9]). If $X = \pi\left(\sum_\iota T_{n_\iota}\right)$ then X is an ℓ^1-normed space, and it is dense in V.

4. The latter example can be generalized by the following way. Each ℓ^1-normed quotient of the direct algebraic sum $\bigoplus_\iota E_\iota$ with its finite dimensional spaces E_ι is an ℓ^1-normed space.

Now let X be an ℓ^1-normed space with its bornology $\{\mathrm{ball}\, Y_\alpha\}$. The space of all linear functionals on X is denoted by $X^\star$ called *the algebraic dual of* X. Similarly, if $T : X \to Y$ is a linear transformation of vector spaces then $T^\star : Y^\star \to X^\star$, $\langle x, T^\star y\rangle = \langle Tx, y\rangle$ denotes the algebraic adjoint to T linear transformation. The space of all linear transformations from X to Y is denoted by $\mathcal{L}(X, Y)$. *The bornological dual* X' *of* X [18] (see also [8]) is defined as the space of all linear functionals $f : X \to \mathbb{C}$ such that $d_\alpha(f) = \sup |f(\mathrm{ball}\, Y_\alpha)| < \infty$ for all α. It turns out to be a polynormed space with its defining family of seminorms $\{d_\alpha\}$. This space can also be thought as the dual space of the inductive limit $\varinjlim \{Y_\alpha\}$ of Banach spaces, which is equipped with the uniform convergence over all $\mathrm{ball}\, Y_\alpha$ topology. It follows that X' is a complete polynormed space. Indeed, if (f_λ) is a Cauchy net in X' then it converges to a linear functional $f : X \to \mathbb{C}$ with respect to the point-converges topology. Since $f_\lambda|\,\mathrm{ball}\, Y_\alpha \to f|\,\mathrm{ball}\, Y_\alpha$ uniformly, it follows that $f|Y_\alpha \in Y_\alpha^*$ for all α. Hence $f \in X'$.

5. Multinormed W^*-algebras

In this section we develop a multinormed W^*-algebras and their link to the central completions of von Neumann algebras.

5.1. The inverse w^*-limits of von Neumann algebras. A *w^*-homomorphism* between von Neumann algebras is defined as a weak* continuous $*$-homomorphism. By *a multinormed* W^*-algebra $\mathcal{M}_\mathcal{E}$ we mean an inverse limit $\mathcal{M}_\mathcal{E} = \varprojlim \{\mathcal{M}_\iota, \varphi_{\iota\kappa}\}$ of von Neumann algebras $\mathcal{M}_\iota$ such that all connecting maps $\varphi_{\iota\kappa} : \mathcal{M}_\kappa \to \mathcal{M}_\iota$ $(\iota \le \kappa)$ are w^*-homomorphisms (Fragoulopoulou [14]) called *the inverse w^*-limit of von Neumann algebras.* We left the terminology *locally W^*-algebra* for an inverse limit of von Neumann algebras. If $\|x\|_\iota = \|\varphi_\iota(x)\|$, $x \in \mathcal{M}_\mathcal{E}$ then $\{\|\cdot\|_\iota\}$ is a defining family of C^*-seminorms on $\mathcal{M}_\mathcal{E}$, where $\varphi_\iota : \mathcal{M}_\mathcal{E} \to \mathcal{M}_\iota$ is the canonical projection, and $\mathcal{M}_\iota = \mathcal{M}_\mathcal{E}/J_\iota$.

Lemma 5.1. *If $\mathcal{M}_\mathcal{E}$ is a multinormed W^*-algebra and $\mathcal{M} = \mathfrak{b}(\mathcal{M}_\mathcal{E})$ is the C^*-algebra of all its bounded elements, then $\mathcal{M}$ is a von Neumann algebra and all ideals I_ι are w^*-closed in $\mathcal{M}$.*

Proof. First note that $\mathcal{M}_\iota = \mathcal{M}_\mathcal{E}/J_\iota = \mathcal{M}/I_\iota$ thanks to Lemma 4.1. So, each $\mathcal{M}/I_\iota$ is a von Neumann algebra and let $\mathcal{N} = \overset{\infty}{\bigoplus_\iota} \mathcal{M}/I_\iota$ (ℓ_∞-direct sum) be the direct sum of von Neumann algebras. Then $\mathcal{N}$ is a von Neumann algebra with its predual $\mathcal{N}_* = \overset{1}{\bigoplus_\iota} (\mathcal{M}/I_\iota)_*$ (ℓ_1-direct sum) (see Proposition 2.2). Consider the $*$-homomorphism $\varphi : \mathcal{M} \to \mathcal{N}$, $\varphi(b) = (\varphi_\iota(b))_\iota$. Using Lemma 4.1, we derive that

$$\|x\| = \sup_\iota \|x\|_\iota = \sup_\iota \|\varphi_\iota(x)\|_\iota = \sup_\iota \|\varphi_\iota(x)\| = \|\varphi(x)\|$$

for all $x \in \mathcal{M}$, that is, φ is an isometry. Actually $\varphi(\mathcal{M})$ is w^*-closed in $\mathcal{N}$. Indeed, take a net $(\varphi(x_\lambda))$ in $\varphi(\mathcal{M})$ such that $u = (u_\iota)_\iota = w^*\text{-}\lim_\lambda \varphi(x_\lambda) \in \mathcal{N}$. If $\prod_\iota \sigma(\mathcal{M}_\iota, (\mathcal{M}_\iota)_*)$ is the direct product of w^*-topologies in $\prod_\iota \mathcal{M}_\iota$ and $\bigoplus_\iota (\mathcal{M}_\iota)_*$ is the direct algebraic sum, then $\prod_\iota \sigma(\mathcal{M}_\iota, (\mathcal{M}_\iota)_*) = \sigma(\prod_\iota \mathcal{M}_\iota, \bigoplus_\iota (\mathcal{M}_\iota)_*)$ (see [27, 4.4 Corollary 1]). Hence

$$\prod\nolimits_\iota \sigma(\mathcal{M}_\iota, (\mathcal{M}_\iota)_*) \,|\mathcal{N} = \sigma\left(\mathcal{N}, \bigoplus\nolimits_\iota (\mathcal{M}_\iota)_*\right) \subseteq \sigma(\mathcal{N}, \mathcal{N}_*).$$

It follows that $u_\iota = w^*\text{-}\lim_\lambda \varphi_\iota(x_\lambda)$ in $\mathcal{M}_\iota$ for each ι. Assume $\iota \leq \kappa$. Taking into account that the connecting mapping $\varphi_{\iota\kappa} : \mathcal{M}_\kappa \to \mathcal{M}_\iota$ is w^*-homomorphism, we conclude that

$$u_\iota = w^*\text{-}\lim_\lambda \varphi_\iota(x_\lambda) = w^*\text{-}\lim_\lambda \varphi_{\iota\kappa}\varphi_\kappa(x_\lambda) = \varphi_{\iota\kappa}\left(w^*\text{-}\lim_\lambda \varphi_\kappa(x_\lambda)\right) = \varphi_{\iota\kappa}(u_\kappa).$$

Hence $a = (u_\iota) \in \varprojlim \{\mathcal{M}_\iota, \varphi_{\iota\kappa}\} = \mathcal{M}_\mathcal{E}$ and $\varphi_\iota(a) = u_\iota$ for all ι. Using again Lemma 4.1, we obtain that $\sup_\iota \|a\|_\iota = \sup_\iota \|\varphi_\iota(a)\|_\iota = \sup_\iota \|u_\iota\| = \|u\|_\mathcal{N} < \infty$, that is, $a \in \mathcal{M}$. So, $\varphi(a) = (\varphi_\iota(a))_\iota = u \in \varphi(\mathcal{N})$. Hence $\mathcal{M}$ is a von Neumann algebra.

Finally, note that $\varphi_\iota : \mathcal{M} \to \mathcal{M}/I_\iota$ is a w^*-homomorphism, for $\langle y_\iota, \varphi_\iota(x)\rangle = \langle y_\iota, \varphi(x)\rangle$ for all $y_\iota \in (\mathcal{M}/I_\iota)_*$ and $x \in \mathcal{M}$. Hence $I_\iota = \ker \varphi_\iota$ is w^*-closed in $\mathcal{M}$. □

Proposition 5.1. *Let $A_\mathcal{E}$ be a multinormed C^*-algebra. The algebra $A_\mathcal{E}$ is a multinormed W^*-algebra if and only if it is a central completion of a von Neumann algebra.*

Proof. First assume that $A_\mathcal{E}$ is a multinormed W^*-algebra. By Lemma 5.1, the C^*-algebra $\mathcal{M}$ of all its bounded elements turns out to be a von Neumann algebra with the weakly closed ideals I_ι. Then $I_\iota = \mathcal{M}f_\iota$ for a certain central projection $f_\iota \in \mathcal{M}$ by Lemma 3.3. Put $e_\iota = 1 - f_\iota$. Since $\mathcal{M} \wedge_\iota f_\iota = \cap_\iota \mathcal{M}f_\iota = \cap_\iota I_\iota = \{0\}$, it follows that $\vee_\iota e_\iota = 1 - \wedge_\iota f_\iota = 1$. The restriction of the mapping $\varphi_\iota : \mathcal{M} \to \mathcal{M}/I_\iota$ to $\mathcal{M}e_\iota$ is a $*$-isomorphism, therefore it is an isometry. Using Lemma 4.1, we obtain that

$$\|x\|_\iota = \|\varphi_\iota(x)\|_\iota = \|\varphi_\iota(x)\| = \|\varphi_\iota(xe_\iota)\| = \|xe_\iota\|$$

for all $x \in \mathcal{M}$. So, $\{\|\cdot\|_\iota\}$ is the defining family of C^*-seminorms of the central topology associated with the central projections (e_ι). It remains to note that $\mathcal{M}$ is dense in the complete algebra $A_\mathcal{E}$. Hence $A_\mathcal{E}$ is a central completion of the von Neumann algebra $\mathcal{M}$.

Conversely, assume that $A_\mathcal{E}$ is a central completion of a von Neumann algebra $\mathcal{M}$ associated with central projections $\mathcal{E} = (e_\iota)$ of $\mathcal{M}$ with $\vee_\iota e_\iota = 1$. We write $\iota \leq \kappa$ whenever $e_\iota \leq e_\kappa$. The latter means that $\|\cdot\|_\iota \leq \|\cdot\|_\kappa$ thanks to Lemma 4.2. Put $A_\iota = \mathcal{M}e_\iota$. If $\iota \leq \kappa$ then $\varphi_{\iota\kappa} : A_\kappa \to A_\iota$, $\varphi_{\iota\kappa}(x) = xe_\iota$ is a w^*-homomorphism by Remark 2.2. By Lemma 4.2, $\{A_\iota, \varphi_{\iota\kappa}\}$ is an inverse system and $C = \varprojlim \{A_\iota, \varphi_{\iota\kappa}\}$ is a multinormed W^*-algebra. Obviously, C is the central completion of $\mathcal{M}$. Hence $A_\mathcal{E} = C$ and $A_\mathcal{E}$ is a multinormed W^*-algebra. □

Corollary 5.1. *If $\mathcal{M}_\mathcal{E}$ is a multinormed W^*-algebra such that $\mathfrak{b}(\mathcal{M}_\mathcal{E})$ is a factor then $\mathcal{M}_\mathcal{E}$ is a von Neumann algebra.*

If $\mathcal{M}=\bigoplus_{\iota}^{\infty}\mathcal{M}_{\iota}$ is an infinite direct sum of factors then we have a canonical family (e_{ι}) of its central orthogonal projections in $\mathcal{M}$ such that $\sum_{\iota}e_{\iota}=1$. If $\mathcal{M}_{\mathcal{E}}$ is the central completion of $\mathcal{M}$ associated with this family then $\mathcal{M}_{\mathcal{E}}$ is a multinormed W^{*}-algebra (Proposition 5.1). Actually, $\mathcal{M}_{\mathcal{E}}=\prod_{\iota}\mathcal{M}_{\iota}$. Therefore $\mathfrak{b}(\mathcal{M}_{\mathcal{E}})=\mathcal{M}$. Obviously $\mathcal{M}$ is not simple and $\mathcal{M}_{\mathcal{E}}$ is not a normed C^{*}-algebra (see Problem 4.17 from [16]).

Theorem 5.1. *Let $A_{\mathcal{E}}$ be a multinormed C^{*}-algebra. Then $A_{\mathcal{E}}$ is a multinormed W^{*}-algebra if and only if it is a bornological dual X' of a certain ℓ_{1}-normed space X. In this case X is unique up to an isometry.*

Proof. By Proposition 5.1, $A_{\mathcal{E}}$ is a multinormed W^{*}-algebra iff it is a central completion of a certain von Neumann algebra $\mathcal{M}$. If $A_{\mathcal{E}}=\mathcal{M}_{\mathcal{E}}$ is a completion of $\mathcal{M}$ with respect to the central topology determined by a domain $\mathcal{E}$ in $\mathcal{M}$, then $\mathcal{M}_{\mathcal{E}}=(\mathcal{M}_{*\mathcal{E}})'$ by [11], that is, it is the bornological dual of the certain ℓ_{1}-normed space $\mathcal{M}_{*\mathcal{E}}$. Conversely, assume that $A_{\mathcal{E}}=X'$ for a certain ℓ_{1}-normed space X. By Definition 4.1, there is a locally ℓ_{1}-quotient mapping $\pi:\bigoplus_{\iota}E_{\iota}\to X$. Let Y be the norm-completion of X and let $Y_{\alpha}=\pi\left(\bigoplus_{\iota\in\alpha}^{1}E_{\iota}\right)$, which is a Banach subspace of X. Since $\pi:\bigoplus_{\iota\in\alpha}^{1}E_{\iota}\to Y_{\alpha}$ is the quotient mapping, it follows that $\pi_{\alpha}^{*}:Y_{\alpha}^{*}\to\bigoplus_{\iota\in\alpha}^{\infty}E_{\iota}^{*}$ is an isometry, where $\pi_{\alpha}=\pi|\bigoplus_{\iota\in\alpha}^{1}E_{\iota}$. By assumption $\{d_{\alpha}\}$ is a defining family of C^{*}-seminorms of $A_{\mathcal{E}}$. Then

$$\begin{aligned}d_{\alpha}(x)&=\|x|Y_{\alpha}\|=\|\pi_{\alpha}^{*}(x|Y_{\alpha})\|\\&=\max_{\iota\in\alpha}\|\pi_{\iota}^{*}(x|Y_{\iota})\|=\max_{\iota\in\alpha}\|x|Y_{\iota}\|=\max_{\iota\in\alpha}d_{\iota}(x),\quad x\in A_{\mathcal{E}}\end{aligned}$$

for all α. Thus $\{d_{\iota}\}$ is a defining family of C^{*}-seminorms on $A_{\mathcal{E}}$. Consider the linear mapping $\psi:Y^{*}\to\mathcal{M}$, $\psi(f)=f|X$, where $\mathcal{M}$ is the C^{*}-algebra of all bounded elements in $A_{\mathcal{E}}$. If $f\in Y^{*}$ then $\sup_{\alpha}d_{\alpha}(\psi(f))=\sup_{\alpha}\|f|Y_{\alpha}\|\leq\|f\|=\|f|X\|\leq\sup_{\alpha}d_{\alpha}(\psi(f))$, that is, $\psi(f)\in\mathcal{M}$ and $\|\psi(f)\|=\|f\|$. If $b\in\mathcal{M}$ then $b:X\to\mathbb{C}$ is norm bounded. Therefore it has a unique bounded extension $f\in Y^{*}$, that is, $\psi(f)=b$. Thus ψ is an isometric isomorphism and $Y=\mathcal{M}_{*}$ is the predual of the C^{*}-algebra $\mathcal{M}$. In particular, $\mathcal{M}$ is a W^{*}-algebra. Further, one can easily verify that $I_{\iota}=Y_{\iota}^{\perp}$. Then I_{ι} is a $\sigma(Y^{*},Y)$-closed ideal in $\mathcal{M}$. In particular, $I_{\iota}=\mathcal{M}f_{\iota}$ for some central projection f_{ι}, and $Y_{\iota}=Y_{\iota}^{\perp\perp}=(\mathcal{M}f_{\iota})^{\perp}=e_{\iota}\mathcal{M}_{*}$ (see Lemma 3.7), which is a complemented Banach subspace in $\mathcal{M}_{*}$, where $e_{\iota}=1-f_{\iota}$. Using Lemma 4.1, we obtain that $d_{\iota}(x)=d_{\iota}(\varphi_{\iota}(x))=\|\varphi_{\iota}(x)\|$, where $\varphi_{\iota}:\mathcal{M}\to\mathcal{M}/I_{\iota}$ is the quotient $*$-homomorphism. Since $\mathcal{M}=\mathcal{M}e_{\iota}\oplus\mathcal{M}f_{\iota}$, it follows that $\varphi_{\iota}:\mathcal{M}e_{\iota}\to\mathcal{M}/I_{\iota}$ is a $*$-isomorphism. Therefore $\varphi_{\iota}:\mathcal{M}e_{\iota}\to\mathcal{M}/I_{\iota}$ is an isometry and $d_{\iota}(x)=\|\varphi_{\iota}(x)\|=\|xe_{\iota}\|=\|x\|_{\iota}$ for all $x\in\mathcal{M}$. Hence $\{d_{\iota}\}$ is a family of C^{*}-seminorms associated with the central projections (e_{ι}) in $\mathcal{M}$. Note that $\vee_{\iota}e_{\iota}=1$. Indeed, since $\mathcal{M}\wedge_{\iota}f_{\iota}=\cap_{\iota}I_{\iota}=\cap_{\iota}Y_{\iota}^{\perp}=(\sum_{\iota}Y_{\iota})^{\perp}=\{0\}$, it follows that $\vee_{\iota}e_{\iota}=1-\wedge_{\iota}f_{\iota}=1$. Thus $A_{\mathcal{E}}$ is a central completion of $\mathcal{M}$.

Finally, assume that $A_{\mathcal{E}}=X'=Z'$ for some ℓ_{1}-normed spaces X and Z. Since their completions are the preduals of the von Neumann algebra $\mathcal{M}$, we conclude that X and Z are dense subspaces in the same Banach space Y [26, 1.13.3], where Y is the predual of $\mathcal{M}$. Further, let $X = \cup_\alpha Y_\alpha$ and $Z = \cup_v Z_v$ be the relevant unions. By assumption, the families $\{d_\alpha\}$ and $\{d_v\}$ of C^*-seminorms on A are equivalent. Then for each α there corresponds v such that $d_\alpha \leq C_{\alpha v} d_v$ and viceversa. Thus $Z_v^\perp = I_v \subseteq I_\alpha = Y_\alpha^\perp$ which in turn implies that $Y_\alpha \subseteq Z_v$ (see Lemma 4.2). Hence $X = Z$. □

The algebraic sum $\sum_\iota \mathcal{M}_* e_\iota$ in the predual $\mathcal{M}_*$ of a von Neumann algebra $\mathcal{M}$ is denoted by $\mathcal{M}_{*\mathcal{E}}$, which is a normed space. Note that $\mathcal{M}_* = \mathcal{M}_* \vee \mathcal{E} = \langle \cup_{e\in\mathcal{E}} \mathcal{M}_* e\rangle = \left(\sum_{e\in\mathcal{E}} \mathcal{M}_* e\right)^- = (\mathcal{M}_{*\mathcal{E}})^-$, that is, $\mathcal{M}_{*\mathcal{E}}$ is dense in $\mathcal{M}_*$. It can be proven that $X = \mathcal{M}_{*\mathcal{E}}$ whenever $A_\mathcal{E} = \mathcal{M}_\mathcal{E}$ [11].

5.2. Noncommutative continuous functions. Let H be a Hilbert space and let $(p_\iota)_{\iota\in\Xi}$ be a family of projections in $\mathcal{B}(H)$ such that $\vee_\iota p_\iota = 1$. The family $\mathcal{E} = \{p_\alpha : \alpha \in \Lambda\}$ is called *a quantum domain in* H, where $p_\alpha = \vee_{\iota\in\alpha} p_\iota$. The commutant $\mathcal{E}'$ in $\mathcal{B}(H)$ is a unital von Neumann algebra on H equipped with the family $\|x\|_\alpha = \|xp_\alpha\|$, $x \in \mathcal{E}'$, $\alpha \in \Lambda$ of C^*-seminorms. The completion of $\mathcal{E}'$ with respect to this topology is denoted by $\mathcal{E}'_\mathcal{D}$, where $\mathcal{D}$ indicates *the union space* $\cup_\alpha H_\alpha$ $(H_\alpha = p_\alpha(H))$ *of the domain* $\mathcal{E}$. Thus $\mathcal{E}'_\mathcal{D}$ is a unital multinormed C^*-algebra. If $\mathcal{E}$ consists of mutually commuting projections then we say that $\mathcal{E}$ is *a domain in* H. We also define the unital multinormed C^*-algebra

$$C^*_\mathcal{E}(\mathcal{D}) = \{x \in L(\mathcal{D}) : p_\alpha x \subseteq xp_\alpha, xp_\alpha \in \mathcal{B}(H), \alpha \in \Lambda\}$$

of *all noncommutative continuous functions on* $\mathcal{E}$, where $L(\mathcal{D})$ is the algebra of all linear transformations on $\mathcal{D}$.

Lemma 5.2. *The algebra $\mathcal{E}'_\mathcal{D}$ is a central completion of $\mathcal{E}'$ associated with the uniquely defined central projections $e = (e_\iota)$, $e_\iota \geq p_\iota$, $\iota \in \Xi$ in $\mathcal{E}'$. In particular, $\mathcal{E}'_\mathcal{D}$ is a multinormed W^*-algebra, which is identified with the algebra $C^*_\mathcal{E}(\mathcal{D})$ of all noncommutative continuous functions on $\mathcal{E}$.*

Proof. First note that $\mathcal{E}'_\mathcal{D}$ is identified with a unital subalgebra in $L(\mathcal{D})$. By its very definition, $\mathcal{E}'_\mathcal{D} = \varprojlim \{\mathcal{E}'/I_\alpha\}$, where $I_\alpha = \ker \|\cdot\|_\alpha \subseteq \mathcal{E}'$. In particular, each $u \in \mathcal{E}'_\mathcal{D}$ is an equivalence class of a compatible family (u_α) from $\mathcal{E}'$. We identify u with the linear mapping $u \in L(\mathcal{D})$, $u(x) = u_\alpha(x)$ if $x \in H_\alpha$. If $x \in H_\alpha \cap H_\beta$ then $u_\alpha(x) = u_\gamma p_\alpha(x) = u_\gamma p_\beta(x) = u_\beta(x)$, where $\gamma = \alpha \cup \beta$. Moreover, if (v_α) is another compatible family representing u then $u_\alpha = v_\alpha \,(\mathrm{mod}\, I_\alpha)$ for all α. Therefore $u_\alpha(x) = v_\alpha(x)$, $x \in H_\alpha$. Thus u is a well defined linear transformation on $\mathcal{D}$. If $u(x) = 0$ for all $x \in \mathcal{D}$ then $u_\alpha p_\alpha = 0$, that is, $\|u\|_\alpha = 0$ for all α. Hence $u = 0$. Thus $\mathcal{E}'_\mathcal{D} \subseteq L(\mathcal{D})$.

Let us prove that $\mathcal{E}'$ is the C^*-algebra of all bounded elements of $\mathcal{E}'_\mathcal{D}$. Since $\sup_\alpha \|u\|_\alpha \leq \|u\|$, it follows that each $u \in \mathcal{E}'$ is a bounded element of $\mathcal{E}'_\mathcal{D}$. Conversely, if $b \in \mathcal{E}'_\mathcal{D}$ is a bounded element then $\|b(\zeta)\| = \|b_\alpha(\zeta)\| \leq \|b\|_\alpha \|\zeta\| \leq \sup_\beta \|b\|_\beta \|\zeta\|$ for each $\zeta \in H_\alpha$. Thus b is bounded on $\mathcal{D}$, and it has a unique extension $b \in \mathcal{B}(H)$. If $\zeta \in H_\beta$ then we have $p_\alpha b(\zeta) = p_\alpha b_\beta(\zeta) = b_\beta p_\alpha(\zeta) = bp_\alpha(\zeta)$ for all α. Since $\mathcal{D}$ is dense in H and b is bounded, it follows that $p_\alpha b = bp_\alpha$ for all α, that is, $b \in \mathcal{E}'$.

Now consider the ideal $I_\iota \subseteq \mathcal{E}'$. If $b = \lim_\lambda b_\lambda$ (SOT) of a certain net (b_λ) in I_ι, then $bp_\iota(\zeta) = \lim_\lambda b_\lambda p_\iota(\zeta) = 0$, $\zeta \in H$, that is, $b \in I_\iota$. Thus I_ι is a strongly closed ideal of von Neumann algebra $\mathcal{E}'$. Therefore $I_\iota = \mathcal{E}' f_\iota$ for the uniquely determined projection $f_\iota \in \mathcal{E}' \cap \mathcal{E}''$. Put $e_\iota = 1 - f_\iota \in \mathcal{E}' \cap \mathcal{E}''$. The quotient homomorphism $\pi : \mathcal{E}' e_\iota \to \mathcal{E}'/I_\iota$ turns out to be a $*$-isomorphism. In particular, it is an isometry. Take $b \in \mathcal{E}'$. Since $b = be_\iota + bf_\iota$, we have $bp_\iota = be_\iota p_\iota + bf_\iota p_\iota$, $bf_\iota \in I_\iota$. Then $0 = \|bf_\iota\|_\iota = \|bf_\iota p_\iota\|$, that is, $bp_\iota = be_\iota p_\iota$. Using Lemma 4.1, we derive that

$$\|b\|_\iota = \|bp_\iota\| = \|be_\iota p_\iota\| = \|be_\iota\|_\iota = \|\pi(be_\iota)\|_\iota = \|\pi(be_\iota)\| = \|be_\iota\|.$$

In particular, $\|b\|_\alpha = \|be_\alpha\|$ thanks to Lemma 4.2. Note that $\|p_\alpha - e_\alpha p_\alpha\| = \|1 - e_\alpha\|_\alpha = \|(1 - e_\alpha) e_\alpha\| = 0$, that is, $p_\alpha \le e_\alpha$. In particular, $1 = \vee_\iota e_\iota$. Thus the multinormed topology in $\mathcal{E}'$ is just the central topology associated with the family of its central projections (e_ι). Using Proposition 5.1, we conclude that $\mathcal{E}'_\mathcal{D}$ is a multinormed W^*-algebra.

Finally, $C^*_\mathcal{E}(\mathcal{D})$ is a multinormed C^*-algebra equipped with the family of C^*-seminorms $\|x\|'_\alpha = \|xp_\alpha\|$, $T \in C$, $\alpha \in \Lambda$ [4]. If $\alpha \subseteq \beta$ then the canonical mapping $\mathcal{E}'/I_\beta \to \mathcal{E}'/I_\alpha$ is reduced to the mapping $\mathcal{E}' e_\beta \to \mathcal{E}' e_\alpha$, $b \mapsto be_\alpha$. Thus $\mathcal{E}'_\mathcal{D} = \varprojlim \{\mathcal{E}' e_\alpha\}$. As we have noted above each $a = (a_\alpha)_\alpha \in \varprojlim \{\mathcal{E}' e_\alpha\}$ is identified with the linear mapping $a \in L(\mathcal{D})$, $a(\zeta) = a_\alpha(\zeta)$ if $\zeta \in H_\alpha$. If $\zeta \in H_\beta$ then $p_\alpha a(\zeta) = p_\alpha a_\beta(\zeta) = a_\beta p_\alpha(\zeta) = ap_\alpha(\zeta)$, that is, $p_\alpha a \subseteq ap_\alpha$ for all α. Hence $\mathcal{E}'_\mathcal{D} \subseteq C^*_\mathcal{E}(\mathcal{D})$ and $\|a\|_\alpha = \|a_\alpha\| = \|ap_\alpha\| = \|a\|'_\alpha$, $a \in \mathcal{E}'_\mathcal{D}$ for all α. The latter means that $\mathcal{E}'_\mathcal{D}$ is a closed $*$-subalgebra in $C^*_\mathcal{E}(\mathcal{D})$.

Conversely, take a bounded element x from $C^*_\mathcal{E}(\mathcal{D})$. Then $\sup_\alpha \|xp_\alpha\| < \infty$ and x has the unique extension up to $x \in \mathcal{B}(H)$. But $p_\alpha x(\zeta) = xp_\alpha(\zeta)$ for all $\zeta \in \mathcal{D}$. Hence $p_\alpha x = xp_\alpha$ in $\mathcal{B}(H)$, that is, $x \in \mathcal{E}'$. Thus $\mathcal{E}'_\mathcal{D}$ contains all bounded elements of $C^*_\mathcal{E}(\mathcal{D})$, which is dense in $C^*_\mathcal{E}(\mathcal{D})$ [25]. Hence $\mathcal{E}'_\mathcal{D} = C^*_\mathcal{E}(\mathcal{D})$. □

Using Lemma 5.2, we conclude that the strong operator topology (briefly, SOT) is defined on $C^*_\mathcal{E}(\mathcal{D})$ by means of the family $\{p_\zeta : \zeta \in \mathcal{D}\}$ of seminorms $p_\zeta(x) = \|x\zeta\|$, $x \in C^*_\mathcal{E}(\mathcal{D})$, $\zeta \in \mathcal{D}$. If $\zeta \in H_\alpha$ then $p_\zeta(x) = \|xp_\alpha \zeta\| \le \|x\|_\alpha \|\zeta\|$ for all $x \in C^*_\mathcal{E}(\mathcal{D})$, that is, SOT is weaker than the original multinormed topology. If A is a subalgebra in $C^*_\mathcal{E}(\mathcal{D})$ then A' denotes the commutant of A in $C^*_\mathcal{E}(\mathcal{D})$ as usual. Note that A' is strongly closed. Indeed, if $u = \lim_t u_t$ (SOT), $(u_t) \subseteq A'$ and $\zeta \in H_\alpha$, then $au(\zeta) = a_\alpha u(\zeta) = \lim_t a_\alpha u_t(\zeta) = \lim_t u_t a_\alpha(\zeta) = ua(\zeta)$ for all $a \in A$. Note that $\operatorname{span}\{p_\alpha : \alpha \in \Lambda\}^-$ (SOT) $\subseteq C^*_\mathcal{E}(\mathcal{D})'$. The following assertion generalizes the representation theorem for a multinormed C^*-algebra (see [19], [4]) on a quantum domain.

Proposition 5.2. *Let A be a multinormed C^*-algebra with its defining family of C^*-seminorms $\{\|\cdot\|_\iota : \iota \in \Xi\}$. There is a (commutative) domain $\mathcal{E} = \{e_\alpha : \alpha \in \Lambda\}$ in a Hilbert space H and $*$-embedding $\varphi : A \hookrightarrow C^*_\mathcal{E}(\mathcal{D})$ such that $\|\varphi(a)\|_\alpha = \|a\|_\alpha$, $a \in A$, $\alpha \in \Lambda$, where $\mathcal{D}$ is the union space of $\mathcal{E}$.*

Proof. Using [19] and [4], we conclude that A is a closed $*$-subalgebra of $C^*_\mathcal{E}(\mathcal{D})$ for a certain quantum domain $\mathcal{E} = \{p_\alpha : \alpha \in \Lambda\}$ in a Hilbert space H, where $p_\alpha = \vee_{\iota \in \alpha} p_\iota$. Based on Lemma 5.2, we have another commutative domain $\mathcal{E}_e = \{e_\alpha : \alpha \in \Lambda\}$ in H associated with the uniquely defined family of central projections $e = (e_\iota)$, $e_\iota \ge p_\iota$, $\iota \in \Xi$ in $\mathcal{E}'$. Let $\mathcal{D}_e = \cup_\alpha e_\alpha(H)$ be its union space. Note that $\mathcal{D} \subseteq \mathcal{D}_e$. Since $\mathcal{E}_e \subseteq \mathcal{E}''$, it follows that $\mathcal{E}' = \mathcal{E}''' \subseteq \mathcal{E}'_e$ in $\mathcal{B}(H)$. Moreover, if $b \in \mathcal{E}'$ then $\|b\|_\alpha = \|bp_\alpha\| = \|be_\alpha\|$ thanks to Lemma 5.2. It means that the inclusion $\mathcal{E}' \subseteq \mathcal{E}'_e$ is compatible with the relevant central topologies. Using again Lemma 5.2, we conclude that $C^*_{\mathcal{E}}(\mathcal{D}) = \mathcal{E}'_{\mathcal{D}} \subseteq C^*_{\mathcal{E}_e}(\mathcal{D}_e)$. In particular, $A \subseteq C^*_{\mathcal{E}_e}(\mathcal{D}_e)$ is a closed $*$-subalgebra. □

5.3. Local von Neumann algebras. Now let $\mathcal{E} = \{p_\alpha : \alpha \in \Lambda\}$ be a (commutative) domain in a Hilbert space H with its union space $\mathcal{D}$. By Lemma 5.2, $C^*_{\mathcal{E}}(\mathcal{D})$ is a central completion of the commutant $\mathcal{E}'$ in $\mathcal{B}(H)$. Moreover, $\|x\|_\alpha = \|xp_\alpha\|$, $x \in C^*_{\mathcal{E}}(\mathcal{D})$, $\alpha \in \Lambda$ is a defining family of C^*-seminorms on $C^*_{\mathcal{E}}(\mathcal{D})$. If $x \in C^*_{\mathcal{E}}(\mathcal{D})$ then we also write $x = (x_\alpha)$, where $x_\alpha = x|H_\alpha$, $\alpha \in \Lambda$. If A is a $*$-subalgebra in $C^*_{\mathcal{E}}(\mathcal{D})$ then we put $A_\alpha = \{x_\alpha : x \in A\} \subseteq \mathcal{B}(H_\alpha)$.

Lemma 5.3. *Let A be a closed $*$-subalgebra in $C^*_{\mathcal{E}}(\mathcal{D})$. The $*$-subalgebra Ap_α equipped with the C^*-seminorm $\|\cdot\|_\alpha$ turns out to be a C^*-algebra, which is $*$-isomorphic to A_α. Moreover, if $Ap_\alpha \subseteq A$ and A is strongly closed then so is Ap_α.*

Proof. As we know (see [28], [25, Corollary 1.12]) $A/\ker\|\cdot\|_\alpha$ is a C^*-algebra with respect to the induced C^*-norm. But the mapping $A/\ker\|\cdot\|_\alpha \to A_\alpha$, $x^\sim \mapsto x_\alpha$ is a $*$-isomorphism. Hence A_α is a C^*-subalgebra in $\mathcal{B}(H_\alpha)$. Since $Ap_\alpha \to A_\alpha$, $xp_\alpha \mapsto x_\alpha$, is a $*$-isomorphism, it follows that Ap_α is a C^*-algebra.

Finally, assume that A is strongly closed and $Ap_\alpha \subseteq A$. If $\lim_\lambda a_\lambda p_\alpha = b$ (SOT) for a certain net (a_λ) in A, then $b \in A$ and $bp_\alpha(\zeta) = \lim_\lambda a_\lambda p_\alpha(p_\alpha(\zeta)) = b(\zeta)$, $\zeta \in \mathcal{D}$. Thus $b = bp_\alpha \in Ap_\alpha$. □

Let us introduce local von Neumann algebras on $\mathcal{D}$.

Definition 5.1. A $*$-subalgebra $A \subseteq C^*_{\mathcal{E}}(\mathcal{D})$ is said to be a local von Neumann algebra on $\mathcal{D}$ if it is strongly closed and $Ap_\alpha \subseteq A$ for all α.

If A is a $*$-subalgebra in $C^*_{\mathcal{E}}(\mathcal{D})$ then its commutant A' in $C^*_{\mathcal{E}}(\mathcal{D})$ is a local von Neumann algebra on $\mathcal{D}$. Indeed, since $\mathcal{E} \subseteq A'$, we conclude that $A'p_\alpha \subseteq A'$ for all α. Moreover, A' is strongly closed.

Lemma 5.4. *If A is a local von Neumann algebra on $\mathcal{D}$ then it has a unit and the algebra A is identified with a unital local von Neumann subalgebra in $C^*_{\mathcal{S}}(\mathcal{O})$ for another domain $\mathcal{S}$ with its union space $\mathcal{O}$, $\mathcal{O} \subseteq \mathcal{D}$.*

Proof. Since A is strongly closed, it follows that A is a closed $*$-subalgebra in $C^*_{\mathcal{E}}(\mathcal{D})$. Using [19, Theorem 2.6], we conclude that there is an approximate unit (u_t) in A. Actually, it can be assumed that (u_t) is an approximate unit of the C^*-algebra $\mathfrak{b}(A)$ (see [25, Corollary 3.12]). Note that $\mathfrak{b}(A) = \{x \in A : \|x\| = \sup_\alpha \|x\|_\alpha < \infty\}$ and it is canonically identified with the C^*-subalgebra of $\mathcal{B}(H)$. Then $\lim_t u_t = q$ (SOT) in $\mathcal{B}(H)$, where q is self-adjoint. But $u_t p_\alpha = p_\alpha u_t$ for all $\alpha \in \Lambda$. Then

$$p_\alpha q(\zeta) = \lim_t p_\alpha u_t(\zeta) = \lim_t u_t p_\alpha(\zeta) = qp_\alpha(\zeta)$$

for all $\zeta \in \mathcal{D}$, that is, $q(\mathcal{D}) \subseteq \mathcal{D}$ and $p_\alpha q|\mathcal{D} \subseteq qp_\alpha$. Hence $q|\mathcal{D} \in C^*_{\mathcal{E}}(\mathcal{D})$. Put $q = q|\mathcal{D}$. Then $q \in A$, for A is strongly closed. If $a \in A$ and $\zeta \in \mathcal{D}$ then $qa(\zeta) = \lim_t u_t a(\zeta)$. But $\lim_t u_t a = a$ in $C^*_{\mathcal{E}}(\mathcal{D})$. In particular, $\lim_t u_t a(\zeta) = a(\zeta)$, $\zeta \in \mathcal{D}$. Thus $qa = a$ for all $a \in A$, that is, q is a unit of A. Moreover, $q_\alpha = qp_\alpha \in Ap_\alpha \subseteq A$, $\alpha \in \Lambda$ (see Definition 5.1), and q_α is a projection such that $q_\alpha \leq p_\alpha$. Consider the quantum domain $\mathcal{S} = \{q_\alpha : \alpha \in \Lambda\}$ in $q(H)$ with its union space $\mathcal{O}$. Actually, it is a domain, for $\mathcal{S}$ is commutative. Moreover, $\mathcal{O} \subseteq \mathcal{D}$ and we have a well defined topological $*$-isomorphism $qC^*_{\mathcal{E}}(\mathcal{D})\,q \to C^*_{\mathcal{S}}(\mathcal{O})$, $u \mapsto u|\mathcal{O}$. If $u = qTq$ then $\|u|\mathcal{O}\|_\alpha = \|uq_\alpha\| = \|up_\alpha\| = \|u\|_\alpha$ for all $\alpha \in \Lambda$. The range of A by means of this mapping is denoted by A_q. If $\zeta \in \mathcal{D}$ then $p_\zeta(T) = \|T\zeta\| = \|Tq(\zeta)\| = p_{q(\zeta)}(T|\mathcal{O})$ for all $T \in qC^*_{\mathcal{E}}(\mathcal{D})\,q$. It follows that A_q is strongly closed in $C^*_{\mathcal{S}}(\mathcal{O})$ and $q|\mathcal{O} = 1_{\mathcal{O}}$. □

The following assertion shows that a local von Neumann algebra is a direct limit of its von Neumann subalgebras.

Proposition 5.3. *Let A be a local von Neumann algebra on $\mathcal{D}$, and let $S = \cup_\alpha Ap_\alpha$. Then A is a multinormed W^*-algebra and $A = S^-(SOT)$ in $C^*_{\mathcal{E}}(\mathcal{D})$. Moreover, $\mathfrak{b}(A)$ is a von Neumann algebra on H and $\mathfrak{b}(A) = S^-(SOT)$ in $\mathcal{B}(H)$. Thus*

$$A = \varinjlim \{Ap_\alpha\} \quad on \quad \mathcal{D}.$$

Proof. Put $B = \mathfrak{b}(A)$. Using Lemma 5.4, we can assume that $1_{\mathcal{D}} \in A$. Then $p_\alpha = 1_{\mathcal{D}} p_\alpha \in Ap_\alpha \subseteq A$ for all $\alpha \in \Lambda$. By Lemma 5.3, each Ap_α is strongly closed and S being the union $\cup_\alpha Ap_\alpha$ of its upward filtered family of $*$-subalgebras is a $*$-subalgebra in B. In particular, each A_α is a unital von Neumann algebra on H_α. We know that $A = \varprojlim \{Ap_\alpha\}$ (see Lemma 5.3). If $\alpha \subseteq \beta$ we have a well defined connecting $*$-homomorphism $\varphi_{\alpha\beta} : Ap_\beta \to Ap_\alpha$, $\varphi_{\alpha\beta}(a) = ap_\alpha$ of von Neumann algebras, which is weak* continuous (see [26, 1.7.7]). Hence A is a multinormed W^*-algebra. Using Lemma 5.1, we conclude that B is strongly closed in $\mathcal{B}(H)$.

If $b \in B$ then $b = \lim_\alpha bp_\alpha$ (SOT) in $\mathcal{B}(H)$. Hence $B = S^-$(SOT) in $\mathcal{B}(H)$. Further, take $a \in A$ and $\zeta \in H_\beta$.

Then $\lim_\alpha p_\zeta(a - ap_\alpha) = \lim_{\alpha,\alpha\supseteq\beta} \|ap_\beta(\zeta) - ap_\alpha p_\beta(\zeta)\| = 0$. Hence $A = S^-$(SOT) in $C^*_{\mathcal{E}}(\mathcal{D})$. □

Corollary 5.2. *Let $A \subseteq C^*_{\mathcal{E}}(\mathcal{D})$ be a $*$-subalgebra such that $1_{\mathcal{D}} \in A$. Then A is a local von Neumann algebra on $\mathcal{D}$ if and only if $A = A''$.*

Proof. If $A = A''$ then A being the commutant of a $*$-subalgebra turns out to be a local von Neumann algebra. Conversely, assume that A is a local von Neumann algebra on $\mathcal{D}$. Using Proposition 5.3, we conclude that B is a unital von Neumann algebra on H, where $B = \mathfrak{b}(A)$. Note that $A' = B'$ in $C^*_{\mathcal{E}}(\mathcal{D})$. Indeed, if $u \in B'$ and $a \in A$ then $a = \lim_\lambda b_\lambda$ in $C^*_{\mathcal{E}}(\mathcal{D})$ for a certain net (b_λ) in B, and $ua = \lim_\lambda ub_\lambda = \lim_\lambda b_\lambda u = au$. Hence $A' = B'$. For a while let us use the notation $C^\sim$ for the commutant in $\mathcal{B}(H)$ of a subset $C \subseteq \mathcal{B}(H)$. Note that $\mathfrak{b}(B') = B^\sim$. Indeed, if $T \in B^\sim$ then $T \in \mathcal{E}^\sim \subseteq C^*_{\mathcal{E}}(\mathcal{D})$, for $\mathcal{E} \subseteq B$ (see Definition 5.1). Hence $T \in B'$.

Take $u \in \mathfrak{b}(A'')$. Then u commutes with all elements from B', in particular from $\mathfrak{b}(B')$, that is, $u \in B^{\sim\sim}$. But $B^{\sim\sim} = B$ thanks to the bicommutant theorem [23, Theorem 4.1.5]. Therefore $u \in B$. Hence $\mathfrak{b}(A'') \subseteq B \subseteq A$. But $\mathfrak{b}(A'')$ is dense in A''. So, $A'' = A$. □

The bicommutant theorem is not true in the general case of a strongly closed $*$-subalgebra.

Example. Let $\mathcal{E} = \{p_n : n \in \mathbb{N}\}$ be a countable domain and let $A = \{\lambda 1_{\mathcal{D}} + \mu p_1 : \lambda, \mu \in \mathbb{C}\}$ be the unital 2-dimensional $*$-subalgebra in $C^*_{\mathcal{E}}(\mathcal{D})$. If $T = \lim_t (\lambda_t 1_{\mathcal{D}} + \mu_t p_1)$ (SOT) for a certain net $(\lambda_t 1_{\mathcal{D}} + \mu_t p_1)$ in A then $T\zeta = \lim_t \lambda_t \zeta$ for each unit vector $\zeta \in \mathcal{D} \cap H_1^{\perp}$. Hence there is $\lambda = \lim_t \lambda_t$ and $T\zeta = \lambda\zeta$. If y is a unit vector in H_1 then $\lim_t \mu_t y = \lim_t \mu_t p_1(y) = Ty - \lambda y$, that is, there is $\mu = \lim_t \mu_t$. Thus $T = \lambda 1_{\mathcal{D}} + \mu p_1 \in A$. So, A is a strongly closed $*$-subalgebra in $C^*_{\mathcal{E}}(\mathcal{D})$. But $A \neq A''$, for $p_2 \in A'' \backslash A$.

That was a serious gap of the papers [21] and [22].

Corollary 5.3. *If $Z(C^*_{\mathcal{E}}(\mathcal{D}))$ is the center of the algebra $C^*_{\mathcal{E}}(\mathcal{D})$ then $Z(C^*_{\mathcal{E}}(\mathcal{D})) = \operatorname{alg}(\mathcal{E})^-$ (SOT), where $\operatorname{alg}(\mathcal{E})$ is the unital algebra generated by $\mathcal{E}$.*

Proof. Assume $A = \operatorname{alg}(\mathcal{E})$. Note that A consists of all polynomials $\sum_\theta \lambda_\theta e_\theta$, where $e_\theta = e_{\alpha_1} \cdots e_{\alpha_n}$, $\lambda_\theta \in \mathbb{C}$. In particular, A is a unital $*$-subalgebra in $C^*_{\mathcal{E}}(\mathcal{D})$ and its SOT closure A^- is a local von Neumann algebra on $\mathcal{D}$. Moreover, $A' = \mathcal{E}'$ and $A \subseteq Z(C^*_{\mathcal{E}}(\mathcal{D}))$. Since $Z(C^*_{\mathcal{E}}(\mathcal{D}))$ is strongly closed, we conclude that $A^- \subseteq Z(C^*_{\mathcal{E}}(\mathcal{D}))$. If $a \in Z(C^*_{\mathcal{E}}(\mathcal{D}))$ and $b \in \mathcal{E}'$. Then $ab = ba$, that is, $a \in \mathcal{E}'' = A''$. Using Corollary 5.2, we conclude that $a \in A^-$. Hence $Z(C^*_{\mathcal{E}}(\mathcal{D})) = A^-$. □

Corollary 5.4. *If A is a local von Neumann algebra then it is a closed linear span of its projections. Moreover, if $a \in A$ with its polar decomposition $a = u|a|$ then $u \in A$.*

Proof. A similar result for von Neumann algebras is well known [23, Ch. 4]. In particular, $\mathfrak{b}(A)$ is a closed linear span of its projections (see Proposition 5.3). But the norm topology in $\mathfrak{b}(A)$ is stronger than the original polynormed topology from A. Hence the linear span of projections is dense in $\mathfrak{b}(A)$. But $\mathfrak{b}(A)$ itself is dense in A (see also [25, Proposition 1.11 (4)]). Therefore the linear span of projections is dense in A too.

Finally, if $a = u|a|$ then $ap_\alpha = up_\alpha |ap_\alpha|$ is the polar decomposition of ap_α in the von Neumann algebra Ap_α. Then $up_\alpha \in Ap_\alpha$ for all α. But $u = \lim_\alpha up_\alpha$ (SOT), therefore $u \in A$. □

5.4. Sakai type theorem for multinormed W^*-algebras. Now we prove Sakai type theorem for multinormed W^*-algebras classifying them as local von Neumann algebras on domains.

Theorem 5.2. *Let A be a multinormed W^*-algebra with its upward filtered family $\{\|\cdot\|_\alpha : \alpha \in \Lambda\}$ of C^*-seminorms. There exists a domain $\mathcal{E} = \{p_\alpha : \alpha \in \Lambda\}$ in a Hilbert space H and $*$-isomorphism $A \to C^*_{\mathcal{E}}(\mathcal{D})$ onto a local von Neumann subalgebra on $\mathcal{D}$ such that $\|a\|_\alpha = \|ap_\alpha\|$, $a \in A$ for all α.*

Proof. By Proposition 5.1, we can assume that A is a central completion of a W^*-algebra B associated with a family $\{e_\iota : \iota \in \Xi\}$ of its central projections such that $\vee_\iota e_\iota = 1$. Using Sakai theorem [26, 1.16.7] (see also [26, 1.15.1]), we can assume that B is a unital von Neumann algebra on a Hilbert space H up to a $*$-isomorphism. In particular, $\mathcal{E} = \{e_\alpha : \alpha \in \Lambda\}$ is a domain in H and $B \subseteq \mathcal{E}'$.

Note that the central topology in $\mathcal{E}'$ associated with (e_ι) is reduced to the original central topology in B. The central completion $\mathcal{E}'_{\mathcal{D}}$ of $\mathcal{E}'$ is the $*$-algebra $C^*_{\mathcal{E}}(\mathcal{D})$ thanks to Lemma 5.2, where $\mathcal{D}$ is the union space of the domain $\mathcal{E}$. Since $C^*_{\mathcal{E}}(\mathcal{D})$ is complete, it follows that A is just the closure of B in $C^*_{\mathcal{E}}(\mathcal{D})$ (see Proposition 5.2). It remains to prove that A is a local von Neumann subalgebra in $C^*_{\mathcal{E}}(\mathcal{D})$. Take $a \in A$. Then $a = \lim_\lambda a_\lambda$ (in A) for a certain net (a_λ) in B. Thus $\lim_\lambda \|a - a_\lambda\|_\alpha = 0$ for each $\alpha \in \Lambda$. In particular, $\lim_\lambda \|ae_\alpha - a_\lambda e_\alpha\| = \lim_\lambda \|a - a_\lambda\|_\alpha = 0$ in $\mathcal{B}(H)$. But $a_\lambda e_\alpha \in B$, for a_λ, $e_\alpha \in B$. Since B is closed in $\mathcal{B}(H)$, it follows that $ae_\alpha \in B$. Hence $Ae_\alpha \subseteq B \subseteq A$ for all $\alpha \in \Lambda$.

Now assume that $a = \lim_\lambda a_\lambda$ (SOT) in $C^*_{\mathcal{E}}(\mathcal{D})$ for a certain net (a_λ) in A. Fix $\alpha \in \Lambda$. Then $\lim_\lambda p_\zeta(a - a_\lambda) = 0$ for all $\zeta \in H_\alpha$. Hence $ae_\alpha = \lim_\lambda a_\lambda e_\alpha$ (SOT) in $\mathcal{B}(H)$. But all $a_\lambda e_\alpha \in B$ and B is strongly closed. Then $ae_\alpha \in B \subseteq A$. Further, let us prove that $a = \lim_\alpha ae_\alpha$ in $C^*_{\mathcal{E}}(\mathcal{D})$. For each $\beta \in \Lambda$, we have $\lim_\alpha \|a - ae_\alpha\|_\beta = \lim_{\alpha,\alpha\supseteq\beta} \|a - ae_\alpha\|_\beta = \lim_{\alpha,\alpha\supseteq\beta} \|ae_\beta - ae_\alpha e_\beta\| = 0$. But A is closed in $C^*_{\mathcal{E}}(\mathcal{D})$. Hence $a \in A$, that is, A is strongly closed. Thus A is a local von Neumann algebra on $\mathcal{D}$ (see Definition 5.1). □

Corollary 5.5. *Let A be a multinormed C^*-algebra. The following conditions are equivalent:*

(i) A is a multinormed W^*-algebra;

(ii) A is a central completion of a W^*-algebra;

(iii) A is the bornological dual X' of a uniquely (up to an isometry) defined ℓ_1-normed space X;

(iv) A is a local von Neumann algebra on the union space $\mathcal{D}$ of a (commutative) domain $\mathcal{E}$.

Proof. It suffices to use Proposition 5.1, Theorems 5.1 and 5.2. □

Example. In the theory of quantum (or local operator) spaces [12], [13], [4] local von Neumann algebras appear by the following way (see [9], [7], [8]). Fix a set J with its partition $J = \bigvee_{\kappa\in\Xi} J_\kappa$. We can identify J with the family $\{J_\kappa\}$ of sets. Assume for each $w \in J$ there corresponds *an atomic algebra* M_{n_w} of all scalar n_w-square matrices, where n_w can be thought as a value of a certain function $n : J \to \mathbb{N}$ at the point w. Then for each member J_κ of the family J relates von Neumann algebra $M_{J_\kappa} = \bigoplus_{w\in J_\kappa}^{\infty} M_{n_w}$. Consider the direct product $\mathfrak{D}_J = \prod_{\kappa\in\Xi} M_{J_\kappa}$, which is a multinormed W^*-algebra. By Theorem 5.2, $\mathfrak{D}_J$ is a local von Neumann algebra. Note that $\mathfrak{D}_J \subseteq C^*_{\mathcal{E}}(\mathcal{D})$, where $\mathcal{E} = \{H_\alpha : \alpha \in \Lambda\}$ is a domain in the Hilbert space $H = \bigoplus_{w\in J} \mathbb{C}^{n_w}$ and $H_\alpha = \bigoplus_{\kappa\in\alpha} N_\kappa$, $N_\kappa = \bigoplus_{w\in J_\kappa} \mathbb{C}^{n_w}$. So, we deal with the orthogonal family (p_κ) of projections in $\mathcal{B}(H)$ such that $\sum_\kappa p_\kappa = 1$. The predual of $\mathfrak{D}_J$ can easily be found. It is the direct sum $\mathcal{T}_J = \bigoplus_{\kappa\in\Xi} \mathcal{T}_{J_\kappa}$ in the space $\mathcal{T}(H)$ of all trace class operators on H, where each $\mathcal{T}_{J_\kappa} = \bigoplus_{w\in J_\kappa}^{1} \mathcal{T}_{n_w}$ is the operator space of all trace class matrices in M_{J_κ} [9]. Note that $\mathcal{T}_J$ is an ℓ_1-normed space (see Definition 4.1). By Theorem 5.1, the space $\mathcal{T}_J$ is unique up to an isometry. Moreover, in this case the bornological dual $\mathcal{T}'_J$ is just the strong dual $(\mathcal{T}_J)'_\beta$ of the polynormed direct sum $\mathcal{T}_J = \bigoplus_{\kappa\in\Xi} \mathcal{T}_{J_\kappa}$. Hence $(\mathcal{T}_J)'_\beta = \mathfrak{D}_J$. The algebra $\mathfrak{D}_J$ plays the main role in the representation theorem and dual realization problem for quantum spaces [9].

6. Background of Banach space valued functions

In this section we provide the presentation with fundamentals of Banach space valued measurable functions that is necessary for our purposes.

6.1. Radon charges. Let $\mathcal{T}$ be a locally compact topological space. The norm dual $C_0(\mathcal{T})^*$ of the C^*-algebra $C_0(\mathcal{T})$ of all complex continuous functions on $\mathcal{T}$ vanishing at infinity is isometrically isomorphic to the Banach space $M(\mathcal{T})$ of all finite Radon charges on $\mathcal{T}$. Recall that a Radon charge is a functional μ on $C_c(\mathcal{T})$ such that $\int f\,|d\mu| = \sup\{|\mu(g)| : g \in C_c(\mathcal{T}), |g| \leq f\}$ (sometimes we briefly write $\int f$) is finite for every $f \in C_c(\mathcal{T})_+$. The total variation of μ is defined as $\int 1\,|d\mu|$. A Radon charge μ is said to be a finite Radon charge if $\int 1\,|d\mu| < \infty$. In this case $\|\mu\| = \sup|\mu(\operatorname{ball} C_0(\mathcal{T}))| = \sup|\mu(\operatorname{ball} C_c(\mathcal{T}))| = \int 1\,|d\mu|$. Thus $\mu \in C_0(\mathcal{T})^*$. Conversely, if $\mu \in C_0(\mathcal{T})^*$ then for every $f \in C_c(\mathcal{T})_+$, we have

$$\int f\,|d\mu| \leq \sup\{\|\mu\|\,\|g\|_\infty : g \in C_c(\mathcal{T}), |g| \leq f\} \leq \|\mu\|\,\|f\|_\infty < \infty,$$

that is, μ is a Radon charge. To prove that it is a finite Radon charge, let us take an increasing net $\{K_\iota\}$ of compact subsets in $\mathcal{T}$ being exhausted. The latter in turn generates an increasing net $\{f_\iota\} \subseteq \operatorname{ball} C_c(\mathcal{T})_+$ such that $f_\iota \nearrow 1$. Therefore $\int 1\,|d\mu| = \lim_\iota \int f_\iota\,|d\mu| \leq \|\mu\| < \infty$, that is, μ is a finite Radon charge. Thus $C_0(\mathcal{T})^* = M(\mathcal{T})$ up to an isometry. In particular, $C(K)^* = M(K)$ for the compact space K. Note that $P(K) = \{\mu \in \operatorname{ball} M(K) : \mu(1) = 1\}$ is the set of all probability measures on K. It is well known (see [24, 2.5.7]) that $P(K)$ is a convex w^*-compact subset of $\operatorname{ball} M(K)$ whose extremal boundary is reduced to the Dirac measures δ_t, $t \in K$ given by $\delta_t(f) = f(t)$, $f \in C(K)$. In particular, every probability measure on K can be approximated pointwise on $C(K)$ by measures with finite support on K thanks to Krein-Milman theorem.

Now let $\int : C_c(\mathcal{T}) \to \mathbb{C}$ be a positive Radon integral with the related Radon measure μ on $\mathcal{T}$. We use the conventual notations $\mathfrak{L}(\mathcal{T})$, $\mathfrak{B}(\mathcal{T})$ and $\mathfrak{N}(\mathcal{T})$ for the classes of measurable, Borel and null functions on $\mathcal{T}$, respectively. We have the $*$-algebra $L(\mathcal{T}) = \mathfrak{L}(\mathcal{T})/\mathfrak{N}(\mathcal{T})$ of classes of measurable functions. In the case of a locally compact, σ-compact space $\mathcal{T}$, one can easily prove that $\mathfrak{L}(\mathcal{T}) = \mathfrak{B}(\mathcal{T}) + \mathfrak{N}(\mathcal{T})$, and $L(\mathcal{T}) = \mathfrak{B}(\mathcal{T})/\mathfrak{N}(\mathcal{T})$. The vector space $\mathfrak{L}^\infty(\mathcal{T})$ of all essentially bounded, measurable functions on $\mathcal{T}$ is equipped with the seminorm $\|x\|_\infty = \operatorname{esssup}|x| = \inf\{r \in \mathbb{R} : \mu\{|x| > r\} = 0\}$, and the quotient norm in $L^\infty(\mathcal{T}) = \mathfrak{L}^\infty(\mathcal{T})/\mathfrak{N}(\mathcal{T})$ is the essential norm. A lifting is an isomorphism $\rho : L^\infty(\mathcal{T}) \to \mathfrak{L}^\infty(\mathcal{T})$ into, whose range consists of bounded measurable functions, and it is a right inverse for the quotient homomorphism $\mathfrak{L}^\infty(\mathcal{T}) \to L^\infty(\mathcal{T})$. It is well known (see [24, 6.5.11]) that $L^\infty(\mathcal{T}) = L^1(\mathcal{T})^*$ up to an isometric isomorphism. Moreover, there is a faithful $*$-representation $M : L^\infty(\mathcal{T}) \to \mathcal{B}(L^2(\mathcal{T}))$, $M_f\zeta = f\zeta$ (see below Corollary 7.1). For all $\zeta, \eta \in \ell^2(L^2(\mathcal{T}))$ we have $\langle M_f, \omega_{\zeta,\eta}\rangle =$

$\sum_n \int f(t)\zeta_n(t)\overline{\eta_n}(t) = \sum_n \int f(t)\theta_n(t)$ with $\theta_n = \zeta_n\overline{\eta_n} \in L^1(\mathcal{T})$. But

$$\sum_n \|\theta_n\|_1 = \sum_n \int |\theta_n(t)| = \sum_n \int |\zeta_n(t)| |\eta_n(t)|$$

$$\leq \sum_n \|\zeta_n\|_2 \|\eta_n\|_2 \leq \|\zeta\| \|\eta\| < \infty,$$

that is, $\theta = \sum_n \theta_n \in L^1(\mathcal{T})$ with $\|\theta\|_1 \leq \sum_n \|\theta_n\|_1 \leq \|\zeta\| \|\eta\|$. It follows that

$$\langle \theta, f \rangle = \int f(t)\theta(t) = \sum_n \int f(t)\theta_n(t) = \langle M_f, \omega_{\zeta,\eta} \rangle$$

by virtue of Lebesgue's theorem (see [24, 6.1.15]). In particular, $p_{\zeta,\eta}(M_f) = |\langle M_f, \omega_{\zeta,\eta}\rangle| = |\langle \theta, f\rangle|$ for all $f \in L^\infty(\mathcal{T})$, which means that M is a w^*-bicontinuous mapping (see Proposition 2.2) of $L^\infty(\mathcal{T})$ onto its range in $\mathcal{B}(L^2(\mathcal{T}))$. In particular, $M(L^\infty(\mathcal{T})) \cap \operatorname{ball}\mathcal{B}(L^2(\mathcal{T})) = \operatorname{ball} M(L^\infty(\mathcal{T})) = M(\operatorname{ball} L^\infty(\mathcal{T}))$ is w^*-compact being homeomorphic with $\operatorname{ball} L^\infty(\mathcal{T})$. Since $M(L^\infty(\mathcal{T})) \cap n$ ball $\mathcal{B}(L^2(\mathcal{T}))$ is w^*-compact for every n, we conclude that $M(L^\infty(\mathcal{T}))$ is a von Neumann algebra based on Krein-Smulian theorem and Corollary 3.2. Thus $L^\infty(\mathcal{T})$ is an abelian von Neumann algebra on $L^2(\mathcal{T})$. Later on (see below Corollary 7.2) we shall see that it is a maximal abelian von Neumann algebra.

6.2. Banach space valued continuous functions. Now let $\mathcal{T}$ be a locally compact topological space, X a Banach space, and let $C_X(\mathcal{T})$ be the vector space of all continuous X-valued functions on $\mathcal{T}$. Confirm that $C_X(\mathcal{T})$ is a Banach space equipped with the uniform norm $\|f\|_\infty = \sup \|f(\mathcal{T})\|$ whenever $\mathcal{T}$ is a compact space. Note that $C_c(\mathcal{T}) \otimes X$ is identified with a subspace of $C_X(\mathcal{T})$ along the linear embedding $\sum_i f_i \otimes x_i \mapsto \sum_i f_i(\cdot)x_i$. Actually, $C_c(\mathcal{T}) \otimes X$ consists of those X-valued functions $g \in C_X(\mathcal{T})$ such that $\operatorname{supp} g$ is compact and $\dim\langle g(\mathcal{T})\rangle < \infty$. Indeed, if g is that sort of function then $g(\cdot) = \sum_{i=1}^n g_i(\cdot)x_i$ for a linearly independent subset $\{x_i\} \subseteq X$. Obviously $g_i \in C(\mathcal{T})$ and $\operatorname{supp} g = \cup_i \operatorname{supp} g_i$, that is, $g_i \in C_c(\mathcal{T})$ and $g = \sum_{i=1}^n g_i \otimes x_i \in C_c(\mathcal{T}) \otimes X$.

A key example of an injective cross norm (see Subsection 2.4) is provided in the following assertion.

Proposition 6.1. *Let K be a compact space, X a Banach space. Then $C_X(K) = C(K) \otimes_\lambda X$ up to an isometric isomorphism of Banach spaces.*

Proof. Take $f \in C_X(K)$ and $\varepsilon > 0$. Since $f(K) \subseteq X$ is a compact subset, it can be covered by $\varepsilon/2$-balls $x_i + \varepsilon/2 \operatorname{ball} X$ centered at $x_i = f(t_i)$, $1 \leq i \leq n$. Put $K_i = f^{-1}(x_i + \varepsilon/2 \operatorname{ball} X)$ and $U_i = f^{-1}(x_i + \varepsilon(\operatorname{ball} X)^\circ)$. Thereby K_i is a closed subset of K whereas U_i is an open one with $K_i \subseteq U_i$ for every i. Note also that $K = \cup_i K_i$. Pick functions $h_i \in C(K)$, $0 \leq h_i \leq 1$, $h_i(K_i) = 1$ and $\operatorname{supp} h_i \subseteq U_i$ (see [24, 1.7.5]), which in turn generate a partition of unity. Namely, put $g_i = h_i\left(\sum_j h_j\right)^{-1}$. Confirm that $0 \leq g_i \leq 1$, $\operatorname{supp} g_i \subseteq U_i$ and $\sum_i g_i = 1$. Take $h = \sum_i g_i \otimes x_i \in C(K) \otimes X$. Then

$$\|f(t) - h(t)\| = \left\| f(t) - \sum_i g_i(t)x_i \right\|$$

$$= \left\| \sum_i g_i(t)(f(t) - x_i) \right\| \le \sum_i g_i(t) \|f(t) - x_i\|$$
$$\le \sum_i g_i(t) \sup \|f(U_i) - x_i\| \le \sum_i g_i(t) \varepsilon = \varepsilon$$

for all $t \in K$. Whence $\|f - h\|_\infty \le \varepsilon$, that is, $C(K) \otimes X$ is dense in $C_X(K)$. We need only to verify that $\|\cdot\|_\infty = \|\cdot\|_\lambda$ over $C(K) \otimes X$. Take $f = \sum_{i=1}^n f_i \otimes x_i \in C(K) \otimes X$. First note that (see Subsections 6.1, 2.4)

$$\|f\|_\lambda = \sup \{|\langle f, \mu \otimes x^* \rangle| : \mu \in \operatorname{ball} M(K), x^* \in \operatorname{ball} X^*\}$$
$$= \sup \left| \sum_i \langle f_i, \mu \rangle \langle x_i, x^* \rangle \right| = \sup \left| \int \sum_i f_i(t) \langle x_i, x^* \rangle \, d\mu \right|$$
$$= \sup \left| \int \langle f(t), x^* \rangle \, d\mu \right| \le \sup \int |\langle f(t), \operatorname{ball} X^* \rangle| \, |d\mu|$$
$$\le \sup \left\{ \int \|f(t)\| \, |d\mu| : \mu \in \operatorname{ball} M(K) \right\}$$
$$\le \|f\|_\infty \sup \left\{ \int 1 \, |d\mu| : \mu \in \operatorname{ball} M(K) \right\} = \|f\|_\infty .$$

Conversely,

$$\|f\|_\infty = \sup \|f(K)\| = \sup |\langle f(K), \operatorname{ball} X^* \rangle|$$
$$= \sup \left\{ \left| \sum_i f_i(t) \langle x_i, x^* \rangle \right| : t \in K, x^* \in \operatorname{ball} X^* \right\}$$
$$= \sup \left| \sum_i \langle f_i, \delta_t \rangle \langle x_i, x^* \rangle \right| = \sup \left| \left\langle \sum_i f_i \otimes x_i, \delta_t \otimes x^* \right\rangle \right| = \sup |\langle f, \delta_t \otimes x^* \rangle|$$
$$\le \sup \{|\langle f, \mu \otimes x^* \rangle| : \mu \in \operatorname{ball} M(K), x^* \in \operatorname{ball} X^*\} = \|f\|_\lambda ,$$

that is, $\|f\|_\lambda = \|f\|_\infty$ for all $f \in C(K) \otimes X$. □

6.3. Banach space valued measurable functions. Now we fix a locally compact topological space $\mathcal{T}$ equipped with a positive Radon measure μ or positive Radon integral $\int : C_c(\mathcal{T}) \to \mathbb{C}$. The class of all absolutely integrable functions on $\mathcal{T}$ is denoted by $\mathfrak{L}^1(\mathcal{T})$, $\mathfrak{M}^1 = \{B \subseteq \mathcal{T} : [B] \in \mathfrak{L}^1(\mathcal{T})\}$ and

$$\mathfrak{M} = \{A \subseteq \mathcal{T} : A \cap C \in \mathfrak{M}^1 \text{ for every compact subset } C \subseteq \mathcal{T}\} \tag{6.1}$$

is the σ-algebra of all measurable subsets of $\mathcal{T}$. Note that $\mathfrak{M}^1 \subseteq \mathfrak{M}$ and $\mathfrak{b}(\mathcal{T}) \subseteq \mathfrak{M}$, where $\mathfrak{b}(\mathcal{T})$ is the σ-algebra of all Borel subsets of $\mathcal{T}$, which is the smallest σ-algebra containing all open (or) closed subsets of $\mathcal{T}$. Recall that a function $f : \mathcal{T} \to \mathbb{C}$ is said to be measurable if $f^{-1}(B) \in \mathfrak{M}$ for every $B \in \mathfrak{b}(\mathbb{C})$, that is, $f^{-1}(\mathfrak{b}(\mathbb{C})) \subseteq \mathfrak{M}$. The $*$-algebra of all measurable functions on $\mathcal{T}$ is denoted by $\mathfrak{L}(\mathcal{T})$.

The following result of mathematical folklore plays a key role in generalization of measurability to Banach space valued functions.

Proposition 6.2. *A function $f : \mathcal{T} \to \mathbb{C}$ is measurable iff for every compact subset $K \subseteq \mathcal{T}$ and $\varepsilon > 0$ there exists a compact subset $K_0 \subseteq K$ such that $\mu(K - K_0) \le \varepsilon$ and $f|K_0 \in C(K_0)$.*

Proof. First assume that $f \in \mathfrak{L}(\mathcal{T})$. Take a compact subset $K \subseteq \mathcal{T}$ and $\varepsilon > 0$. Put $C_n = \{|f| \leq n\} \cap K$. Since $\{|f| \leq n\} \in \mathfrak{M}$, it follows that $C_n \in \mathfrak{M}^1$ (see (6.1)). Since $\mu(K - \cup_n C_n) = 0$, we conclude that $\mu(K - C_n) \leq \varepsilon/2$ for a certain n. Note that $\int^* |f[C_n]| \leq n \int [C_n] < \infty$, therefore $f[C_n] \in \mathfrak{L}^1(\mathcal{T})$ (see [24, 6.2.16]). Using Lusin's theorem (see [24, 6.4.12]), we obtain an open subset $A \subseteq \mathcal{T}$, $\mu(A) \leq \varepsilon/2$ such that $f[C_n][\mathcal{T} - A] \in C_0(\mathcal{T} - A)$. But $f[C_n][\mathcal{T} - A] = f[C_n \cap (\mathcal{T} - A)] = f[C_n - A]$. Put K_0 to be $C_n - A$. Then $\mu(K - K_0) = \mu((K - C_n) \cup (K \cap A)) \leq \mu(K - C_n) + \mu(A) \leq \varepsilon$ and $f|K_0 \in C(K_0)$.

Conversely, suppose f satisfies the property over compact subsets stated in the assertion. Prove that $f \in \mathfrak{L}(\mathcal{T})$. Take $B \in \mathfrak{b}(\mathbb{C})$ and a compact subset $K \subseteq \mathcal{T}$. We have to prove that $A = f^{-1}(B) \cap K \in \mathfrak{M}^1$. By assumption there is a closed subset $K_n \subseteq K$ such that $\mu(K - K_n) \leq n^{-1}$ and $f|K_n \in C(K_n)$ for every n. Then $A_n = f^{-1}(B) \cap K_n = (f|K_n)^{-1}(B) \in \mathfrak{b}(\mathcal{T}) \subseteq \mathfrak{M}$ and $\int^* [A_n] \leq \mu(K_n) < \infty$, therefore $A_n \in \mathfrak{M}^1$. We can assume that $K_n \subseteq K_{n+1}$ for all n, that is, $\lim_n \mu(K_n) = \mu(K)$. In particular, $A_n \subseteq A_{n+1}$ for all n, and $\lim_n \mu(A_n) \leq \mu(K)$. Thus $[A_n] \nearrow [A]$ a.e. on $\mathcal{T}$, and $\lim_n \int [A_n] < \infty$. By Beppo-Levi's theorem (see [24, 6.4.3]), $[A] \in \mathfrak{L}^1(\mathcal{T})$, that is, $A \in \mathfrak{M}^1$. □

Now we fix a Banach space X and consider an X-valued function $f : \mathcal{T} \to X$. Based on Proposition 6.2, we say that f is measurable if for every compact subset $K \subseteq \mathcal{T}$ and $\varepsilon > 0$ there exists a compact subset $K_0 \subseteq K$ such that $\mu(K - K_0) \leq \varepsilon$ and $f|K_0 \in C_X(K_0)$. If $x^* \in X^*$ then $\langle f(\cdot), x^* \rangle \in \mathfrak{L}(\mathcal{T})$ thanks to Proposition 6.2. In this case we say that f is *weakly measurable*. If $X = Y^*$ for a Banach space Y, and $\langle y, f(\cdot) \rangle \in \mathfrak{L}(\mathcal{T})$ for all $y \in Y$, we say that f is *w^*-measurable*. For a compact subset $K \subseteq \mathcal{T}$ and n one can find a closed subset $K_n \subseteq K$, $\mu(K - K_n) \leq n^{-1}$ such that $f|K_n \in C_X(K_n)$. In particular, $f(K_n)$ is separable being a metrizable compact in X ([24, E 1.2.9]). Put $K_\infty = \cup_{n=1}^\infty K_n$. Then $K_\infty \in \mathfrak{M}^1$, $\mu(K - K_\infty) = 0$ and $f(K_\infty) = \cup_{n=1}^\infty f(K_n)$ is separable in X. The class of all X-valued measurable functions on $\mathcal{T}$ is denoted by $\mathfrak{L}_X(\mathcal{T})$.

Proposition 6.3. *An X-valued function $f : \mathcal{T} \to X$ is measurable iff it is weakly measurable and for every compact subset $K \subseteq \mathcal{T}$ there exists a measurable subset $K_\infty \subseteq K$ such that $\mu(K - K_\infty) = 0$ and $f(K_\infty)$ is separable in X.*

Proof. As we have just seen above every $f \in \mathfrak{L}_X(\mathcal{T})$ possesses the property stated in the assertion. Conversely, take a compact subset $K \subseteq \mathcal{T}$ and $\varepsilon > 0$. By assumption, there exists a measurable subset $K_\infty \subseteq K$ such that $\mu(K - K_\infty) = 0$ and $f(K_\infty)$ is separable in X. Since $K_\infty \in \mathfrak{M}^1$, it follows that $\mu(K_\infty - K'_\infty) \leq \varepsilon/2$ for a certain compact subset $K'_\infty \subseteq K_\infty$ (the inner regularity of μ [24, 6.3.1]). Since $f(K_\infty)$ is separable in the metric space X, it follows that $f(K_\infty)$ is second countable (see [24, E 1.2.9]). In particular, so is $f(K'_\infty)$, therefore $f(K'_\infty)$ is separable. Put $Y = \langle f(K'_\infty) \rangle$, which is a closed separable subspace of X. Then Y^* is w^*-separable and $\operatorname{ball} Y^*$ is w^*-metrizable compact (see [24, E 2.5.3]), that is, $\operatorname{ball} Y^*$ is w^*-separable. Take a dense sequence $\{y_n\}$ in Y, and w^*-dense sequence $\{y_n^*\}$ in $\operatorname{ball} Y^*$. For every $y \in Y$ we put $f_y(t) = \|f(t) - y\|$, $t \in \mathcal{T}$. Since all functionals from Y^* can be extended up to functionals from X^*, we conclude that $\langle f(\cdot)[K'_\infty], y^* \rangle \in \mathfrak{L}(\mathcal{T})$ for all $y^* \in Y^*$. But

$$f_y(t) = \sup |\langle f(t) - y, \operatorname{ball} Y^* \rangle| = \sup \{|\langle f(t) - y, y_n^* \rangle| : n \in \mathbb{N}\},$$

therefore $f_y[K'_\infty] = \sup\{|\langle f(\cdot)[K'_\infty] - y, y_n^*\rangle| : n \in \mathbb{N}\} \in \mathfrak{L}(\mathcal{T})$. In particular, $\{f_{y_n}[K'_\infty]\} \subseteq \mathfrak{L}(\mathcal{T})$. By Proposition 6.2, there are compact subsets $K_n \subseteq K'_\infty$ such that $\mu(K'_\infty - K_n) \leq \varepsilon/2^{n+1}$ and $f_{y_n}[K_n] \in C(K_n)$. Put $K_0 = \cap_{n=1}^{\infty} K_n$. Then

$$\begin{aligned}\mu(K - K_0) &= \mu\left((K - K_\infty) \cup \left(K_\infty - K'_\infty\right) \cup \left(K'_\infty - K_0\right)\right)\\ &\leq \mu\left(K_\infty - K'_\infty\right) + \sum_n \mu\left(K'_\infty - K_n\right) \leq \varepsilon\end{aligned}$$

and $\{f_{y_n}[K_0]\} \subseteq C(K_0)$. In this case, for every $y \in Y$ one can choose a subsequence $\{y_{n_k}\}$ convergent to y, therefore

$$\begin{aligned}&\left\|f_y[K_0] - f_{y_{n_k}}[K_0]\right\|_\infty\\ &= \sup\{|\|f(t) - y\| - \|f(t) - y_{n_k}\|| : t \in K_0\} \leq \|y - y_{n_k}\| \to 0,\end{aligned}$$

that is, $f_y[K_0] \in C(K_0)$ being a uniform limit of continuous functions on the compact set K_0. In particular, so is $f_{y_0}[K_0]$ for $y_0 = f(t_0)$, where t_0 is a fixed point from K_0. Since $f_{y_0}[K_0](t_0) = 0$, it follows that $\sup\|f(U \cap K_0) - f(t_0)\| = \sup f_{y_0}[K_0](U \cap K_0) \leq \varepsilon$ for a certain open neighborhood U of t_0 in $\mathcal{T}$, that is, $f|K_0 \in C_X(K_0)$. Whence $f \in \mathfrak{L}_X(\mathcal{T})$. □

Corollary 6.1. *Let $\{f_n\} \subseteq \mathfrak{L}_X(\mathcal{T})$ be a sequence such that $\lim_n f(t) = f(t)$ a.e. on $\mathcal{T}$. Then $f \in \mathfrak{L}_X(\mathcal{T})$.*

Proof. For every $x^* \in X^*$ we certainly have $\langle f(\cdot), x^*\rangle = \lim_n \langle f_n(\cdot), x^*\rangle \in \mathfrak{L}(\mathcal{T})$. Take a compact subset $K \subseteq \mathcal{T}$. Based on Proposition 6.3, for every n one can find $K_n \in \mathfrak{M}$ such that $\mu(K - K_n) = 0$ and $f_n(K_n)$ is separable. Put $K_\infty = \cap_n K_n$. Note that $K_\infty \in \mathfrak{M}$, $\mu(K - K_\infty) = 0$ and $f(K_\infty) \subseteq \langle \cup_n f_n(K_\infty)\rangle$ turns out to be separable. So, $f \in \mathfrak{L}_X(\mathcal{T})$ by Proposition 6.3. □

Based on Proposition 6.3, we also derive that for a separable Banach space X the measurability of $f : \mathcal{T} \to X$ is equivalent to its weak measurability.

Corollary 6.2. *Let X be a separable Banach space and $f : \mathcal{T} \to X^*$ a function. Then f is w^*-measurable iff it is weakly measurable. In this case, for every compact subset $K \subseteq \mathcal{T}$ and $\varepsilon > 0$ there exists a compact subset $K_0 \subseteq K$ such that $\mu(K - K_0) \leq \varepsilon$ and $f|K_0 : K_0 \to X^*$ is a w^*-continuous function with the norm bounded range $f(K_0)$.*

Proof. First confirm that the w^*-topology of $\operatorname{ball} X^*$ is metrizable and X^* is separable in the w^*-topology (see [24, E 2.5.3]). If f is weakly measurable then it is w^*-measurable without any doubt. Conversely, assume that f is w^*-measurable. Take a sequence $\{x_n\} \subseteq \operatorname{ball} X$ with $\{x_n\}^- = \operatorname{ball} X$. Since $\operatorname{ball} X$ is dense in $\operatorname{ball} X^{**}$ with respect to the weak* topology $\sigma(X^{**}, X^*)$ denoted by w^{**}, it follows that $\{x_n\}^{-w^{**}} = \operatorname{ball} X^{**}$. For $x^{**} \in \operatorname{ball} X^{**}$ take a subsequence $\{x_{n_k}\}$ such that $x^{**} = w^{**}\text{-}\lim_k x_{n_k}$. By assumption $\langle x_{n_k}, f(\cdot)\rangle \in \mathfrak{L}(\mathcal{T})$ for all k. Therefore $\langle f(\cdot), x^{**}\rangle = \lim_k \langle x_{n_k}, f(\cdot)\rangle \in \mathfrak{L}(\mathcal{T})$, that is, f is weakly measurable. Note also that $\|f(\cdot)\| = \sup|\langle \operatorname{ball} X, f(\cdot)\rangle| = \sup\{|\langle x_n, f(\cdot)\rangle| : n \in \mathbb{N}\} \in \mathfrak{L}(\mathcal{T})$.

Further, take a compact subset $K \subseteq \mathcal{T}$ and $\varepsilon > 0$. By Proposition 6.2, for every n one can find a compact subset $K_n \subseteq K$ such that $\mu(K - K_n) \leq \varepsilon/2^{n+1}$ and $\langle x_n, f(\cdot)\rangle |K_n \in C(K_n)$. Put $K'_0 = \cap_n K_n$. Then $\mu(K - K'_0) \leq \varepsilon/2$ and $\langle x_n, f(\cdot)\rangle |K'_0 \in C(K'_0)$ for all n. But there exists a compact subset

$K_0'' \subseteq K$ such that $\mu(K - K_0'') \leq \varepsilon/2$ and $\|f(\cdot)\| \,|K_0'' \in C(K_0'')$ as well. Put $K_0 = K_0' \cap K_0''$. Then $\mu(K - K_0) \leq \varepsilon$ and $\langle x_n, f(\cdot)\rangle |K_0, \|f(\cdot)\| \,|K_0 \in C(K_0)$ for all n. In particular, we can assume that $f(K_0) \subseteq \operatorname{ball} X^*$. For every $x \in \operatorname{ball} X$ one can find a subsequence $\{x_{n_m}\}$ such that $x = \lim_m x_{n_m}$. Then $\|\langle x, f(\cdot)\rangle |K_0 - \langle x_{n_m}, f(\cdot)\rangle |K_0\|_\infty \leq \|x - x_{n_m}\| \sup \|f(K_0)\| \leq \|x - x_{n_m}\| \to 0$ as $m \to \infty$, that is, $\langle x, f(\cdot)\rangle |K_0 \in C(K_0)$. Hence $f|K_0 : K_0 \to X^*$ is a w^*-continuous function. □

Corollary 6.3. *Let X be a Banach space, $f : \mathcal{T} \to X^*$ a w^*-measurable function and let $x : \mathcal{T} \to X$ be a measurable function. Then $\langle x(\cdot), f(\cdot)\rangle \in \mathfrak{L}(\mathcal{T})$.*

Proof. Take a compact subset $K \subseteq \mathcal{T}$ and $\varepsilon > 0$. Since $x \in \mathfrak{L}_X(\mathcal{T})$, it follows that there exists a compact subset $K_1 \subseteq K$ such that $\mu(K - K_1) \leq \varepsilon/2$ and $x|K_1 \in C_X(K_1)$. But $C_X(K_1) = C(K_1) \otimes_\lambda X$ (see Proposition 6.1), thereby one can find a sequence $\{x_n\} \subseteq C(K_1) \otimes X$ such that $\lim_n \|x|K_1 - x_n\|_\infty = 0$. But $x_n = \sum_{k=1}^{m_n} f_{nk} \otimes x_{nk}$ with $\{f_{nk}\} \subseteq C(K_1)$ and $\{x_{nk}\} \subseteq X$. Note that $\langle x_n(t), f(t)\rangle = \sum_{k=1}^{m_n} f_{nk}(t) \langle x_{nk}, f(t)\rangle$, $t \in K_1$, and $\lim_n \langle x_n(t), f(t)\rangle = \langle x(t), f(t)\rangle$ for almost all $t \in K_1$. Moreover, $\langle x_n(\cdot), f(\cdot)\rangle [K_1] = \sum_{k=1}^{m_n} f_{nk} \langle x_{nk}, f[K_1]\rangle \in \mathfrak{L}(\mathcal{T})$, for f is w^*-measurable. It follows that $\langle x(\cdot), f(\cdot)\rangle [K_1] \in \mathfrak{L}(\mathcal{T})$. In particular, there exists a compact subset $K_0 \subseteq K_1$ such that $\mu(K_1 - K_0) \leq \varepsilon/2$ and $\langle x(\cdot), f(\cdot)\rangle |K_0 \in C(K_0)$ by Proposition 6.2. Thus we find a compact subset $K_0 \subseteq K$ such that $\mu(K - K_0) \leq \varepsilon$ and $\langle x(\cdot), f(\cdot)\rangle |K_0 \in C(K_0)$. Appealing again to Proposition 6.2, we conclude that $\langle x(\cdot), f(\cdot)\rangle \in \mathfrak{L}(\mathcal{T})$. □

6.4. Banach space valued L^p-functions. As above let $\mathcal{T}$ be a locally compact topological space equipped with a positive Radon measure μ (or Radon integral $\int$), and let X be a Banach space. For every p, $1 \leq p < \infty$ we have the seminorm $\|f\|_p = \left(\int \|f(t)\|^p\right)^{1/p}$ on $C_c(\mathcal{T}) \otimes X$. We define $L_X^p(\mathcal{T})$ to be the Hausdorff completion of the seminormed space $\left(C_c(\mathcal{T}) \otimes X, \|\cdot\|_p\right)$. For every $f \in L_X^p(\mathcal{T})$ there corresponds a sequence $\{f_n\} \subseteq C_c(\mathcal{T}) \otimes X$ that converges to f. By passing to a subsequence, one may assume that $\sum_{n=0}^\infty \|f_{n+1} - f_n\|_p < \infty$ with $f_0 = 0$. Put $g(\cdot) = \sum_{n=0}^\infty \|f_{n+1}(\cdot) - f_n(\cdot)\| \in \mathfrak{L}(\mathcal{T})$. For the partial sum $g_k(\cdot) = \sum_{n=0}^k \|f_{n+1}(\cdot) - f_n(\cdot)\|$ we have $\int g_k^p \leq \left(\sum_{n=0}^k \|f_{n+1} - f_n\|_p\right)^p \leq \left(\sum_{n=0}^\infty \|f_{n+1} - f_n\|_p\right)^p$. By Beppo-Levi's theorem, we conclude that $g \in L_X^p(\mathcal{T})$ and $\left(\int g^p\right)^{1/p} \leq \sum_{n=0}^\infty \|f_{n+1} - f_n\|_p$. In particular, there is a limit $f(t) = \lim_n f_n(t)$ in X (since $\|f_m(t) - f_n(t)\| \leq \sum_{k=n}^m \|f_{k+1}(t) - f_k(t)\|$, $m > n$) for almost all $t \in \mathcal{T}$. By Corollary 6.1, $f \in \mathfrak{L}_X(\mathcal{T})$. Further, $\|f(t) - f_n(t)\| \leq \sum_{k=n}^{m-1} \|f_{k+1}(t) - f_k(t)\| + \|f(t) - f_m(t)\|$ for all $m > n$, therefore $\|f(t) - f_n(t)\|^p \leq g^p(t)$ a.e. on $\mathcal{T}$. By Lebesgue's theorem (see [24, 6.1.15]), $\lim_n \int \|f(t) - f_n(t)\|^p = 0$, that is, the function $f(\cdot)$ gives rise to the element f in $L_X^p(\mathcal{T})$.

Lemma 6.1. *Let $f(\cdot) \in \mathfrak{L}_X(\mathcal{T})$ be a function such that $f(\cdot) = f(\cdot)[K]$ (or $\operatorname{supp} f(\cdot) \subseteq K$) and $f(\cdot)|K \in C_X(K)$. Then $f(\cdot)$ gives rise to the element f of $L_X^p(\mathcal{T})$.*

Proof. For $\varepsilon > 0$ there exists $h = \sum_{k=1}^n h_k \otimes x_k \in C(K) \otimes X$ with $\|f|K - h\|_\infty \leq \varepsilon$ by virtue of Proposition 6.1. Take an open neighborhood $A \subseteq \mathcal{T}$ of K such that

$\mu(A-K) \leq \varepsilon$ (the outer regularity of μ). Moreover, A can be assumed to be a relatively compact subset. We can extend h_k up to $g_k \in C_c(\mathcal{T})$ such that $\operatorname{supp} g_k \subseteq A$ and $\|g_k\|_\infty \leq \|h_k\|_\infty + \varepsilon$. Then $g = \sum_{k=1}^n g_k \otimes x_k \in C_c(\mathcal{T}) \otimes X$ and

$$\|f-g\|_p^p = \int_K \|f(t) - h(t)\|^p + \int_{A-K} \|g(t)\|^p$$

$$\leq \|f|K - h\|_\infty^p \mu(K) + \left(\sum_{k=1}^n \|g_k\|_\infty \|x_k\|\right)^p \mu(A-K)$$

$$\leq \mu(K)\varepsilon^p + M^p \left(\sum_{k=1}^n \|x_k\|\right)^p \varepsilon,$$

where $M = \max\{\|f_k\|_\infty + \varepsilon : 1 \leq k \leq n\}$. Thus g approximates f in $L_X^p(\mathcal{T})$. Whence the function $f(\cdot)$ gives rise to the element of $L_X^p(\mathcal{T})$. □

Proposition 6.4. *Let X be a Banach space.*
Then $L_X^p(\mathcal{T}) = \left\{f(\cdot) \in \mathfrak{L}_X(\mathcal{T}) : \|f\|_p < \infty\right\}$ up the canonical identification.

Proof. We have seen above that every element $f \in L_X^p(\mathcal{T})$ is identified with an X-valued measurable function $f(\cdot)$ such that $\int \|f(t)\|^p < \infty$. Conversely, take that sort of function $f(\cdot)$ with $\|f\|_p \neq 0$. It is well known (see [24, 6.4.11]) that $C_c(\mathcal{T})$ is dense in $L^p(\mathcal{T})$ (that is the case of $X = \mathbb{C}$). In particular, for $\varepsilon > 0$ there is a function $g \in C_c(\mathcal{T})_+$ such that $\int |\|f(t)\| - g(t)|^p \leq \varepsilon$. It follows that

$$\int_{\mathcal{T}-K} \|f(t)\|^p \leq \int_{\mathcal{T}-\operatorname{supp} g} \|f(t)\|^p \leq \int_{\mathcal{T}-\operatorname{supp} g} |\|f(t)\| - g(t)|^p$$

$$+ \int_{\operatorname{supp} g} |\|f(t)\| - g(t)|^p \leq \varepsilon$$

for all compact subsets $K \subseteq \mathcal{T}$, $\operatorname{supp} g \subseteq K$. In particular, $\int_K \|f(t)\|^p \neq 0$ for a compact subset $K \subseteq \mathcal{T}$. Now take a disjoint family $\{K_\iota : \iota \in \Lambda\}$ of compact subsets in $\mathcal{T}$ with $\int_{K_\iota} \|f(t)\|^p \neq 0$. Based on Zorn's lemma, we can assume that $\{K_\iota\}$ is maximal with this property. Since $\sum_{\iota \in F} \|f(\cdot)\|^p [K_\iota] \leq \|f(\cdot)\|^p$ for a finite subset $F \subseteq \Lambda$, it follows that $\sum_\iota \int_{K_\iota} \|f(t)\|^p$ is summable and $\sum_\iota \int_{K_\iota} \|f(t)\|^p = \lim_F \sum_{\iota \in F} \int_{K_\iota} \|f(t)\|^p \leq \int \|f(t)\|^p$. It follows that the family $\{K_\iota\}$ is at most countable, say $\{K_\iota\} = \{K_n\}$, $\Omega = \cup_n K_n \in \mathfrak{M}$ and $\sum_n \int_{K_n} \|f(t)\|^p = \sum_n \int \|f(t)\|^p [K_n] = \int \|f(t)\|^p [\Omega]$ by Fatou's lemma [24, 6.1.13]. Since $\{K_n\}$ is maximal, we conclude that $\int_K \|f(t)\|^p = 0$ for a compact subset $K \subseteq \mathcal{T} - \Omega$. If $K \subseteq \left\{\|f(\cdot)\|^p \geq n^{-1}\right\} \cap (\mathcal{T} - \Omega)$ is a compact subset then $\mu(K) = \int [K] \leq \int n \|f(t)\|^p [K] = n \int_K \|f(t)\|^p = 0$, which in turn implies that $\mu\left(\left\{\|f(\cdot)\|^p \geq n^{-1}\right\} \cap (\mathcal{T} - \Omega)\right) = 0$ (inner regularity). In particular,

$$\mu(\{\|f(\cdot)\|^p \neq 0\} \cap (\mathcal{T} - \Omega)) = \lim_n \mu\left(\left\{\|f(\cdot)\|^p \geq n^{-1}\right\} \cap (\mathcal{T} - \Omega)\right) = 0,$$

that is, $\|f(\cdot)\|^p = 0$ a.e. on $\mathcal{T} - \Omega$, and $\sum_n \int_{K_n} \|f(t)\|^p = \int \|f(t)\|^p$.

Further, for $\varepsilon > 0$ take nonzero $g \in C_c(\mathcal{T})_+$ with $\int |\|f(t)\|^p - g(t)| \leq \varepsilon/2$. Since $f(\cdot) \in \mathfrak{L}_X(\mathcal{T})$, there are compact subsets $K_n' \subseteq K_n$ such that $\mu(K_n - K_n') \leq$

$\varepsilon/\left(\|g\|_\infty 2^{n+2}\right)$ and $f(\cdot)[K_n'] \in C_X(K_n')$. Put $f'(\cdot) = f(\cdot)[\cup_n K_n']$. For $r = \int \|f(t) - f'(t)\|^p$ we have

$$r = \sum_n \int \|f(t)\|^p \left[K_n - K_n'\right] \leq \sum_n \int \left|\|f(t)\|^p - g(t)\right| \left[K_n - K_n'\right]$$

$$+ \sum_n \int g(t) \left[K_n - K_n'\right]$$

$$\leq \int \left|\|f(t)\|^p - g(t)\right| + \|g\|_\infty \sum_n \mu\left(K_n - K_n'\right) \leq \varepsilon/2 + \|g\|_\infty \varepsilon/\left(2\|g\|_\infty\right) = \varepsilon.$$

Thereby we can assume that $f(\cdot)$ is continuous on Ω and $f(\cdot) = 0$ a.e. on $\mathcal{T} - \Omega$. By Proposition 6.1, $C_X(K_n) = C(K_n) \otimes_\lambda X$ and we can choose $\{f_{nk}\} \subseteq C(K_n) \otimes X$ such that $\lim_k \|f(\cdot)[K_n] - f_{nk}\|_\infty = 0$ for all n. Then $\|f(\cdot)[K_n] - f_{nk}\|_p \leq \|f(\cdot)[K_n] - f_{nk}\|_\infty \mu(K_n) \to 0$ as $k \to \infty$. By passing to a subsequence we can assume that $\|f(\cdot)[K_n] - f_{nk}\|_p^p \leq 2^{-k}$ for all n and k. Put $f_k = \sum_{n=1}^k f_{nk}[K_n] \in C_X\left(\cup_{n=1}^k K_n\right)$. For $\varepsilon > 0$ choose k_0 with $k2^{-k} \leq \varepsilon/2$ and $\sum_{n \geq k+1} \int_{K_n} \|f(t)\|^p \leq \varepsilon/2$ for all $k \geq k_0$. Then

$$\|f - f_k\|_p^p = \sum_{n=1}^k \int_{K_n} \|f(t) - f_{nk}(t)\|^p$$

$$+ \sum_{n \geq k+1} \int_{K_n} \|f(t)\|^p \leq \sum_{n=1}^k \|f[K_n] - f_{nk}\|_p^p + \varepsilon/2$$

$$\leq k2^{-k} + \varepsilon/2 \leq \varepsilon$$

for all $k \geq k_0$. Thus we can assume that $f(\cdot)$ is an X-valued function which is continuous on a compact subset K and vanishing out of K. Using Lemma 6.1, we conclude that $f(\cdot)$ gives rise to the element of $L_X^p(\mathcal{T})$. □

Now let $X = H$ be a Hilbert space with its orthonormal basis $(\epsilon_i)_{i \in I}$. Based on the Hilbert space tensor product construction considered in Subsection 2.4, we derive that $L^2(\mathcal{T}) \otimes_\sigma H = \bigoplus_I L^2(\mathcal{T})$. Thus every $\zeta \in L^2(\mathcal{T}) \otimes_\sigma H$ admits a unique expansion $\zeta = \sum_i \zeta_i \otimes \epsilon_i$ with $(\zeta_i)_i \in \bigoplus_I L^2(\mathcal{T})$. In particular, $L^2(\mathcal{T}) \otimes_\sigma H$ consists of H-valued functions $\zeta(\cdot) = \sum_i \zeta_i(\cdot)\epsilon_i$ with $\sum_i \int |\zeta_i(t)|^2 < \infty$.

Corollary 6.4. *Let H be a Hilbert space. Then $L_H^2(\mathcal{T}) = L^2(\mathcal{T}) \otimes_\sigma H$ up to an isometric isomorphism.*

Proof. Take an H-valued function $\zeta(\cdot) \in L^2(\mathcal{T}) \otimes_\sigma H$. Then

$$\|\zeta\|_\sigma^2 = \sum_{i \in I} \int |\zeta_i(t)|^2 = \sum_{n=1}^\infty \int |\zeta_{i_n}(t)|^2,$$

that is, $\zeta_i(\cdot) = 0$ a.e. on $\mathcal{T}$ for all $i \in I - \{i_n\}$. In particular, $\zeta(t) = \sum_{n=1}^\infty \zeta_{i_n}(t)\epsilon_{i_n}$ with $\|\zeta(t)\|^2 = \sum_{n=1}^\infty |\zeta_{i_n}(t)|^2$ for almost all t. Since $\zeta_{i_n}(\cdot)\epsilon_{i_n} \in \mathfrak{L}_H(\mathcal{T})$ for every n (use for example Proposition 6.3), it follows that $\zeta(\cdot) \in \mathfrak{L}_H(\mathcal{T})$ by virtue of Corollary 6.1. By Fatou's lemma, $\int \|\zeta(t)\|^2 = \int \sum_{n=1}^\infty |\zeta_{i_n}(t)|^2 = \sum_{n=1}^\infty \int |\zeta_{i_n}(t)|^2 < \infty$, that is, $\|\zeta\|_2 < \infty$. It remains to use Proposition 6.4 to conclude that $\zeta \in L_H^2(\mathcal{T})$.

Conversely, if $\zeta \in L^2_H(\mathcal{T})$ then ζ is represented by $\zeta(\cdot) \in \mathfrak{L}_H(\mathcal{T})$ with $\|\zeta\|_2 < \infty$ thanks to Proposition 6.4. Then $\zeta_i(\cdot) = (\zeta(\cdot), \epsilon_i) \in \mathfrak{L}(\mathcal{T})$ and $\sum_{i\in F} \|\zeta_i\|_2^2 = \sum_{i\in F} \int |\zeta_i(t)|^2 = \int \sum_{i\in F} |(\zeta(t), \epsilon_i)|^2 \le \int \|\zeta(t)\|^2 < \infty$ for every finite subset $F \subseteq I$. Whence $\zeta(\cdot) = \sum_i \zeta_i(\cdot)\epsilon_i \in \bigoplus_I L^2(\mathcal{T}) = L^2(\mathcal{T}) \otimes_\sigma H$. □

Now let us look at the case of $L^1_X(\mathcal{T})$ space. Based on Proposition 2.4, we obtain an isometric isomorphism $\Phi : (L^1(\mathcal{T}) \otimes_\pi X)^* \to \mathcal{B}(X, L^\infty(\mathcal{T}))$,$\langle f, \Phi(\varphi)x\rangle = \langle f \otimes x, \varphi\rangle$ for all $f \in L^1(\mathcal{T})$, $x \in X$. Note that $\Phi(\varphi) : X \to L^1(\mathcal{T})^* = L^\infty(\mathcal{T})$ is a bounded linear operator with the norm $\|\varphi\|$. Since $\Phi(\varphi)x \in L^\infty(\mathcal{T})$, it follows that $\langle f \otimes x, \varphi\rangle = \int f(t)\rho(\Phi(\varphi)x)(t)$ for all $f \in L^1(\mathcal{T})$, where ρ is a lifting for the algebra $L^\infty(\mathcal{T})$ (see Subsection 6.1). Thus $\rho(\Phi(\varphi)x)$ is a bounded measurable function on $\mathcal{T}$. Let us define a function $\varphi(\cdot) : \mathcal{T} \to X^*$, $\langle x, \varphi(t)\rangle = \rho(\Phi(\varphi)x)(t)$. Note that $|\langle x, \varphi(t)\rangle| \le \|\rho(\Phi(\varphi)x)\| = \|\Phi(\varphi)x\|_\infty \le \|\Phi(\varphi)\| \|x\| \le \|\varphi\| \|x\|$ for all t, that is, $\langle x, \varphi(\cdot)\rangle \in L^\infty(\mathcal{T})$ for all x. In particular, $\varphi(\cdot) : \mathcal{T} \to X^*$ is a w^*-measurable function (see Subsection 6.3) such that $\|\varphi(t)\| \le \|\varphi\|$ for all t, and $\langle f \otimes x, \varphi\rangle = \int f(t)\langle x, \varphi(t)\rangle = \int \langle (f \otimes x)(t), \varphi(t)\rangle$ for all $f \in L^1(\mathcal{T})$. Thus

$$\langle w, \varphi\rangle = \int \langle w(t), \varphi(t)\rangle \quad \text{for all } w \in L^1(\mathcal{T}) \otimes X. \tag{6.2}$$

For $x(\cdot) \in L^1_X(\mathcal{T})$ (see Proposition 6.4) the function $\langle x(\cdot), \varphi(\cdot)\rangle$ is measurable by Corollary 6.3. Moreover, $\int |\langle x(t), \varphi(t)\rangle| \le \int \|x(t)\| \|\varphi(t)\| \le \|\varphi(\cdot)\|_\infty \|x(\cdot)\|_1$, that is, $x(\cdot) \mapsto \langle x(\cdot), \varphi\rangle = \int \langle x(t), \varphi(t)\rangle$ is a bounded linear functional on $L^1_X(\mathcal{T})$. We use this observation in the following assertions.

Proposition 6.5. *Let X be a Banach space. Then $L^1(\mathcal{T}) \otimes_\pi X = L^1_X(\mathcal{T})$ up to an isometric isomorphism.*

Proof. First note that $L^1(\mathcal{T}) \otimes X$ is dense in both spaces $L^1(\mathcal{T}) \otimes_\pi X$ and $L^1_X(\mathcal{T})$ (see Proposition 6.4). Moreover, $\|f \otimes x\|_1 = \int \|f(t)x\| = \|x\| \int |f(t)| = \|f\|_1 \|x\|$ for all $f \in L^1(\mathcal{T})$ and $x \in X$. It follows that $\|\cdot\|_1$ is a cross-norm on $L^1(\mathcal{T}) \otimes X$, therefore $\|z\|_1 \le \|z\|_\pi$ for all $z \in L^1(\mathcal{T}) \otimes X$. Conversely, take a nonzero $z \in L^1(\mathcal{T}) \otimes X$. Then $\|z\|_\pi = \langle z, \varphi\rangle$ for a certain $\varphi \in (L^1(\mathcal{T}) \otimes_\pi X)^*$, $\|\varphi\| = 1$. As we have just seen above φ defines a w^*-measurable function $\varphi(\cdot) : \mathcal{T} \to X^*$, $\|\varphi(t)\| \le \|\varphi\|$ for all $t \in \mathcal{T}$. Using (6.2), we derive that

$$\|z\|_\pi = \langle z, \varphi\rangle = \int \langle z(t), \varphi(t)\rangle \le \int \|z(t)\| \|\varphi(t)\| \le \|z\|_1 \|\varphi\| = \|z\|_1,$$

that is, $\|\cdot\|_1 = \|\cdot\|_\pi$ on $L^1(\mathcal{T}) \otimes X$. Based on the density fact just mentioned above, we conclude that $L^1(\mathcal{T}) \otimes_\pi X = L^1_X(\mathcal{T})$ up to an isometric isomorphism. □

Corollary 6.5. *Let X be a Banach space. If $\varphi \in L^1_X(\mathcal{T})^*$ then it defines a bounded, w^*-measurable function $\varphi(\cdot) : \mathcal{T} \to X^*$ and $\|\varphi\| = \|\varphi(\cdot)\|_\infty$. Conversely, if $\varphi(\cdot) : \mathcal{T} \to X^*$ is a bounded, w^*-measurable function then $\langle x(\cdot), \varphi\rangle = \int \langle x(t), \varphi(t)\rangle$ gives rise to a bounded linear functional $\overline{\varphi}$ on $L^1_X(\mathcal{T})$ with the norm at most $\|\varphi\|_\infty$. In this case, for every $x \in X$ we have $\langle x, \overline{\varphi}(t)\rangle = \langle x, \varphi(t)\rangle$ for almost all $t \in \mathcal{T}$. In the separable case of X, we have $\overline{\varphi}(t) = \varphi(t)$ for almost all $t \in \mathcal{T}$, and $\|\overline{\varphi}\| = \|\varphi(\cdot)\|_\infty$.*

Proof. If $\varphi \in L^1_X(\mathcal{T})^*$ then based on Proposition 6.5, we derive that $\varphi \in \left(L^1(\mathcal{T}) \otimes_\pi X\right)^*$ and there is a well defined bounded, w^*-measurable functions $\varphi : \mathcal{T} \to X^*$ with the property (6.2). In this case, $\|\varphi(t)\| \leq \|\varphi\|$ for all t, that is, $\|\varphi\|_\infty \leq \|\varphi\|$. Moreover,

$$\|\varphi\| = \sup\left|\left\langle \operatorname{ball} L^1_X(\mathcal{T}), \varphi\right\rangle\right| = \sup\left|\left\langle \operatorname{ball}\left(L^1(\mathcal{T}) \otimes X\right), \varphi\right\rangle\right|$$

$$\leq \sup\left\{\int |\langle z(t), \varphi(t)\rangle| : z \in \operatorname{ball}\left(L^1(\mathcal{T}) \otimes X\right)\right\}$$

$$\leq \sup\left\{\int \|z(t)\| \|\varphi(t)\| : z \in \operatorname{ball}\left(L^1(\mathcal{T}) \otimes X\right)\right\} \leq \|\varphi\|_\infty,$$

that is, $\|\varphi\| = \|\varphi\|_\infty$. Conversely, take a bounded, w^*-measurable functions $\varphi : \mathcal{T} \to X^*$. For $x(\cdot) \in L^1_X(\mathcal{T})$ the function $\langle x(\cdot), \varphi(\cdot)\rangle$ is measurable by Corollary 6.3, and $\langle x(\cdot), \varphi\rangle = \int \langle x(t), \varphi(t)\rangle$ is a well defined bounded linear functional $\overline{\varphi}$ on $L^1_X(\mathcal{T})$ with the norm at most $\|\varphi\|_\infty$. Then we have a well defined bounded, w^*-measurable function $\overline{\varphi}(\cdot) : \mathcal{T} \to X^*$ by $\langle x, \overline{\varphi}(t)\rangle = \rho(\Phi(\overline{\varphi})x)(t)$. But $\int f(t)\rho(\Phi(\varphi)x)(t) = \langle f \otimes x, \overline{\varphi}\rangle = \int f(t)\langle x, \varphi(t)\rangle$ for all $f \in L^1(\mathcal{T})$, which in turn implies that $\langle x, \overline{\varphi}(t)\rangle = \langle x, \varphi(t)\rangle$ a.e. on $\mathcal{T}$, that is, $\langle x, \overline{\varphi}(\cdot)\rangle = \langle x, \varphi(\cdot)\rangle$ in $L^\infty(\mathcal{T})$ for every $x \in X$. Finally, if X is separable then $\langle x_n, \overline{\varphi}(t)\rangle = \langle x_n, \varphi(t)\rangle$, $t \in \mathcal{T} - N_n$ for all n, where $\{x_n\}$ is a dense subset of $\mathcal{T}$, and $N_n \in \mathfrak{M}$ with $\mu(N_n) = 0$. Put $N = \cup_n N_n$. Then $\langle x_n, \overline{\varphi}(t)\rangle = \langle x_n, \varphi(t)\rangle$, $t \in \mathcal{T} - N$ for all n, that is, $\overline{\varphi}(t) = \varphi(t)$ for all $t \in \mathcal{T} - N$. Moreover, $\|\overline{\varphi}\| = \|\overline{\varphi}(\cdot)\|_\infty = \|\varphi(\cdot)\|_\infty$. □

7. $\mathcal{B}(H)$-valued measurable functions

In this section we discuss $\mathcal{B}(H)$-measurability of functions defined on a locally compact topological space $\mathcal{T}$ equipped with a positive Radon integral.

7.1. Decomposable operators. First confirm that $\mathcal{B}(H)$ is a Banach space and Banach space valued measurability applicable in this case as well. But the following example shows that $*$-algebra structure on $\mathcal{B}(H)$ can seriously be ignored in this case.

Suppose $\mathcal{T} = [0,1]$ equipped with Lebesgue's integral, $H = \ell^2[0,1]$ with its canonical basis $\{\epsilon_t\}_{t\in[0,1]}$, and let $x : \mathcal{T} \to \mathcal{B}(H)$, $x(t) = \epsilon_0 \odot \epsilon_t$, where $(\epsilon_0 \odot \epsilon_t)\zeta = (\zeta, \epsilon_t)\epsilon_0$ is the one-rank operator on H. Since $\|\zeta\|^2 = \sum_t |(\zeta, \epsilon_t)|^2$ for every $\zeta \in H$, it follows that $(\zeta, \epsilon_t) = 0$ for all t but countably many of them. In particular, $x(t)\zeta = 0$ for almost all $t \in [0,1]$, that is, the function $x(\cdot)\zeta : \mathcal{T} \to H$ is measurable. But $(x(t)^*\zeta, \eta) = (\zeta, x(t)\eta) = (\epsilon_t, \eta)(\zeta, \epsilon_0) = ((\zeta, \epsilon_0)\epsilon_t, \eta) = ((\epsilon_t \odot \epsilon_0)\zeta, \eta)$, $\zeta, \eta \in H$, that is, $x(t)^*\zeta = (\epsilon_t \odot \epsilon_0)\zeta$, $t \in [0,1]$. In particular, $x(t)^*\epsilon_0 = \epsilon_t$ and $\|x(t)^*\epsilon_0 - x(s)^*\epsilon_0\| = \|\epsilon_t - \epsilon_s\| = \sqrt{2}$ for all $s, t \in [0,1]$. Thus the function $x(\cdot)^*\epsilon_0$ can not be continuous on every infinite compact subset of $[0,1]$. By Proposition 6.2, $x(\cdot)^*\epsilon_0$ is not a Lebesgue measurable function.

Based on the example just provided we say that a function $x(\cdot) : \mathcal{T} \to \mathcal{B}(H)$ is *measurable* if both H-valued functions $x(\cdot)\zeta$ and $x(\cdot)^*\zeta$ are measurable for every $\zeta \in H$. Recall (see Subsection 2.3) that $\mathcal{B}(H)$ possesses the w^*-topology $\sigma\left(\mathcal{B}(H), \mathcal{B}^1(H)\right)$ being the dual of the Banach algebra $\mathcal{B}^1(H)$.

Lemma 7.1. *Let $x(\cdot):\mathcal{T}\to\mathcal{B}(H)$ be a function such that $x(\cdot)\zeta$ is measurable for every $\zeta\in H$. Then $x(\cdot)$ is a w^*-measurable function, and for every measurable H-valued function $\zeta:\mathcal{T}\to H$ the function $x(\cdot)\zeta(\cdot):\mathcal{T}\to H$, $t\mapsto x(t)\zeta(t)$ is measurable.*

Proof. Based on Proposition 2.2, we can take a w^*-continuous functional $\omega_{\zeta,\eta}:\mathcal{B}(H)\to\mathbb{C}$ obtained from $\zeta,\eta\in\ell^2(H)$. Then $\langle x(t),\omega_{\zeta,\eta}\rangle=\sum_{n=1}^{\infty}(x(t)\zeta_n,\eta_n)$, $t\in\mathcal{T}$. For every n, $(x(\cdot)\zeta_n,\eta_n)\in\mathfrak{L}(\mathcal{T})$, therefore $\langle x(\cdot),\omega_{\zeta,\eta}\rangle\in\mathfrak{L}(\mathcal{T})$ (see also Corollary 6.1). Consequently, $x(\cdot)$ is a w^*-measurable function.

Now let $\zeta:\mathcal{T}\to H$ be a measurable H-valued function. Then $\zeta_\eta:\mathcal{T}\to\mathcal{B}^1(H)$, $\zeta_\eta(t)=\zeta(t)\odot\eta$ is measurable, for $\|\zeta_\eta(t)-\zeta_\eta(s)\|_1=\|\zeta(t)-\zeta(s)\|\,\|\eta\|$, $t,s\in\mathcal{T}$ (see Subsection 2.3). But

$$(x(t)\zeta(t),\eta)=\operatorname{tr}(x(t)\zeta(t)\odot\eta)=\operatorname{tr}((\zeta(t)\odot\eta)x(t))$$

$$=\langle\zeta(t)\odot\eta,x(t)\rangle=\langle\zeta_\eta(t),x(t)\rangle,t\in\mathcal{T},$$

that is, $(x(\cdot)\zeta(\cdot),\eta)=\langle\zeta_\eta(\cdot),x(\cdot)\rangle$. Using Corollary 6.3, we derive that $x(\cdot)\zeta(\cdot)$ is weakly measurable. Further, take a compact subset $K\subseteq\mathcal{T}$. Using Proposition 6.3, we obtain a measurable subset $K_0\subseteq K$ such that $\mu(K-K_0)=0$ and $\zeta(K_0)\subseteq Y$ for a closed separable subspace $Y\subseteq H$. Take a dense sequence $\{\eta_n\}$ from Y. By assumption, $\{x(\cdot)\eta_n\}\subseteq\mathfrak{L}_H(\mathcal{T})$, that is, there are measurable subsets $K_n\subseteq K$ such that $\mu(K-K_n)=0$ and $x(K_n)\eta_n$ are separable. Put $K_\infty=\cap_{n=0}^{\infty}K_n$. Then $K_\infty\in\mathfrak{M}^1$, $\mu(K-K_\infty)=0$ and all $x(K_\infty)\eta_n$ are separable. Thus we can assume that $\cup_n x(K_\infty)\eta_n\subseteq Z$ for a closed separable subspace $Z\subseteq H$. Then

$$\begin{aligned}x(\cdot)\zeta(\cdot)(K_\infty)&\subseteq x(K_\infty)\zeta(K_\infty)\subseteq x(K_\infty)\zeta(K_0)\subseteq x(K_\infty)Y=x(K_\infty)\langle\{\eta_n\}\rangle\\&\subseteq\langle x(K_\infty)\{\eta_n\}\rangle=\langle\cup_n x(K_\infty)\eta_n\rangle\subseteq Z.\end{aligned}$$

Using Proposition 6.3 ones again, we conclude that $x(\cdot)\zeta(\cdot)\in\mathfrak{L}_H(\mathcal{T})$. □

Remark 7.1. As above assume that $x(\cdot)\zeta\in\mathfrak{L}_H(\mathcal{T})$ for every $\zeta\in H$. If H is separable then $\|x(\cdot)\|\in\mathfrak{L}(\mathcal{T})$. Indeed, take a sequence $\{\zeta_n\}\subseteq\operatorname{ball}H$ which is dense in $\operatorname{ball}H$. Since $x(\cdot)\zeta_n\in\mathfrak{L}_H(\mathcal{T})$, we have $\|x(\cdot)\zeta_n\|\in\mathfrak{L}(\mathcal{T})$ for all n. Then $\|x(\cdot)\|=\sup\{\|x(\cdot)\zeta_n\|:n\in\mathbb{N}\}\in\mathfrak{L}(\mathcal{T})$.

Now assume that $x(\cdot):\mathcal{T}\to\mathcal{B}(H)$ is a measurable function such that $\|x(\cdot)\|\in\mathfrak{L}^\infty(\mathcal{T})$. By Lemma 7.1, $x(\cdot)\zeta(\cdot)\in\mathfrak{L}_H(\mathcal{T})$ for every $\zeta(\cdot)\in\mathfrak{L}_H(\mathcal{T})$. If $\zeta(\cdot)\in L^2_H(\mathcal{T})$ (see Proposition 6.4), then for every null set $N\in\mathfrak{M}^1$ we have

$$\int\|x(t)\zeta(t)\|^2\le\int\|x(t)\|^2\|\zeta(t)\|^2$$

$$=\int\|x(t)\|^2[\mathcal{T}-N]\|\zeta(t)\|^2\le\|[\mathcal{T}-N]x(\cdot)\|_\infty^2\int\|\zeta(t)\|^2,$$

which in turn implies that $\|x(\cdot)\zeta(\cdot)\|_2\le\operatorname{esssup}\|x(\cdot)\|\,\|\zeta(\cdot)\|_2=\|x(\cdot)\|_\infty\|\zeta(\cdot)\|_2$. In particular, $x(\cdot)\zeta(\cdot)\in L^2_H(\mathcal{T})$ thanks to Proposition 6.4. We obtain a well defined bounded linear operator

$$M_x:L^2_H(\mathcal{T})\to L^2_H(\mathcal{T}),\quad M_x(\zeta(\cdot))=x(\cdot)\zeta(\cdot)$$

such that $\|M_x\| \leq \|x(\cdot)\|_\infty$. The operator M_x is called a decomposable operator, which is used to denote by $\int^{\oplus} x(t)$. Note that $\left(\int^{\oplus} x(t)\right)^* = \int^{\oplus} x^*(t)$, for

$$(M_x\zeta(\cdot), \eta(\cdot)) = \int (x(t)\zeta(t), \eta(t)) = \int (\zeta(t), x(t)^* \eta(t)) = (\zeta(\cdot), M_{x^*}\eta(\cdot))$$

for all $\zeta(\cdot), \eta(\cdot) \in L^2_H(\mathcal{T})$, where $x^*(t) = x(t)^*$ is measurable with the same norm function $\|x(\cdot)^*\| = \|x(\cdot)\|$ from $L^\infty(\mathcal{T})$. Further, $L^2_H(\mathcal{T}) = L^2(\mathcal{T}) \otimes_\sigma H$ by Corollary 6.4, and if $x(t) = f(t)1$ for a certain $f \in L^\infty(\mathcal{T})$ then $(M_x\zeta(\cdot))(t) = x(t)\zeta(t) = f(t)\zeta(t) = (\pi(f)\zeta(\cdot))(t)$, $\zeta(\cdot) \in L^2_H(\mathcal{T})$, where $\pi : \mathcal{B}\left(L^2(\mathcal{T})\right) \to \mathcal{B}\left(L^2(\mathcal{T}) \otimes_\sigma H\right)$, $\pi(y) = y \otimes 1$ is the diagonal representation from Subsection 2.5, and we identify $L^\infty(\mathcal{T})$ with the abelian von Neumann algebra of multiplication operators M_f from $\mathcal{B}\left(L^2(\mathcal{T})\right)$ (see Subsection 6.1).

Lemma 7.2. *Let $x(\cdot) : \mathcal{T} \to \mathcal{B}(H)$ be a measurable function such that $\|x(\cdot)\| \in L^\infty(\mathcal{T})$. If H is separable then $\left\|\int^{\oplus} x(t)\right\| = \|x(\cdot)\|_\infty$.*

Proof. As we have just seen above $\|M_x\| \leq \|x(\cdot)\|_\infty$. For $\varepsilon > 0$ take a compact subset $C \subseteq L = \{t \in \mathcal{T} : \|x(t)\| > \|x(\cdot)\|_\infty - \varepsilon\}$ with $\mu(C) > 0$. Choose a countable dense subset $\{\zeta_n, \eta_n\} \subseteq \operatorname{ball} H$ of unit vectors such that $\|x\| = \sup\{|(x\zeta_n, \eta_n)| : n \in \mathbb{N}\}$ (see Remark 7.1) for every $x \in \mathcal{B}(H)$. Using Lemma 7.1, we conclude that $L_n = \{t \in \mathcal{T} : |(x(t)\zeta_n, \eta_n)| > \|x(\cdot)\|_\infty - \varepsilon\} \in \mathfrak{M}$ for all n, and $L = \cup_n L_n$. Define $K_1 = L_1$, $K_2 = L_2 \cap (L - K_1)$ and $K_n = L_n \cap (L - (K_1 \cup \cdots \cup K_{n-1}))$ for $n > 1$. Then L is the disjoint union of all K_n from $\mathfrak{M}$. Put $\zeta_n(\cdot) = [K_n]\zeta_n$ and $\eta_n(\cdot) = [K_n]\eta_n$ which are functions from $\mathfrak{L}_H(\mathcal{T})$. Put $\zeta'(\cdot) = \sum_n \zeta_n(\cdot)$ and $\eta'(\cdot) = \sum_n \eta_n(\cdot)$, which are functions from $\mathfrak{L}_H(\mathcal{T})$ by Corollary 6.1. Note that $\|[C]\zeta'(\cdot)\|_2^2 = \int [C]\|\zeta'(t)\|^2 = \sum_n \int [C \cap K_n]\|\zeta_n\|^2 = \sum_n \mu(C \cap K_n) = \mu(C)$. Similarly, $\|[C]\eta'(\cdot)\|_2 = \mu(C)^{1/2}$. Thus there are functions $\overline{\zeta}(\cdot)$, $\overline{\eta}(\cdot) \in L^2_H(\mathcal{T})$ such that $\left\|\overline{\zeta}(\cdot)\right\|_2 = \|\overline{\eta}(\cdot)\|_2 = \mu(C)^{1/2}$ and

$$\left|\left(x(t)\overline{\zeta}(t), \overline{\eta}(t)\right)\right| = \sum_n [C \cap K_n] |(x(t)\zeta_n, \eta_n)| > \|x(\cdot)\|_\infty - \varepsilon$$

for all $t \in C$. Put $f(t) = \dfrac{\left|\left(x(t)\overline{\zeta}(t), \overline{\eta}(t)\right)\right|}{\left(x(t)\overline{\zeta}(t), \overline{\eta}(t)\right)}[C]$, $f \in L^\infty(\mathcal{T})$ and $\zeta(\cdot) = f(\cdot)\overline{\zeta}(\cdot)$, $\eta(\cdot) = \overline{\eta}(\cdot) \in L^2_H(\mathcal{T})$. Then

$$\begin{aligned}
&\left(\left(\int^{\oplus} x(t)\right)\zeta(\cdot), \eta(\cdot)\right) \\
&= \int (x(t)\zeta(t), \eta(t)) = \int f(t)\left(x(t)\overline{\zeta}(t), \overline{\eta}(t)\right) \\
&= \int \left|\left(x(t)\overline{\zeta}(t), \overline{\eta}(t)\right)\right|[C] \geq (\|x(\cdot)\|_\infty - \varepsilon)\mu(C) \\
&= (\|x(\cdot)\|_\infty - \varepsilon)\|\zeta(\cdot)\|_2\|\eta(\cdot)\|_2,
\end{aligned}$$

which in turn implies that $\left\|\int^{\oplus} x(t)\right\| \geq \|x(\cdot)\|_\infty - \varepsilon$ for every $\varepsilon > 0$. □

Remark 7.2. The equality from Lemma 7.2 is not true in the non-separable case. Indeed, take $\mathcal{T} = [0,1]$ equipped with the Lebesgue's integral, $H = \ell^2[0,1]$ with its canonical basis $(\epsilon_t)_{t\in[0,1]}$, and $x(\cdot) : \mathcal{T} \to \mathcal{B}(H)$ to be the function $x(t) = \epsilon_t \odot \epsilon_t$. As above, $x(t)\zeta = x(t)^*\zeta = 0$ for almost all t. Thus $x(\cdot)$ is measurable, $\int^{\oplus} x(t) = 0$, whereas $\|x(\cdot)\|_\infty = \inf\{\sup\|x(\mathcal{T}-N)\| : \mu(N) = 0\} = 1$.

Corollary 7.1. *The mapping $M : L^\infty(\mathcal{T}) \to \mathcal{B}(L^2(\mathcal{T}))$, $f \mapsto M_f$ is an isometry.*

Proof. It suffices put $H = \mathbb{C}$ in Lemma 7.2. Namely, a measurable function $f : \mathcal{T} \to \mathcal{B}(\mathbb{C}) = \mathbb{C}$ with $|f| \in L^\infty(\mathcal{T})$ is reduced to a function $f(\cdot) \in \mathfrak{L}^\infty(\mathcal{T})$. Moreover, $\int^{\oplus} f(t) = M_f$ and $\|M_f\| = \|f\|_\infty$ thanks to Lemma 7.2. □

In the general case, put $\mathcal{A}_H = \pi(L^\infty(\mathcal{T}))$ called the diagonal algebra, and denote $\mathcal{D}_H$ to be the set of all decomposable operators on $L^2_H(\mathcal{T})$ obtained from bounded, $\mathcal{B}(H)$-valued measurable functions on $\mathcal{T}$. As above we have $\mathcal{A}_H \subseteq \mathcal{D}_H$. Moreover, $\mathcal{D}_H \subseteq \mathcal{A}'_H$. Indeed, $(M_x\pi(f)\zeta)(t) = x(t)f(t)\zeta(t) = f(t)x(t)\zeta(t) = f(t)(M_x\zeta)(t) = (\pi(f)M_x\zeta)(t)$, $t \in \mathcal{T}$ for all $f \in L^\infty(\mathcal{T})$ and $M_x \in \mathcal{D}_H$.

Proposition 7.1. *Let H be a Hilbert space. Then $\mathcal{D}_H = \mathcal{A}'_H$.*

Proof. First note that every function from $C_c(\mathcal{T})$ can be written like $f\overline{g}$ for some $f, g \in C_c(\mathcal{T})$. Take $\zeta, \eta \in H$ and $x \in \mathcal{A}'_H$. If $K = (\operatorname{supp} f) \cup (\operatorname{supp} g)$ then

$$\begin{aligned}&(x(f\otimes\zeta), g\otimes\eta)\\ &= (x\pi(f)([K]\otimes\zeta), \pi(g)[K]\otimes\eta) = (\pi(f\overline{g})x([K]\otimes\zeta), [K]\otimes\eta).\end{aligned}$$

If $f_1\overline{g_1} = f_2\overline{g_2}$ and $K = \cup_i(\operatorname{supp} f_i) \cup (\operatorname{supp} g_i)$, then

$$\begin{aligned}&(x(f_1\otimes\zeta), g_1\otimes\eta)\\ &= (\pi(f_1\overline{g_1})x([K]\otimes\zeta), [K]\otimes\eta) = (\pi(f_2\overline{g_2})x([K]\otimes\zeta), [K]\otimes\eta)\\ &= (x(f_2\otimes\zeta), g_2\otimes\eta),\end{aligned}$$

that is, $f\overline{g} \mapsto (x(f\otimes\zeta), g\otimes\eta)$ is a well defined linear functional on $C_c(\mathcal{T})$. Consider the polar decompositions $f = u|f|$, $g = v|g|$ with $u, v \in L^\infty(\mathcal{T})$. Using Corollary 6.4, we obtain that

$$\begin{aligned}|(x(f\otimes\zeta), g\otimes\eta)| &= \left|\left(\pi(u\overline{v})x\left(|fg|^{1/2}\otimes\zeta\right), |fg|^{1/2}\otimes\eta\right)\right|\\ &\le \|\pi(u\overline{v})\|\|x\|\left\||fg|^{1/2}\otimes\zeta\right\|_2\left\||fg|^{1/2}\otimes\eta\right\|_2 = \|x\|\left\||fg|^{1/2}\right\|_2^2\|\zeta\|\|\eta\|\\ &= \|x\|\|f\overline{g}\|_1\|\zeta\|\|\eta\|.\end{aligned}$$

Since $C_c(\mathcal{T})$ is dense in $L^1(\mathcal{T})$, it follows that the latter mapping extends to a bounded linear functional $h_{\zeta,\eta}$ on $L^1(\mathcal{T})$. Hence $h_{\zeta,\eta} \in L^\infty(\mathcal{T})$ such that

$$(x(f\otimes\zeta), g\otimes\eta) = \int f(t)\overline{g}(t)h_{\zeta,\eta}(t), \quad \|h_{\zeta,\eta}\|_\infty \le \|x\|\|\zeta\|\|\eta\|. \tag{7.1}$$

Using the lifting $\rho : L^\infty(\mathcal{T}) \to \mathfrak{L}^\infty(\mathcal{T})$ we can assume $h_{\zeta,\eta}(\cdot) = \rho(h_{\zeta,\eta}) \in \mathfrak{L}^\infty(\mathcal{T})$ which is everywhere on $\mathcal{T}$ defined, bounded measurable function. In particular, $(\zeta, \eta) \mapsto h_{\zeta,\eta}(t)$ is a bounded sesquilinear form on H, therefore $h_{\zeta,\eta}(t) = (x(t)\zeta, \eta)$ for a bounded operator $x(t) \in \mathcal{B}(H)$ such that $\|x(t)\| \le \|x\|$ thanks to (7.1). In particular, $x(\cdot)\zeta$ is weakly measurable for every $\zeta \in H$, and $(x(f\otimes\zeta), g\otimes\eta) = \int(x(t)(f\otimes\zeta)(t), (g\otimes\eta)(t))$ for all $f, g \in C_c(\mathcal{T})$.

Prove that $x(\cdot)\zeta \in \mathfrak{L}_H(\mathcal{T})$ for all $\zeta \in H$. Based on Proposition 6.3, we need only to verify separability condition. Take a compact subset $K \subseteq \mathcal{T}$ and $[K] \otimes \zeta \in L^2_H(\mathcal{T})$. Since $x \in \mathcal{B}\left(L^2_H(\mathcal{T})\right)$, it follows that $x([K]\otimes\zeta) \in L^2_H(\mathcal{T})$. In particular, $x([K]\otimes\zeta)(\cdot) \in \mathfrak{L}_H(\mathcal{T})$, and using Proposition 6.3, we obtain $K_\infty \in \mathfrak{M}$ such that $\mu(K - K_\infty) = 0$ and $x([K]\otimes\zeta)(K_\infty) \subseteq Y$ for a separable subspace $Y \subseteq H$. Take a projection e onto Y, and a sequence $(\psi_n)_n \subseteq C_c(\mathcal{T})$ such that $\lim_n \psi_n = [K]$ in $L^2(\mathcal{T})$. Thus $(1-e)x([K]\otimes\zeta)(K_\infty) = \{0\}$, and

$$\begin{aligned}&\int \left|f(t)\,\overline{g}(t)\,h_{\zeta,(1-e)\eta}(t)\,([K]-\psi_n)(t)\right| \\ &\leq \left(\int \left|f(t)\,\overline{g}(t)\,h_{\zeta,(1-e)\eta}(t)\right|^2\right)^{1/2}\left(\int \left|([K]-\psi_n)(t)\right|^2\right)^{1/2} \\ &\leq \left\|h_{\zeta,(1-e)\eta}\right\|_\infty \|f\overline{g}\|_2 \,\|[K]-\psi_n\|_2 \to 0.\end{aligned}$$

Moreover, $\left\|[K]g - g\overline{\psi_n}\right\|_2^2 = \int \left|[K]g(t) - g(t)\overline{\psi_n}(t)\right|^2 \leq \|g\|_\infty^2 \|[K]-\psi_n\|_2^2 \to 0$, that is, $\lim_n g\overline{\psi_n} = [K]g$ in $L^2(\mathcal{T})$. In particular, $\lim_n g\overline{\psi_n}\otimes(1-e)\eta = [K]g\otimes(1-e)\eta$ in $L^2_H(\mathcal{T})$. Note also that

$$\begin{aligned}[K](t)\,x(f\otimes\zeta)(t) &= (\pi([K])\,x(f\otimes\zeta))(t) \\ &= (x\pi([K])(f\otimes\zeta))(t) = x([K]f\otimes\zeta)(t) \\ &= x\pi(f)([K]\otimes\zeta)(t) = \pi(f)\,x([K]\otimes\zeta)(t) = f(t)\,x([K]\otimes\zeta)(t)\end{aligned}$$

for all $t \in \mathcal{T}$. Put $\alpha_K = \int_K f(t)\,\overline{g}(t)\,h_{\zeta,(1-e)\eta}(t)$. Then

$$\begin{aligned}\alpha_K &= \lim_n \int f(t)\,\overline{g}(t)\,h_{\zeta,(1-e)\eta}(t)\,\psi_n(t) = \lim_n \left(x(f\otimes\zeta), g\overline{\psi_n}\otimes(1-e)\eta\right) \\ &= (x(f\otimes\zeta), [K]g\otimes(1-e)\eta) = \int_K (x(f\otimes\zeta)(t), (1-e)\,g(t)\,\eta) \\ &= \int ([K](t)\,x(f\otimes\zeta)(t), (1-e)\,g(t)\,\eta) \\ &= \int f(t)\,(x([K]\otimes\zeta)(t), (1-e)\,g(t)\,\eta) \\ &= \int f(t)\,((1-e)(x([K]\otimes\zeta)(t)), g(t)\,\eta) \\ &= \int_{K_\infty} f(t)\,((1-e)(x([K]\otimes\zeta)(t)), g(t)\,\eta) = 0\end{aligned}$$

for all $f, g \in C_c(\mathcal{T})$. Hence we get $h_{\zeta,(1-e)\eta}[K] = 0$ in $L^\infty(\mathcal{T})$, which in turn implies that

$$\begin{aligned}\rho([K])\,((1-e)\,x(\cdot)\,\zeta, \eta) &= \rho([K])\,(x(\cdot)\,\zeta, (1-e)\,\eta) \\ &= \rho([K])\,h_{\zeta,(1-e)\eta}(\cdot) = \rho([K])\,\rho\left(h_{\zeta,(1-e)\eta}\right) \\ &= \rho\left([K]\,h_{\zeta,(1-e)\eta}\right) = 0\end{aligned}$$

in $\mathfrak{L}^\infty(\mathcal{T})$. But $\rho([K])(t) = [K](t)$ for all $t \in K - N$, where $N \in \mathfrak{M}^1$ with $\mu(N) = 0$. It follows that $((1-e)\,x(t)\,\zeta, \eta) = 0$ for all $t \in K - N$ and for all ζ, η. In particular, $(1-e)\,x(t)\,\zeta = 0$ for all $t \in K - N$, or $x(K-N)\,\zeta \subseteq Y$, that is, $x(K-N)\,\zeta$ is separable. Based on Proposition 6.3, we conclude that $x(\cdot)\,\zeta \in \mathfrak{L}_H(\mathcal{T})$ for all $\zeta \in H$. Since the lifting ρ is a $*$-homomorphism, the same argument applicable for the function $x(\cdot)^*\zeta$. Thus $x(\cdot) : \mathcal{T} \to \mathcal{B}(H)$

is a bounded, measurable function. Thereby we get a decomposable operator $\int^{\oplus} x(t) \in \mathcal{D}_H$. Finally,

$$\left(\left(\int^{\oplus} x(t)\right)(f \otimes \zeta), g \otimes \eta\right) = \int (f(t)\, x(t)\, \zeta, g(t)\, \eta)$$

$$= \int f(t)\, \overline{g}(t)\, h_{\zeta,\eta}(t) = (x(f \otimes \zeta), g \otimes \eta)$$

for all $f, g \in C_c(\mathcal{T})$ and $\zeta, \eta \in H$. Using density of $C_c(\mathcal{T}) \otimes H$ in $L^2(\mathcal{T})$ (see Corollary 6.4), we conclude that $x = \int^{\oplus} x(t) \in \mathcal{D}_H$. Whence $\mathcal{D}_H = \mathcal{A}_H'$. $\square$

Corollary 7.2. *The C^*-algebra $L^\infty(\mathcal{T})$ is acting as a maximal abelian von Neumann algebra on $L^2(\mathcal{T})$.*

Proof. We have seen above in Subsection 6.1 that $L^\infty(\mathcal{T})$ being identified with $M(L^\infty(\mathcal{T}))$ is a von Neumann algebra on $L^2(\mathcal{T})$ up to a w^*-homeomorphism (see also Corollary 7.1). Note that $L^\infty(\mathcal{T}) = M(L^\infty(\mathcal{T})) = \mathcal{A}_{\mathbb{C}}$. Using Proposition 7.1, we obtain that $L^\infty(\mathcal{T})' = \mathcal{D}_{\mathbb{C}}$. But every measurable function $x : \mathcal{T} \to \mathcal{B}(\mathbb{C})$ with $\|x(\cdot)\| \in L^\infty(\mathcal{T})$ merely means that $x \in L^\infty(\mathcal{T})$ and $\left\|\int^{\oplus} x(t)\right\| = \|x\|_\infty$ thanks to Lemma 7.2. Hence $\mathcal{D}_{\mathbb{C}} = M(L^\infty(\mathcal{T})) = L^\infty(\mathcal{T})$, that is, $L^\infty(\mathcal{T})' = L^\infty(\mathcal{T})$. If $\mathcal{L}$ is an abelian algebra in $\mathcal{B}(L^2(\mathcal{T}))$ such that $L^\infty(\mathcal{T}) \subseteq \mathcal{L}$ then $\mathcal{L} \subseteq \mathcal{L}' \subseteq L^\infty(\mathcal{T})' = L^\infty(\mathcal{T})$, which means that $L^\infty(\mathcal{T})$ is a maximal abelian von Neumann algebra. $\square$

7.2. $\mathcal{M}$-valued measurable functions. Now let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra. Since $L^\infty(\mathcal{T}) \subseteq \mathcal{B}(L^2(\mathcal{T}))$ is a von Neumann algebra (maximal, abelian), we can generate a new von Neumann algebra $L^\infty(\mathcal{T})\overline{\otimes}\mathcal{M}$ on $\mathcal{B}(L^2_H(\mathcal{T}))$ (see Subsection 3.5). Recall that we have diagonal algebra $\mathcal{A}_H$ on $L^2_H(\mathcal{T})$ which consists of the operators $\pi(g) = M_g \otimes 1$, $g \in L^\infty(\mathcal{T})$. For $h \in L^\infty(\mathcal{T})$ and $x \in \mathcal{M}$ we have $(M_h \otimes x)(M_g \otimes 1) = M_{hg} \otimes x = (M_g \otimes 1)(M_h \otimes x)$, that is, $M_h \otimes x \in \mathcal{A}_H'$. Based on Proposition 7.1, we derive that $L^\infty(\mathcal{T})\overline{\otimes}\mathcal{M} \subseteq \mathcal{D}_H$. In particular, $M_h \otimes x$ defines $\mathcal{B}(H)$-valued bounded, measurable function $x(\cdot)$ such that $((M_h \otimes x)(f \otimes \zeta), g \otimes \eta) = \int f(t)\, \overline{g}(t)\, h_{\zeta,\eta}(t)$ with $h_{\zeta,\eta}(t) = h(t)(x\zeta, \eta) = (x(t)\, \zeta, \eta)$ (see (7.1)) for all $f, g \in C_c(\mathcal{T})$ and $\zeta, \eta \in H$. It follows that $x(\cdot) = h(\cdot)\, x$ and $\|x(\cdot)\|_\infty = \|h\|_\infty \|x\| = \|M_h\| \|x\| = \|M_h \otimes x\|$ thanks to Corollary 7.1. Moreover, $x(t) \in \mathcal{M}$ for all t. Put

$$\mathcal{D}_{\mathcal{M}} = \left\{\int^{\oplus} x(t) \in \mathcal{D}_H : x(t) \in \mathcal{M}, \quad t \in \mathcal{T}\right\},$$

which is a $*$-subalgebra of $\mathcal{D}_H$. As we have just seen above $L^\infty(\mathcal{T}) \otimes \mathcal{M} \subseteq \mathcal{D}_{\mathcal{M}}$. Namely, for every $M_h \otimes x \in L^\infty(\mathcal{T}) \otimes \mathcal{M}$ we have $M_h \otimes x = \int^{\oplus} h(t)\, x \in \mathcal{D}_{\mathcal{M}}$ and $\|M_h \otimes x\| = \|h(\cdot)\, x\|_\infty$. In the general case, the norm of a decomposable operator $\int^{\oplus} x(t)$ can be strictly less than $\|x(\cdot)\|_\infty$ (see Remark 7.2).

Lemma 7.3. *The $*$-algebra $\mathcal{D}_{\mathcal{M}}$ is complete, that is, it is a C^*-algebra.*

Proof. Take a sequence $x_n = \int^{\oplus} x_n(t)$, $n \in \mathbb{N}$ from $\mathcal{D}_{\mathcal{M}}$ such that $\lim_n x_n = x$ for some $x \in \mathcal{B}(L^2_H(\mathcal{T}))$. By Proposition 7.1, $x \in \mathcal{D}_H$. Take $\zeta, \eta \in H$. Note that the functions $(x_n(\cdot)\zeta,\eta)$ and $(x(\cdot)\zeta,\eta)$ are bounded, measurable functions from $\mathfrak{L}^{\infty}(\mathcal{T})$, and (using the lifting property and (7.1))

$$\begin{aligned}
\|((x-x_n)(\cdot)\zeta,\eta)\|_{\infty} &= \|((x-x_n)(\cdot)\zeta,\eta)\|_{L^{\infty}(\mathcal{T})} \\
&= \sup\left|\left\langle \operatorname{ball} L^1(\mathcal{T}),((x-x_n)(\cdot)\zeta,\eta)\right\rangle\right| \\
&= \sup\left\{\left|\sum_{i=1}^{n}\lambda_i\int[K_i](t)((x(t)-x_n(t))\zeta,\eta)\right| : \sum_{i=1}^{n}|\lambda_i|\mu(K_i)\leq 1\right\} \\
&= \sup\left\{\left|\sum_{i=1}^{n}\lambda_i((x-x_n)([K_i]\otimes\zeta),[K_i]\otimes\eta)\right| : \sum_{i=1}^{n}|\lambda_i|\mu(K_i)\leq 1\right\} \\
&\leq \sup\left\{\sum_{i=1}^{n}|\lambda_i|\,\|x-x_n\|\,\|[K_i]\otimes\zeta\|_2\,\|[K_i]\otimes\eta\|_2 : \sum_{i=1}^{n}|\lambda_i|\mu(K_i)\leq 1\right\} \\
&= \|x-x_n\|\,\|\zeta\|\,\|\eta\|\sup\left\{\sum_{i=1}^{n}|\lambda_i|\,\|[K_i]\|_2^2 : \sum_{i=1}^{n}|\lambda_i|\mu(K_i)\leq 1\right\} \\
&= \|x-x_n\|\,\|\zeta\|\,\|\eta\|\to 0,\quad \text{for large } n.
\end{aligned}$$

It follows that $p_{\zeta,\eta}(x(t)-x_n(t)) = |((x-x_n)(t)\zeta,\eta)|\to 0$, that is, $x(t) =$WOT-$\lim_n x_n(t)$. But $(x_n(t))\subseteq\mathcal{M}$ and $\mathcal{M}$ is WOT-closed, therefore $x(t)\in\mathcal{M}$ for every t. The latter means that $x\in\mathcal{D}_{\mathcal{M}}$ either. □

Consider the predual $\mathcal{M}_*$ of the von Neumann algebra $\mathcal{M}$ (see Proposition 2.2). Based on Corollary 6.5, we obtain that every $\varphi\in L^1_{\mathcal{M}_*}(\mathcal{T})^*$ defines a bounded, w^*-measurable function

$$\varphi(\cdot):\mathcal{T}\to(\mathcal{M}_*)^*=\mathcal{M},\quad \langle f\otimes a,\varphi\rangle$$
$$=\int f(t)\langle a,\varphi(t)\rangle,\quad f\otimes a\in C_c(\mathcal{T})\otimes\mathcal{M}_*\subseteq L^1_{\mathcal{M}_*}(\mathcal{T}),$$

and $\|\varphi\| = \|\varphi(\cdot)\|_{\infty}$. In particular, we obtain the decomposable operator $\int^{\oplus}\varphi(t)$ from $\mathcal{D}_{\mathcal{M}}$ such that $\left\|\int^{\oplus}\varphi(t)\right\|\leq\|\varphi(\cdot)\|_{\infty}=\|\varphi\|$. Thus there is a well defined bounded linear operator

$$\Psi: L^1_{\mathcal{M}_*}(\mathcal{T})^*\to\mathcal{D}_{\mathcal{M}},\quad \Psi(\varphi)=\int^{\oplus}\varphi(t)$$

with the norm at most 1. Conversely, if $x(\cdot):\mathcal{T}\to\mathcal{M}$ is a bounded, w^*-measurable function then $\langle y(\cdot),\varphi\rangle=\int\langle y(t),x(t)\rangle$, $y(\cdot)\in L^1_{\mathcal{M}_*}(\mathcal{T})$ gives rise to a bounded linear functional φ on $L^1_{\mathcal{M}_*}(\mathcal{T})$ with the norm at most $\|x(\cdot)\|_{\infty}$. The functional φ in turn defines a bounded, w^*-measurable function $\varphi(\cdot):\mathcal{T}\to(\mathcal{M}_*)^*=\mathcal{M}$ and $\|\varphi(\cdot)\|_{\infty}=\|\varphi\|$. Moreover, for every $a\in\mathcal{M}_*$ we have $\langle a,\varphi(t)\rangle=\langle a,x(t)\rangle$ for almost all $t\in\mathcal{T}$.

Lemma 7.4. *If $x=\int^{\oplus}x(t)\in\mathcal{D}_{\mathcal{M}}$ then it defines a functional $\varphi\in L^1_{\mathcal{M}_*}(\mathcal{T})^*$ with in turn defines a bounded, w^*-measurable function $\varphi(\cdot):\mathcal{T}\to\mathcal{M}$ such that $\int^{\oplus}\varphi(t)=x$ and $\|\varphi\|=\|\varphi(\cdot)\|_{\infty}\leq\|x(\cdot)\|_{\infty}$. In particular, the linear mapping Ψ is an isomorphism of Banach spaces.*

Proof. By assumption $x(\cdot) : \mathcal{T} \to \mathcal{M}$ is a bounded, measurable function. In this case $\left(x\widetilde{\zeta}\right)(t) = x(t)\widetilde{\zeta}(t)$ for all $\widetilde{\zeta}(\cdot) \in L^2_H(\mathcal{T})$. Pick $\zeta, \eta \in \ell^2(H)$. Then $\langle \omega_{\zeta,\eta}, x(\cdot)\rangle = \sum_{n=1}^{\infty}(x(\cdot)\zeta_n, \eta_n)$. Since $x(\cdot)\zeta_n \in \mathfrak{L}_H(\mathcal{T})$, we have $(x(\cdot)\zeta_n, \eta_n) \in \mathfrak{L}(\mathcal{T})$ for all n, which in turn implies that $\langle \omega_{\zeta,\eta}, x(\cdot)\rangle \in \mathfrak{L}(\mathcal{T})$. So, $\langle a, x(\cdot)\rangle \in \mathfrak{L}(\mathcal{T})$ for every $a \in \mathcal{M}_*$, that is, $x(\cdot) : \mathcal{T} \to (\mathcal{M}_*)^*$ is a w^*-measurable function. Using Corollary 6.5, we obtain a functional $\varphi \in L^1_{\mathcal{M}_*}(\mathcal{T})^*$ such that $\|\varphi\| \le \|x(\cdot)\|_\infty$ and $\langle y(\cdot), \varphi\rangle = \int \langle y(t), x(t)\rangle$ for all $y(\cdot) \in L^1_{\mathcal{M}_*}(\mathcal{T})$. The latter in turn defines a bounded, w^*-measurable function $\varphi(\cdot) : \mathcal{T} \to \mathcal{M}$ such that $\langle a, \varphi(\cdot)\rangle = \langle a, x(\cdot)\rangle$ in $L^\infty(\mathcal{T})$ for every $a \in \mathcal{M}_*$, and $\|\varphi(\cdot)\|_\infty = \|\varphi\| \le \|x(\cdot)\|_\infty$. The latter function in turn defines the operator $\int^\oplus \varphi(t) \in \mathcal{D}_\mathcal{M}$ such that $\left(\left(\int^\oplus \varphi(t)\right)\widetilde{\zeta}\right)(t) = \varphi(t)\widetilde{\zeta}(t)$ for all $\widetilde{\zeta}(\cdot) \in L^2_H(\mathcal{T})$. But, for $f, g \in C_c(\mathcal{T})$ and $\zeta, \eta \in H$, we have

$$\begin{aligned}\left(\left(\int^\oplus \varphi(t)\right) f\otimes\zeta, g\otimes\eta\right) &= \int (\varphi(t) f(t)\zeta, g(t)\eta) = \int (f\overline{g})(t)(\varphi(t)\zeta, \eta)\\ &= \int (f\overline{g})(t)\langle \omega_{\zeta,\eta}, \varphi(t)\rangle = \int (f\overline{g})(t)\langle \omega_{\zeta,\eta}, x(t)\rangle\\ &= \int (f\overline{g})(t)(x(t)\zeta, \eta) = \int (x(t) f(t)\zeta, g(t)\eta)\\ &= \left(\left(\int^\oplus x(t)\right)(f\otimes\zeta), g\otimes\eta\right).\end{aligned}$$

Using the density of $C_c(\mathcal{T}) \otimes H$ in $L^2_H(\mathcal{T})$, we derive that $\Psi(\varphi) = \int^\oplus \varphi(t) = \int^\oplus x(t) = x$. By Lemma 7.3, $\mathcal{D}_\mathcal{M}$ is complete, thereby Ψ is a bounded bijection between Banach spaces $L^1_{\mathcal{M}_*}(\mathcal{T})^*$ and $\mathcal{D}_\mathcal{M}$. Using the open mapping theorem, we conclude that Ψ is an isomorphism of Banach spaces. □

Note that $L^1_{\mathcal{M}_*}(\mathcal{T})^*$ possesses the w^*-topology being the dual space of the Banach space $L^1_{\mathcal{M}_*}(\mathcal{T})$. Similarly, $\mathcal{D}_\mathcal{M}$ can be equipped with the w^*-topology $\sigma\left(\mathcal{B}\left(L^2_H(\mathcal{T})\right), \mathcal{B}^1\left(L^2_H(\mathcal{T})\right)\right)$ (see Proposition 2.2).

Lemma 7.5. *The C^*-algebra $\mathcal{D}_\mathcal{M}$ is a von Neumann algebra. In particular, $L^\infty(\mathcal{T})\overline{\otimes}\mathcal{M} \subseteq \mathcal{D}_\mathcal{M}$.*

Proof. Based on Lemma 7.4, we conclude that $\operatorname{ball}\mathcal{D}_\mathcal{M}$ is identified (up to a homeomorphism) with a closed, convex subset B of $n\operatorname{ball}L^1_{\mathcal{M}_*}(\mathcal{T})^*$ for some n. But $n\operatorname{ball}L^1_{\mathcal{M}_*}(\mathcal{T})^*$ is w^*-compact, and B is w^*-closed (being norm-closed) subset of $n\operatorname{ball}L^1_{\mathcal{M}_*}(\mathcal{T})^*$, therefore it is a w^*-compact subset, and $\Psi(B) = \operatorname{ball}\mathcal{D}_\mathcal{M}$. Note that $C_c(\mathcal{T}) \otimes H$ is the linear span of $S = \{f\otimes\zeta : f \in C_c(\mathcal{T}), \zeta \in H\}$, and it is dense in $L^2_H(\mathcal{T})$ (see Corollary 6.4). Furthermore the w^*-topology on $\operatorname{ball}\mathcal{D}_\mathcal{M}$ is reduced to WOT-topology, which can be given by means of the seminorms $p_{f\otimes\zeta,g\otimes\eta}$, $f\otimes\zeta, g\otimes\eta \in S$ thanks to Lemma 2.2. Similarly, by Lemma 2.1, the w^*-topology on $n\operatorname{ball}L^1_{\mathcal{M}_*}(\mathcal{T})^*$ is given by means of the seminorms $p_{h\otimes a}$, $h \in C_c(\mathcal{T})$, $a \in \mathcal{M}_*$ (see also Proposition 6.5). Take $\varphi \in L^1_{\mathcal{M}_*}(\mathcal{T})^*$, which in turn defines the decomposable operator $\int^\oplus \varphi(t) \in \mathcal{D}_\mathcal{M}$ by Lemma 7.4. But

$$\left\langle \omega_{f\otimes\zeta,g\otimes\eta}, \int^\oplus \varphi(t)\right\rangle = \left(\left(\int^\oplus \varphi(t)\right) f\otimes\zeta, g\otimes\eta\right) = \int (\varphi(t) f(t)\zeta, g(t)\eta)$$

$$= \int (f\overline{g})(t)(\varphi(t)\zeta,\eta) = \int (f\overline{g})(t)\langle\omega_{\zeta,\eta},\varphi(t)\rangle = \langle(f\overline{g})\otimes\omega_{\zeta,\eta},\varphi\rangle,$$

which means that $p_{f\otimes\zeta,g\otimes\eta}(\Psi(\varphi)) = p_{f\overline{g}\otimes\omega_{\zeta,\eta}}(\varphi)$, that is, Ψ is weakly continuous on every bounded subset. In particular, Ψ implements a w^*-homeomorphism of B onto $\operatorname{ball}\mathcal{D}_{\mathcal{M}}$. Hence $\mathcal{D}_{\mathcal{M}}\cap\operatorname{ball}\mathcal{B}\left(L^2_H(\mathcal{T})\right)$ is w^*-compact. Using Krein-Smulian theorem, we derive that $\mathcal{D}_{\mathcal{M}}$ is w^*-closed (or WOT-closed), $*$-algebra.

Finally, we have seen above that $L^\infty(\mathcal{T})\otimes\mathcal{M}\subseteq\mathcal{D}_{\mathcal{M}}$. But $L^\infty(\mathcal{T})\overline{\otimes}\mathcal{M}$ is the von Neumann algebra generated by $L^\infty(\mathcal{T})\otimes\mathcal{M}$, and $\mathcal{D}_{\mathcal{M}}$ is WOT-closed, therefore $L^\infty(\mathcal{T})\overline{\otimes}\mathcal{M}\subseteq\mathcal{D}_{\mathcal{M}}$. □

Alternatively, one can use Sakai theorem to have a conclusion that $\mathcal{D}_{\mathcal{M}}$ is a von Neumann algebra being the dual space (see Lemma 7.4).

Theorem 7.1. *Let $\mathcal{M}\subseteq\mathcal{B}(H)$ be a von Neumann algebra.*
Then $L^\infty(\mathcal{T})\overline{\otimes}\mathcal{M}=\mathcal{D}_{\mathcal{M}}$.

Proof. We have just seen in Lemma 7.5, that $L^\infty(\mathcal{T})\overline{\otimes}\mathcal{M}\subseteq\mathcal{D}_{\mathcal{M}}$. In particular, the restriction mapping of functionals generates the bounded linear mapping $(\mathcal{D}_{\mathcal{M}})_*\to(L^\infty(\mathcal{T})\overline{\otimes}\mathcal{M})_*$ onto. Prove that it is an isomorphism. By Lemma 7.4, $L^1_{\mathcal{M}_*}(\mathcal{T})=(\mathcal{D}_{\mathcal{M}})_*$. Take $y(\cdot)\in L^1_{\mathcal{M}_*}(\mathcal{T})$ from the kernel of the previous mapping of preduals. Then $\langle y(\cdot),M_h\otimes x\rangle=0$ for all $h\in L^\infty(\mathcal{T})$ and $x\in\mathcal{M}$. Recall that $M_h\otimes x\in\mathcal{D}_{\mathcal{M}}$ with its $\mathcal{M}$-valued measurable function $h(\cdot)x$, that is, $M_h\otimes x=\int^\oplus h(t)x$. Moreover,

$$\int\langle y(t),x\rangle h(t)=\int\langle y(t),h(t)x\rangle=\left\langle y(\cdot),\int^\oplus h(t)x\right\rangle=0$$

and $\langle y(\cdot),x\rangle\in\mathfrak{L}(\mathcal{T})$, $\int|\langle y(t),x\rangle|\leq\int\|y(t)\|\|x\|=\|y(\cdot)\|_1\|x\|$, that is, $\langle y(\cdot),x\rangle\in L^1(\mathcal{T})$. Thus for every $h\in L^\infty(\mathcal{T})=L^1(\mathcal{T})^*$ we have $\langle\langle y(\cdot),x\rangle,h\rangle=\int\langle y(t),x\rangle h(t)=0$, which means that $\langle y(\cdot),x\rangle=0$. Further, take a compact subset $K\subseteq\mathcal{T}$. By Proposition 6.3, there exists a measurable subset $K_\infty\subseteq K$ such that $\mu(K-K_\infty)=0$ and $y(K_\infty)$ is separable in $\mathcal{M}_*$. Put $Y=\langle y(K_\infty)\rangle$, which is a closed separable subspace of $\mathcal{M}_*$, and $X=Y^*$. Then $\operatorname{ball}X$ is w^*-separable (see [24, E 2.5.3]). Take a w^*-dense sequence $\{x_n\}$ in $\operatorname{ball}X$. Then $\|y(t)\|=\sup|\langle y(t),\operatorname{ball}X\rangle|=\sup|\langle y(t),\{x_n\}\rangle|$. But for every n we have $\langle y(t),x_n\rangle=0$ for almost all $t\in K_\infty$. It follows that $\|y(t)\|=0$ for almost all $t\in K_\infty$, which in turn implies that $\|y(t)\|=0$ for almost all $t\in K$. For $\varepsilon>0$ one can find a compact subset $K\subseteq\mathcal{T}$ such that $\|y(\cdot)\|_1-\varepsilon\leq\int_K\|y(t)\|=0$ (see to the proof of Proposition 6.4). Hence $y(\cdot)=0$. Thus $(\mathcal{D}_{\mathcal{M}})_*=(L^\infty(\mathcal{T})\overline{\otimes}\mathcal{M})_*$ up to an isomorphism, therefore $L^\infty(\mathcal{T})\overline{\otimes}\mathcal{M}=\mathcal{D}_{\mathcal{M}}$. □

Corollary 7.3. *Let H be a Hilbert space with its basis $(\epsilon_i)_{i\in I}$. Then $\mathcal{D}_H=L^\infty(\mathcal{T})\overline{\otimes}\mathcal{B}(H)=M_I(L^\infty(\mathcal{T}))$. In particular, for every bounded measurable function $x(\cdot):\mathcal{T}\to\mathcal{B}(H)$ we have*

$$\int^\oplus x(t)=SOT\text{-}\sum_{i,j\in I}(x(\cdot)\epsilon_j,\epsilon_i)\otimes u_{ij},$$

where $u_{ij}=\epsilon_i\odot\epsilon_j\in\mathcal{B}_f(H)$ and $(x(\cdot)\epsilon_j,\epsilon_i)\in L^\infty(\mathcal{T})$ for all $i,j\in I$.

Proof. One suffices to apply Theorem 7.1 and Proposition 3.2. Take $x=\int^\oplus x(t)\in\mathcal{D}_H$. Using (3.2), we derive that $x=$SOT-$\sum_{i,j}x_{ij}\otimes u_{ij}$, where $x_{ij}=U_i^*xU_j\in$

$L^{\infty}(\mathcal{T}) \subseteq \mathcal{B}\left(L^2(\mathcal{T})\right)$. Note that $x_{ij}\zeta(\cdot) = U_i^* x(\cdot)\zeta(\cdot)\epsilon_j = \zeta(\cdot)U_i^* x(\cdot)\epsilon_j = \zeta(\cdot)U_i^* \sum_k (x(\cdot)\epsilon_j, \epsilon_k)\epsilon_k = \zeta(\cdot)(x(\cdot)\epsilon_j, \epsilon_i)$ for all $\zeta(\cdot) \in L^2(\mathcal{T})$. In particular, $x_{ij} = (x(\cdot)\epsilon_j, \epsilon_i)$ for all $i, j \in I$. □

8. Multinormed L^{∞}-algebras

In this section we develop a multinormed version of L^{∞}-algebras.

8.1. Locally essentially bounded functions. In this subsection we introduce multinormed W^*-algebra $L^{\infty}(\mathcal{T})_{\mathcal{E}}$ as a central completion of the von Neumann algebra $L^{\infty}(\mathcal{T})$ determined by means of a measurable covering $\mathcal{E}$ of $\mathcal{T}$.

Let $\mathcal{T}$ be a locally compact equipped with a positive Radon integral $\int$: $C_c(\mathcal{T}) \to \mathbb{C}$, and let μ be the related Radon measure on $\mathcal{T}$. Consider a family $\mathcal{E} = (E_\iota)_{\iota\in\Xi}$ of measurable subsets in $\mathcal{T}$ whose union is almost covering $\mathcal{T}$, that is, $\mathcal{E} \subseteq \mathfrak{M}$ and $\mu(\mathcal{T} - \cup_\iota E_\iota) = 0$. As above the set of all finite subsets of Ξ is denoted by Λ. Taking finite unions $E_\alpha = \cup_{\iota\in\alpha} E_\iota$ from the family $\mathcal{E}$ we can always assume that $\mathcal{E}$ is upward filtered. We say that $\mathcal{E}$ is *a measurable covering of* $\mathcal{T}$ if for every nonempty compact subset $C \subseteq \mathcal{T}$ we have $C \cap E_\iota = \varnothing$ for all ι but at most countable many of them. In this case, $C = (\cup_{n=1}^{\infty} C \cap E_{\alpha_n}) \cup (C \cap (\mathcal{T} - \cup_\iota E_\iota))$ for a certain (increasing) countable subset $\{\alpha_n\} \subseteq \Lambda$. Since $C \cap (\mathcal{T} - \cup_\iota E_\iota)$ is a null subset, it follows that $\mu(C) = \lim_n \mu(C \cap E_{\alpha_n})$. Every countable almost covering $\mathcal{E}$ is a measurable covering. A typical example of a measurable covering in a locally compact, σ-compact topological spaces is a countable family $(C_n)_n$ of open, relatively compact subsets C_n such that $C_n^- \subseteq C_{n+1}$ for every n (see [24, 1.7.7]).

Now let $\mathcal{E}$ be a measurable covering of $\mathcal{T}$. For every ι define the seminorm $\|x\|_{\iota,\infty} = \|x \cdot e_\iota\|_\infty$ on $L^{\infty}(\mathcal{T})$, where $e_\iota = [E_\iota]$. We obtain the family $\left\{\|\cdot\|_{\iota,\infty}\right\}$ of C^*-seminorms on $L^{\infty}(\mathcal{T})$. Note that $e_\alpha = \vee_{\iota\in\alpha} e_\iota = [E_\alpha]$ for every $\alpha \in \Lambda$. Thus $\mathcal{E}$ is identified with a domain in the von Neumann algebra $\mathcal{M} = L^{\infty}(\mathcal{T})$ with the predual $\mathcal{M}_* = L^1(\mathcal{T})$.

Lemma 8.1. *If* $f \in \mathfrak{L}^1(\mathcal{T})$ *then* $(fe_\alpha)_{\alpha\in\Lambda} \subseteq \mathfrak{L}^1(\mathcal{T})$ *and* $\int f = \lim_\alpha \int fe_\alpha$.

Proof. We can assume that $f \in \mathfrak{L}^1(\mathcal{T})_+$. Since $(fe_\alpha)_{\alpha\in\Lambda} \subseteq \mathfrak{L}(\mathcal{T})_+$ and $fe_\alpha \leq f$ for all α, we conclude that $(fe_\alpha)_{\alpha\in\Lambda} \subseteq \mathfrak{L}^1(\mathcal{T})_+$. Take $\varepsilon > 0$. As in the proof of Proposition 6.4, there is a function $g \in C_c(\mathcal{T})_+$ such that $\int |f-g| \leq \varepsilon/3$, which in turn implies that $\int_{\mathcal{T}-K} f \leq \int_{\mathcal{T}-K} |f-g| + \int_K |f-g| \leq \varepsilon/3$ for $K = \operatorname{supp}(g)$. We assume that $g \neq 0$. Further, put $K_\alpha = K \cap E_\alpha \in \mathfrak{M}^1$, $\alpha \in \Lambda$. By its very definition, $K_\alpha = \varnothing$ for all α but at most countable family K_{α_n}, $n \in \mathbb{N}$. In this case, $\mu(K) = \lim_n \mu(K_{\alpha_n})$, and $\mu(K) - \mu(K_{\alpha_n}) \leq \varepsilon/(3\|g\|_\infty)$ for all $n \geq N$. For every $\alpha \geq \alpha_N$ we have $K_{\alpha_N} \subseteq K_\alpha$ and

$$\begin{aligned}\int f - fe_\alpha &= \int_{\mathcal{T}-E_\alpha} f \leq \int_{K-K_\alpha} f + \int_{\mathcal{T}-K} f \leq \int_{K-K_\alpha} |f-g| + \int_{K-K_\alpha} g + \varepsilon/3 \\ &\leq \int_{K-K_\alpha} g + \int |f-g| + \varepsilon/3 \leq \int_{K-K_\alpha} g + 2\varepsilon/3 \\ &\leq \|g\|_\infty \mu(K-K_\alpha) + 2\varepsilon/3 \leq \|g\|_\infty \mu(K-K_{\alpha_N}) + 2\varepsilon/3 \leq \varepsilon,\end{aligned}$$

that is, $\int f - fe_\alpha \leq \varepsilon$ for all $\alpha \geq \alpha_N$. Whence $\int f = \lim_\alpha \int fe_\alpha$. □

The following assertion follows from Corollary 4.1.

Lemma 8.2. *For every $x \in L^{\infty}(\mathcal{T})$ the equality $\|x\|_{\infty} = \sup\left\{\|x\|_{\iota,\infty} : \iota \in \Lambda\right\}$ holds, that is, $\|\cdot\|_{\infty} = \sup\left\{\|\cdot\|_{\iota,\infty} : \iota \in \Lambda\right\}$ on $L^{\infty}(\mathcal{T})$.*

Proof. Take $x \in \mathfrak{L}^{\infty}(\mathcal{T})$ and put $a = \sup\left\{\|x\|_{\iota,\infty} : \iota \in \Lambda\right\}$, $x_{\iota} = x \cdot e_{\iota}$, $n \in \mathbb{N}$. It is obvious that $\|x\|_{\iota,\infty} \leq \|x\|_{\infty}$ for all ι, that is, $a \leq \|f\|_{\infty}$. Since $L^{\infty}(\mathcal{T}) = L^{1}(\mathcal{T})^{*}$, it follows that $\|x\|_{\infty} = \sup\left|\left\langle \operatorname{ball} L^{1}(\mathcal{T}), x\right\rangle\right|$. Using Lemma 8.1 and Lemma 4.2, we derive that

$$\|x\|_{\infty} - \varepsilon \leq |\langle f, x\rangle| - \varepsilon/2 = \left|\int fx\right| - \varepsilon/2 = \lim_{\beta}\left|\int fxe_{\beta}\right| - \varepsilon/2 \leq \left|\int fxe_{\alpha}\right|$$
$$\leq \|xe_{\alpha}\|_{\infty}\|f\|_{1} \leq \|x\|_{\alpha,\infty} = \sup\left\{\|x\|_{\iota,\infty} : \iota \in \alpha\right\} \leq a$$

for some $f \in \operatorname{ball} L^{1}(\mathcal{T})$ and $\alpha \in \Lambda$. Consequently, $\|x\|_{\infty} = a$. ☐

Based on Lemma 8.2, we obtain that the completion of $\left(L^{\infty}(\mathcal{T}), \left\{\|\cdot\|_{\iota,\infty}\right\}\right)$, which is a multinormed W^{*}-algebra (see Subsection 5.1) denoted by $L^{\infty}(\mathcal{T})_{\mathcal{E}}$.

Theorem 8.1. *The algebra $L^{\infty}(\mathcal{T})_{\mathcal{E}}$ is represented by means of $\mathcal{E}$-locally essentially bounded functions, that is, those functions $f \in \mathfrak{L}(\mathcal{T})$ such that $\operatorname{esssup}|f|E_{\iota}| < \infty$ for all ι. Moreover, $\mathfrak{b}(L^{\infty}(\mathcal{T})_{\mathcal{E}}) = L^{\infty}(\mathcal{T})$.*

Proof. First note that $L^{\infty}(\mathcal{T}) / \ker\|\cdot\|_{\alpha,\infty} = L^{\infty}(\mathcal{T})e_{\alpha}$. The connecting W^{*}-homomorphisms $\varphi_{\alpha\beta} : L^{\infty}(\mathcal{T})e_{\beta} \to L^{\infty}(\mathcal{T})e_{\alpha}$ are reduced to the restriction mappings for all $\alpha \leq \beta$, and $L^{\infty}(\mathcal{T})_{\mathcal{E}} = \varprojlim\{L^{\infty}(\mathcal{T})e_{\alpha}, \varphi_{\alpha\beta}\}$. Take $f = (f_{\alpha})_{\alpha} \in L^{\infty}(\mathcal{T})_{\mathcal{E}}$ with $f_{\alpha} = f_{\alpha}e_{\alpha}$ for all α. Fix a lifting $\rho : L^{\infty}(\mathcal{T}) \to \mathfrak{L}^{\infty}(\mathcal{T})$, and put $e_{\alpha}' = \rho(e_{\alpha})$, $f_{\alpha}' = \rho(f_{\alpha})$, where $e_{\alpha}' = [E_{\alpha}']$ for some $E_{\alpha}' \in \mathfrak{M}$. Since $e_{\alpha} = e_{\alpha}'$ in $L^{\infty}(\mathcal{T})$, it follows that $E_{\alpha} - N_{\alpha} = E_{\alpha}' - N_{\alpha}$ for a certain null subset N_{α}. For $\alpha \leq \beta$ we have $e_{\beta}'e_{\alpha}' = e_{\alpha}'$ or $E_{\alpha}' \subseteq E_{\beta}'$, and $f_{\beta}'e_{\alpha}' = \rho(f_{\beta})e_{\alpha}' = \rho(f_{\beta}e_{\alpha}) = \rho(f_{\alpha}e_{\alpha}) = \rho(f_{\alpha}) = f_{\alpha}'$. In particular, there is a well defined function f' on $\mathcal{T}$ such that $f'(t) = \lim_{\alpha} f_{\alpha}'(t)$, $t \in \mathcal{T}$. Take a compact subset $K \subseteq \mathcal{T}$. For every α we have $K \cap E_{\alpha} = (K \cap E_{\alpha} \cap N_{\alpha}) \cup (K \cap (E_{\alpha} - N_{\alpha})) = (K \cap E_{\alpha} \cap N_{\alpha}) \cup (K \cap E_{\alpha}' - N_{\alpha})$ and $K \cap E_{\alpha}' = (K \cap E_{\alpha}' \cap N_{\alpha}) \cup (K \cap E_{\alpha}' - N_{\alpha})$, that is, $K \cap E_{\alpha}$ and $K \cap E_{\alpha}'$ can be separated up to a null set. But $K = \cup_{n} K \cap E_{\alpha_{n}}$ for an increasing sequence $\{\alpha_{n}\} \subseteq \Lambda$, therefore K and $K' = \cup_{n} K \cap E_{\alpha_{n}}'$ are separated up to a null set. In particular, $f'(t) = \lim_{n} f_{\alpha_{n}}'(t)$ for almost all $t \in K$. Actually, $f'(t) = \lim_{n} f_{\alpha_{n}}'(t)$ for all $t \in K'$. Note that $\lim_{n} f_{\alpha_{n}}' \in \mathfrak{L}(\mathcal{T})$, and K' may not be a compact subset. For $\varepsilon > 0$ choose a compact subset $K'' \subseteq K'$ such that $\mu(K' - K'') \leq \varepsilon/2$ (the inner regularity of μ). By Proposition 6.2, one can find a compact subset $K_{0} \subseteq K''$ such that $\mu(K'' - K_{0}) \leq \varepsilon/2$ and $f'|K_{0} = \lim_{n} f_{\alpha_{n}}'|K_{0} \in C(K_{0})$. But $\mu(K - K_{0}) = \mu(K - K') + \mu(K' - K'') + \mu(K'' - K_{0}) \leq \varepsilon$. Appealing again to Proposition 6.2, we obtain that $f' \in \mathfrak{L}(\mathcal{T})$. Moreover, $\|f'\|_{\alpha,\infty} = \|f' \cdot e_{\alpha}\|_{\infty} = \|f' \cdot e_{\alpha}'\|_{\infty} = \|\rho(f \cdot e_{\alpha})\|_{\infty} = \|f \cdot e_{\alpha}\|_{\infty} = \|f\|_{\alpha,\infty}$ for all α. Conversely, every $\mathcal{E}$-locally essentially bounded function f defines the element $(fe_{\alpha})_{\alpha} \in L^{\infty}(\mathcal{T})_{\mathcal{E}}$. Finally, using Lemma 8.2, we conclude that $\mathfrak{b}(L^{\infty}(\mathcal{T})_{\mathcal{E}}) = L^{\infty}(\mathcal{T})$. ☐

Now we describe the bornological predual $L^{\infty}(\mathcal{T})_{*\mathcal{E}}$ of the multinormed W^*-algebra $L^{\infty}(\mathcal{T})_{\mathcal{E}}$. By its very definition, $L^{\infty}(\mathcal{T})_{*\mathcal{E}} = \sum_{\iota} L^1(\mathcal{T}) e_{\iota} = \cup_{\alpha} L^1(\mathcal{T}) e_{\alpha}$ (see Subsection 4.2). The latter ℓ^1-normed space is denoted by $L^1(\mathcal{T})_{\mathcal{E}}$.

Lemma 8.3. *For every projection $e = [E] \in L^{\infty}(\mathcal{T})$ the equality*

$$L^1(\mathcal{T}) e = \left\{ g \in L^1(\mathcal{T}) : \|g\|_1 = \int_E |g| \right\}$$

holds. Moreover, $\|fe\|_{\infty} = \sup \left\{ \left| \int gf \right| : \int_E |g| = 1 \right\}$ for every $f \in L^{\infty}(\mathcal{T})$.

Proof. Take $g \in L^1(\mathcal{T}) e$. By Lemma 3.7, $L^1(\mathcal{T}) e = ((1-e) L^{\infty}(\mathcal{T}))^{\perp}$, that is, $0 = \langle g, (1-e) f \rangle$ or $\langle g, f \rangle = \int_E gf$ for all $f \in L^{\infty}(\mathcal{T})$. Then

$$\begin{aligned} \int_E |g| \leq \|g\|_1 &= \sup |\langle g, \operatorname{ball} L^{\infty}(\mathcal{T}) \rangle| = \sup \left| \int_E g \operatorname{ball} L^{\infty}(\mathcal{T}) \right| \\ &\leq \sup \int_E |g| \, \|\operatorname{ball} L^{\infty}(\mathcal{T})\|_{\infty} \leq \int_E |g|, \end{aligned}$$

that is, $\|g\|_1 = \int_E |g|$. Conversely, if $\|g\|_1 = \int_E |g|$ for some $g \in L^1(\mathcal{T})$, then $|\langle g, f \rangle| = \left| \int gf \right| = \left| \int gf (1-e) \right| = \left| \int gf [\mathcal{T} \backslash E] \right| \leq \|f\|_{\infty} \int_{\mathcal{T} \backslash E} |g| = 0$ for all $f \in (1-e) L^{\infty}(\mathcal{T})$, that is, $g \in ((1-e) L^{\infty}(\mathcal{T}))^{\perp} = L^1(\mathcal{T}) e$. Finally, using Corollary 3.14, we have $(eL^{\infty}(\mathcal{T}))_* = L^1(\mathcal{T}) e$ up to an isometric isomorphism. It follows that $\|fe\|_{\infty} = \left\| f | L^1(\mathcal{T}) e \right\| = \sup \left| \langle \operatorname{ball} L^1(\mathcal{T}) e, f \rangle \right| = \sup \left\{ \left| \int fg \right| : \int_E |g| = 1 \right\}$. □

Theorem 8.2. *Let $\mathcal{E}$ be a measurable covering of $\mathcal{T}$. Then $L^{\infty}(\mathcal{T})_{*\mathcal{E}}$ consists of those $g \in L^1(\mathcal{T})$ such that*

$$\|g\|_1 = \inf \left\{ \sum_{\iota \in \beta} \int_{E_{\iota}} |g_{\iota}| : g = \sum_{\iota \in \beta} g_{\iota} e_{\iota} \right\} = \int_{E_{\alpha}} |g|$$

for some $\alpha \in \Lambda$.

Proof. The first equality follows from Theorem 8.1, whereas the second one follows from Lemma 8.3. □

Thus $L^1(\mathcal{T})_{\mathcal{E}}$ is represented by means of those $f \in \mathfrak{L}^1(\mathcal{T})$ such that $\int_{\mathcal{T} \backslash E_{\alpha}} |g| = 0$ for some α. Note that $L^1(\mathcal{T})_{\mathcal{E}}$ is an ℓ^1-normed space with its locally quotient mapping $\pi_{\mathcal{T}} : \bigoplus_{\iota} L^1(E_{\iota}) \to L^1(\mathcal{T})_{\mathcal{E}}$ acting as an identity mapping over each $L^1(E_{\iota})$. In particular, it possesses a natural bornology $\left\{ \operatorname{ball} L^1(\mathcal{T}) e_{\alpha} \right\}$ and the bornological dual $L^1(\mathcal{T})'_{\mathcal{E}}$ is reduced to $L^{\infty}(\mathcal{T})_{\mathcal{E}}$ [11], that is, $L^1(\mathcal{T})_{\mathcal{E}} = L^{\infty}(\mathcal{T})_{*\mathcal{E}}$. If the covering is reduced to the trivial one $\mathcal{E} = (\mathcal{T})$ then $L^{\infty}(\mathcal{T})_{\mathcal{E}} = L^{\infty}(\mathcal{T})$, $L^1(\mathcal{T})_{\mathcal{E}} = L^1(\mathcal{T})$ and we obtain the classical result $L^1(\mathcal{T})^* = L^{\infty}(\mathcal{T})$. Confirm that $L^1(\mathcal{T})_{\mathcal{E}}$ is dense in $L^1(\mathcal{T})$ by Lemma 8.1 (or Lebesgue's convergence theorem). Thus locally essentially bounded functions generate a bit narrow class of locally integrable functions (see [31]). In the case of the relatively compact covering $\mathcal{E} = (C_n)_n$ we obtain that $L^1(\mathcal{T})_{\mathcal{E}} \neq L^1_{\text{loc}}(\mathcal{T})$. The description of the multinormed completions of $\mathcal{D}_{\mathcal{M}}$ were obtained in [11].

References

[1] J.-P. Antoine, A. Inoue, C. Trapani, Partial $*$-algebras their operator realizations, Math Appl. **533**, Kluwer Acad. Dordrecht (2003).

[2] F. Bagarello, A. Inoue, C. Trapani, *Bicommutant of reduced unbounded operators*, Proc. AMS., **137** (2009), 3709-3716.

[3] D. P. Blecher, *The standart dual of an operator space,* Pacific J. Math., **152** (2) (1992) 15-30.

[4] A. A. Dosiev, *Local operator spaces, unbounded operators and multinormed C^*-algebras,* J. Funct. Anal., **255** (2008) 1724–1760.

[5] A. A. Dosiev, *Quantized moment problem,* Comptes Rendus Math. **344** (1) (2007) 627-630.

[6] A. A. Dosi, *Local operator algebras, fractional positivity and quantum moment problem,* Trans. AMS, **363** (2) (2011) 801-856.

[7] A. A. Dosi, *Noncommutative Mackey theorem,* Inter. J. Math. **22** (4) (2011) 535-544.

[8] A. A. Dosi, *Quotients of quantum bornological spaces*, Taiwan. J. Math. **15** (2011) (to appear).

[9] A. A. Dosi, *Quantum duality, unbounded operators and inductive limits,* J. Mathematical Physics **51** (6) (2010) 1-43.

[10] A. A. Dosi, *Multinormed W^*-algebra and unbounded operators,* Proc. AMS, **140** (2012) 4187-4202.

[11] A. A. Dosi, *Multinormed W^*-algebra of type I, Preprint (2016).*

[12] E. G. Effros, C. Webster, *Operator analogues of locally convex spaces,* Operator Algebras and Applications (Samos 1996), (NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 495), Kluwer (1997).

[13] E. G. Effros, S. Winkler, *Matrix convexity: operator analogues of the Bilpolar and Hanh-Banach theorems,* J. Funct. Anal., **144** (1997) 117-152.

[14] M. Fragoulopoulou, *On locally W^*-algebras,* Yokohama Math. J., **34** (1986) 35-51.

[15] M. Fragoulopoulou, Topological algebras with involution, Math. Stud. **200** (2005).

[16] R. Harti, G. Lukacs, *Bounded and Unitary elements in pro-C^*-algebras,* Appl. Categor. Struct. **14** (2006) 151-164.

[17] A. Ya. Helemskii, Banach and polynormed algebras: general theory, representations, homology, Nauka, Moscow, 1989.

[18] H. Hogbe-Nlend, *Bornologies and functional analysis: introductory course on the theory of duality topology-bornology and its use in functional analysis,* Elsevier (1977).

[19] A. Inoue, *Locally C^*-algebras,* Memoi. Fac. Scien. Kyushu Univ. Ser. A, **25** (2) (1971) 197-235.

[20] A. Inoue, Tomita-Takesaki Theory in Algebras of Unbounded Operators, Lect. Notes Math. **1699** (1998).

[21] M. Joiţa, *Locally von Neumann algebras I,* Bull. Math. Soc. Sci. Math. Roumanie, **42** (90) (1999) 51-64.

[22] M. Joiţa, *Locally von Neumann algebras II,* Rendconti d. Cir. Matem. di Palermo LI (II) (2002) 83-94.

[23] G. J. Murphy, C^*-algebras and operator theory, Acad. Press, INC. (1990).

[24] G. Pederson, Analysis Now, Graduate Text in Math. 1988.

[25] N. C. Phillips, *Inverse limits of C^*-algebras,* J. Operator Theory, **19** (1988) 159-195.

[26] S. Sakai, C^*-algebras and W^*-algebras, Springer-Verlag, 1971.

[27] H. Schaefer, Topological vector spaces, Springer-Verlag, 1970.

[28] K. Schmudgen, *Über LMC^*-Algebren,* Math. Nachr., **68** (1975) 167-182.

[29] K. Schmudgen, Unbounded operator algebras and representation theory, Birkhauser Verlag (1990).

[30] M. Takesaki, Theory of operator algebras I, Encycl. Math. Scien. **124** Springer (2002).
[31] R. Welland, *Generalized local integrability,* Proc. AMS,**13** (4) (1962), 635-639.

Anar Dosi
Middle East Technical University Northern Cyprus Campus, Guzelyurt, KKTC, Mersin 10, Turkey
E-mail address: dosiev@yahoo.com

Received: June 10, 2016; Accepted: November 23, 2016