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# HIGHER ORDER CONDITIONS IN NONDIFFERENTIABLE PROGRAMMING PROBLEMS

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**Abstract**. In the paper, using the classes of $S-(\alpha,\beta,\nu,\delta,\omega)$ and $S-(\beta,\delta)$ locally Lipschitz mappings at the point, higher order necessary conditions of the extremum are received for extreme problems in the presence of restrictions.

## 1. Introduction

Research of smooth extreme problems with restriction (the problem on conditional extremum) is based on Lagrange's principle offered by J.L.Lagrange at the close of the 18th century. Strict justification of the Lagrange principle for the wide class of extreme problems demanded serious efforts of many mathematicians and was generally finished in the second half of the XX century. The convex extreme problem with restriction is well studied in the books [4, 9]. Nonsmooth extreme problems with restriction are considered in the book [3], and in classes of locally Lipschitz functions necessary conditions of the extremum of the first order are received. In the present work necessary conditions of the extremum of any order for nonsmooth and, in particular, for smooth extreme problems in the presence of restrictions are proved. Let's note that when receiving necessary conditions of the extremum, the essential role has the classes of $S-(\alpha,\beta,\nu,\delta,\omega)$ and $S-(\beta,\delta)$ locally Lipschitz mappings at the point (see [5]-[8]).

The work consists of three sections. In Section 2, a number of properties of $S-(\alpha,\beta,\nu,\delta,\omega)$ and $S-(\beta,\delta)$ locally Lipschitz mappings at the point are studied. In Section 3, a number of properties of the approximate cone is studied. In Section 4, by means of Lagrange's function and the approximate cone necessary conditions of the extremum of higher order in the presence of restrictions are received. Let's note that in Section 4 the necessary condition of the extremum is received where the regularity at the minimum point is not required. In particular, from here follows necessary conditions of the extremum of second order for the classical problem on the conditional extremum and for the mathematical programming problem(see [1], p.237).

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## 2. Class of locally Lipschitz functions of higher order

Let $X$ and $Y$ be real Banach spaces, $F : X \to Y$, $S : X \to Y$, f: X→R, $\varphi : X \to R$, $\alpha > 0$, $\nu > 0$, $\beta \geq \alpha\nu$, $\delta > 0$, $K > 0$, $o : R_+ \to R_+$ and $\omega : R_+ \to R_+$, where $o(0) = 0$, $\omega(0) = 0$, $R_+ = [0, +\infty)$. Let's put $B = \{y \in X : \ \|y\| \leq 1\}$, $B(x, \delta) = \{y \in X : \ \|y - x\| \leq \delta\}$.

The mapping $F$ is said to be $S-(\alpha, \beta, \nu, \delta, \omega)$ locally Lipschitz with the constant $K$ at the point $\bar{x} \in X$, if $F$ satisfies the condition

$$\|F(\bar{x} + x + y) - F(\bar{x} + x) - S(x + y) + S(x)\|$$
$$\leq K \|y\|^{\nu} \left(\|x\|^{\beta - \alpha\nu} + \|y\|^{\frac{\beta - \alpha\nu}{\alpha}}\right) + \omega(\|x\|)$$

at $x, y \in \delta B$. If $\omega(t) \equiv 0$, then the mapping $F$ is said to be $S - (\alpha, \beta, \nu, \delta)$ locally Lipschitz with the constant $K$ at the point $\bar{x}$ (see [7]). If $\omega(t) \equiv 0$ and $S(x) \equiv 0$, then the mapping $F$ is said to be $(\alpha, \beta, \nu, \delta)$ locally Lipschitz with the constant $K$ at the point $\bar{x}$.

If there is a function $o : R_+ \to R_+$, where $\lim\limits_{t \downarrow 0} \frac{o(t)}{t} = o$, such that $\omega(\|x\|) = o(\|x\|^{\beta})$, then $S - (\alpha, \beta, \nu, \delta, \omega)$ locally Lipschitz with the constant $K$ at the point $\bar{x}$ mapping $F$ we call $S - (\alpha, \beta, \nu, \delta, o(\beta))$ locally Lipschitz with the constant $K$ at the point $\bar{x}$.

Let's consider generalization of the class $S - (\alpha, \beta, \nu, \delta, \omega)$ locally Lipschitz mapping with the constant $K$ at the point $\bar{x}$. Let $\nu > 0$, $\mu > 0$, $\sigma > 0$, $K > 0$. We call the mapping $F$ $S - [\mu, \sigma, \nu, \delta, \omega]$ locally Lipschitz with the constant $K$ at the point $\bar{x} \in X$, if $F$ satisfies the condition

$$\|F(\bar{x} + x + y) - F(\bar{x} + x) - S(x + y) + S(x)\| \leq K \|y\|^{\nu} (\|x\|^{\mu} + \|y\|^{\sigma}) + \omega(\|x\|)$$

at $x, y \in \delta B$. Further we consider that $\mu \geq \beta - \alpha\nu$, $\sigma \geq \frac{\beta - \alpha\nu}{\alpha}$, where $\alpha > 0$, $\nu > 0$, $\beta \geq \nu$.

We call the mapping $F : X \to Y$ satisfying the condition

$$\|F(\bar{x} + y) - F(\bar{x}) - S(y)\| \leq K \|y\|^{\beta}$$

at $y \in \delta B$, $S - (\beta, \delta)$ locally Lipschitz with the constant $K$ at the point $\bar{x}$.

If there is a function $o : R_+ \to R_+$, where $\lim\limits_{t \downarrow 0} \frac{o(t)}{t} = 0$, such that

$$\|F(\bar{x} + y) - F(\bar{x}) - S(y)\| \leq o(\|y\|^{\beta})$$

at $y \in \delta B$, we call the mapping $F : X \to Y$ $S - (o(\beta), \delta)$ locally Lipschitz at the point $\bar{x}$.

If $F(x) = f(x)$, we will put $S(x) = \varphi(x)$.

If the function $f : X \to R$ satisfies the condition $f(\bar{x}+y) - f(\bar{x}) - \varphi(y) \leq K \|y\|^{\beta}$ at $y \in \delta B$, we call the function f $\varphi-(\beta, \delta)$ locally semi-Lipschitz with the constant $K$ at the point $\bar{x}$.

If there is a function $o : R_+ \to R_+$, where $\lim\limits_{t \downarrow 0} \frac{o(t)}{t} = 0$, such that

$$f(\bar{x} + y) - f(\bar{x}) - \varphi(y) \leq o(\|y\|^{\beta})$$

at $y \in \delta B$, we will call the function f $\varphi - (o(\beta), \delta)$ locally semi-Lipschitz at the point $\bar{x}$.

Further we consider that $S(0) = 0$ and $\varphi(0) = 0$ (if $S(0) \neq 0$, it is necessary to consider the function $\tilde{S}(x) = S(x) - S(0)$).

In [5]-[7] $S - (\alpha, \beta, \nu, \delta, \omega)$ locally Lipschitz mapping with the constant $K$ at the point are defined and a number of their properties are studied.

**Lemma 2.1.** *Let $X$ and $Y$ be normed spaces, $x_0 \in X$, the derivative $F'(z)$ in Frechet's sense exist at $z \in x_0 + 2\delta B$, also there be $L > 0$ such that $\|F'(u) - F'(v)\| \leq L\|u - v\|$ at $u, v \in x_0 + 2\delta B$. Then*

$$\left\|F(x_0 + x + y) - F(x_0 + x) - F'(x_0)(y)\right\| \leq L\|y\|(\|x\| + \|y\|)$$

*at $x, y \in \delta B$.*

Lemma 2.1 is proved in [7] (see lemma 4.4.3).

Let $M \subset X$, $x_0 \in M$. Let's put $d_M(x) = \inf\{\|y - x\| : y \in M\}$. It is easily checked that

$$|d_M^n(x_0 + x + y) - d_M^n(x_0 + x)| \leq 2 \cdot 9^{n-1}\|y\|(\|x\|^{n-1} + \|y\|^{n-1})$$

at $x,\ y \in X$.

If the function $f$ satisfies the Lipschitz condition in $\alpha$- neighborhood of the point $x_0$, then by lemma 2.4.2[5] for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(x) \leq f(x_0) + \sup_{p \in \partial f(x_0)} \langle p, x - x_0 \rangle + \varepsilon\|x - x_0\|$$

at $x \in B(x_0, \delta)$, where $\delta \leq \alpha$, $\partial f(x_0)$ is Clark's subdifferential. If $\varepsilon = \frac{1}{n}$, where $n \in N$, then there exists $\delta_n > 0$ such that

$$f(x) \leq f(x_0) + \sup_{p \in \partial f(x_0)} \langle p, x - x_0 \rangle + \frac{1}{n}\|x - x_0\| \tag{2.1}$$

at $x \in B(x_0, \delta_n)$. We consider that $\delta_n > \delta_{n+1}$. Having put $o(t) = \frac{t}{n}$ at $t \in (\delta_{n+1}, \delta_n]$ from (2.1) we have that

$$f(x_0 + z) \leq f(x_0) + \sup_{p \in \partial f(x_0)} \langle p, z \rangle + o(\|z\|)$$

at $z \in B(0, \delta_1)$.

In particular from here we have that if $f : X \to R$ is a continuous convex function, there is the function $o : R_+ \to R_+$, where $\lim\limits_{t \downarrow 0} \frac{o(t)}{t} = 0$ and $\delta > 0$ such that

$$f(x_0 + x) - f(x_0) - f'(x_0; x) \leq o(\|x\|)$$

at $x \in B(0, \delta)$. Then from proposition 4.3.4[2] we have that

$$0 \leq f(x_0 + x) - f(x_0) - f'(x_0; x) \leq o(\|x\|)$$

at $x \in B(0, \delta)$.

Further we will denote $I = \{0, 1, \ldots, m\}$, $J = \{1, \ldots, m\}$.

## 3. Number of properties of the approximate cone

Let $X$ be a Banach space, $C \subset X$, $x_0 \in C$, $o(x,0) = 0$, $o_1(x,0) = 0$, $o_2(x,0) = 0$. Let's put (see [7])

$K_{\alpha,\mu}(x_0; C, \varphi) = \{x \in X : \exists\lambda_x > 0, \exists o_1(x,\lambda) : [0,\lambda_x] \to X, \exists o_2(x,\lambda) : [0,\lambda_x] \to R_+$, where $\frac{o_1(x,\lambda)}{\lambda^\alpha} \to 0$ and $\frac{o_2(x,\lambda)}{\lambda^\mu} \to 0$ at $\lambda \downarrow 0$, that $x_0 + \lambda x + o_1(x,\lambda) \in C$, $\varphi(\lambda x + o_1(x,\lambda)) \le o_2(x,\lambda)$ at $\lambda \in [0,\lambda_x]\}$,

$T_{\alpha,\mu}(x_0; C, \varphi) = \{x \in X : \exists o(x,\lambda) : [0,\lambda_x] \to R_+$, where $\frac{o(x,\lambda)}{\lambda^\mu} \to 0$ at $\lambda \downarrow 0$ and $\exists \lambda_i \downarrow 0$, $\exists \upsilon_i \in X$, where $\frac{1}{\lambda_i^{\alpha-1}} \|\upsilon_i - x\| \to 0$ at $i \to +\infty$, that $x_0 + \lambda_i \upsilon_i \in C$, $\varphi(\lambda_i \upsilon_i) \le o(x,\lambda_i)$ at all $i\}$,

$K_\mu(x_0; C, \varphi) = \{x \in X : \exists\lambda_x > 0, \exists o_2(x,\lambda) : [0,\lambda_x] \to R_+$, where $\frac{o_2(x,\lambda)}{\lambda^\mu} \to 0$ at $\lambda \downarrow 0$, that $x_0 + \lambda x \in C$, $\varphi(\lambda x) \le o_2(x,\lambda)$ at $\lambda \in [0,\lambda_x]\}$,

$K_\alpha(x_0; C) = \{x \in X : \exists\lambda_x > 0, \exists o(x,\lambda) : [0,\lambda_x] \to X$, where $\frac{o(x,\lambda)}{\lambda^\alpha} \to 0$ at $\lambda \downarrow 0$, that $x_0 + \lambda x + o(x,\lambda) \in C$ for $\lambda \in [0,\lambda_x]\}$, $K_C(x_0) \equiv K_1(x_0; C)$.

**Lemma 3.1.** *Let $X$ and $Y$ be real Banach spaces, $C = \{x \in C_1 : f_i(x) \le 0$, $i = 1,\dots,m$, $F(x) = 0\}$, where $f_i : X \to R$, $i \in J$, $f_i(x_0) = 0$ at $i \in J$, $F : X \to Y$, $C_1 \subset X$, the function $f_i$ satisfy $\varphi_i - (\beta,\delta)$ locally Lipschitz condition with the constant $K_i$ at the point $x_0$, $\varphi_i$ and $\varphi$ be a continuous positively homogeneous function of degree $\alpha$, $\beta > \alpha$, $\mu \ge \alpha \ge 1$ and the mapping $F$ be strictly differentiable and regular at the point $x_0 \in C$, i.e. $ImF'(x_0) = Y$. Then $K_{\alpha,\mu}(x_0; C, \varphi) \subset \{x \in K_\alpha(x_0; C_1) : \varphi_i(x) \le 0,\ i \in J,\ \varphi(x) \le 0,\ F'(x_0)x = 0\}$.*

*Proof.* Let $x \in K_{\alpha,\mu}(x_0; C, \varphi)$. Then there exists $\exists\lambda_x > 0$, $\exists o_1(x,\lambda) : [0,\lambda_x] \to X$, $\exists o_2(x,\lambda) : [0,\lambda_x] \to R_+$, *where* $\frac{o_1(x,\lambda)}{\lambda^\alpha} \to 0$, $\frac{o_2(x,\lambda)}{\lambda^\mu} \to 0$ at $\lambda \downarrow 0$, that $f_i(x_0 + \lambda x + o_1(x,\lambda)) \le 0$ at $i \in J$, $F(x_0 + \lambda x + o_1(x,\lambda)) = 0$, $x_0 + \lambda x + o_1(x,\lambda) \in C_1$ and $\varphi(\lambda x + o_1(x,\lambda)) \le o_2(x,\lambda)$ at $\lambda \in [0,\lambda_x]$. Besides let $\lambda_x > 0$ such that $\|\lambda x + o_1(x,\lambda)\| \le \delta$ at $\lambda \in [0,\lambda_x]$. From equality $F(x_0 + \lambda x + o_1(x,\lambda)) = 0$ according to Lyusternik's theorem we have that $F'(x_0)x = 0$. As the functions $f_i$ satisfy $\varphi_i - (\beta,\delta)$ locally Lipschitz condition with the constant $K_i$ at the point $x_0$, then

$$|f_i(x_0 + y) - f_i(x_0) - \varphi_i(y)| \le K_i \|y\|^\beta$$

at $y \in \delta B$. Therefore

$$|f_i(x_0 + \lambda x + o_1(x,\lambda)) - f_i(x_0) - \varphi_i(\lambda x + o_1(x,\lambda))| \le K_i \|\lambda x + o_1(x,\lambda)\|^\beta$$

at $\lambda \in [0,\lambda_x]$. From here we receive

$$\varphi_i(\lambda x + o_1(x,\lambda)) \le f_i(x_0 + \lambda x + o_1(x,\lambda)) + K_i \|\lambda x + o_1(x,\lambda)\|^\beta$$

at $\lambda \in [0,\lambda_x]$. As $\varphi_i$ is a positively homogeneous function of degree $\alpha$, then $\lambda^\alpha \varphi_i(x + \frac{o_1(x,\lambda))}{\lambda^\alpha}) \le K_i \|\lambda x + o_1(x,\lambda)\|^\beta$ at $\lambda \in [0,\lambda_x]$. By the condition, $\varphi_i$ is a continuous function and $\beta > \alpha$, from here we receive $\varphi_i(x) \le 0$ at $i \in J$.

By the condition $\varphi(\lambda x + o_1(x,\lambda)) \le o_2(x,\lambda)$ at $\lambda \in [0,\lambda_x]$, where $\frac{o_2(x,\lambda)}{\lambda^\mu} \to 0$ at $\lambda \downarrow 0$. Therefore $\lambda^\alpha \varphi(x + \frac{o_1(x,\lambda))}{\lambda^\alpha}) \le o_2(x,\lambda)$ at $\lambda \in [0,\lambda_x]$. From here we receive $\varphi(x) \le 0$.

As $x_0 + \lambda x + o_1(x,\lambda) \in C_1$ at $\lambda \in [0,\lambda_x]$, by definition $x \in K_\alpha(x_0; C_1)$. The lemma is proved.

**Lemma 3.2.** *Let $X$ and $Y$ be Banach spaces, $C = \{x \in C_1 : f_i(x) \leq 0,$ $i = 1, \ldots, m,$ $F(x) = 0\}$, where $f_i : X \to R$, $i \in J$, $F : X \to Y$, $C_1 \subset X$, the functions $f_i$ satisfy $\varphi_i - (\beta, \delta)$ locally Lipschitz condition with the constant $K_i$ at the point $x_0$, $\varphi_i$ and $\varphi$ be continuous positively homogeneous functions of degree $\alpha$, the mapping $F$ satisfy $S - (\beta, \delta)$ locally Lipschitz condition with the constant $K$ at the point $x_0$, $S$ be a continuous positively homogeneous mapping of degree $\alpha$, $\beta > \alpha$, $\mu \geq \alpha \geq 1$. Then*

$$K_{\alpha,\mu}(x_0; C, \varphi) \subset \{x \in K_\alpha(x_0; C_1) : \varphi_i(x) \leq 0,\ i \in J,\ \varphi(x) \leq 0,\ \ S(x) = 0\}.$$

*Proof.* Let $x \in K_{\alpha,\mu}(x_0; C, \varphi)$. Then there exists $\exists \lambda_x > 0$, $\exists o_1(x, \lambda) : [0, \lambda_x] \to X$, $\exists o_2(x, \lambda) : [0, \lambda_x] \to R_+$, where $\frac{o_1(x,\lambda)}{\lambda^\alpha} \to 0$, $\frac{o_2(x,\lambda)}{\lambda^\mu} \to 0$ at $\lambda \downarrow 0$, that $f_i(x_0 + \lambda x + o_1(x, \lambda)) \leq 0$ at $i \in J$, $F(x_0 + \lambda x + o_1(x, \lambda)) = 0$, $x_0 + \lambda x + o_1(x, \lambda) \in C_1$ and $\varphi(\lambda x + o_1(x, \lambda)) \leq o_2(x, \lambda)$ at $\lambda \in [0, \lambda_x]$. Besides let $\lambda_x > 0$ such that $\|\lambda x + o_1(x, \lambda)\| \leq \delta$ at $\lambda \in [0, \lambda_x]$.

As the mapping $F : X \to Y$ satisfies $S - (\beta, \delta)$ locally Lipschitz condition with the constant $K$ at the point $x_0$, then

$$\|F(x_0 + y) - F(x_0) - S(y)\| \leq K \|y\|^\beta$$

at $y \in \delta B$. Therefore

$$\|F(x_0 + \lambda x + o_1(x, \lambda)) - F(x_0) - S(\lambda x + o_1(x, \lambda))\| \leq K \|\lambda x + o_1(x, \lambda)\|^\beta$$

at $\lambda \in [0, \lambda_x]$. From here we receive

$$\|S(\lambda x + o_1(x, \lambda))\| \leq \|F(x_0 + \lambda x + o_1(x, \lambda))\| + K \|\lambda x + o_1(x, \lambda)\|^\beta \qquad (3.1)$$

at $\lambda \in [0, \lambda_x]$. As $F(x_0 + \lambda x + o_1(x, \lambda)) = 0$ and $S$ is a continuous positively homogeneous mapping of degree $\alpha$, and $\beta > \alpha$, then from (3.1) we have that

$$\lambda^\alpha \left\| S(x + \frac{o_1(x, \lambda))}{\lambda^\alpha}) \right\| \leq K \|\lambda x + o_1(x, \lambda)\|^\beta$$

at $\lambda \in [0, \lambda_x]$. From here we receive $\|S(x)\| = 0$.

The remaining part of the proof repeats the proof of lemma 3.1. The lemma is proved.

Further we consider that $\alpha_0 \geq 0$, $\alpha_1 \geq 0, \ldots, \alpha_m \geq 0$ and $y^* \in Y^*$ simultaneously are non zero. The case $\alpha_0 = \alpha_1 = \cdots = \alpha_m = 0$ and $y^* = 0$ is trivial.

Vector $v \in X$ is called hypertangent to the set $C$ at the point $x_0 \in C$, if for some $\varepsilon > 0$ and for all $y \in (x_0 + \varepsilon B) \bigcap C$, $\omega \in v + \varepsilon B$, $t \in (0, \varepsilon)$ the ratio $y + t\omega \in C$ is fulfilled(see [3]). The set of all hypertangents to the set $C$ at the point $x_0 \in C$ is denoted by $I_C(x_0)$. If $\bar{x} \in I_C(x_0)$, by definition $I_C(x_0)$ there exists $\alpha_0 > 0$ such that $x_0 + t\bar{x} + r(t) \in C$ at $t \in [0, \alpha_0]$, where $r(t) \to 0$ at $t \downarrow 0$.

**Lemma 3.3.** *Let $X$ and $Y$ be Banach spaces, $C = \{x \in C_1 : f_i(x) \leq 0,$ $i = 1, \ldots, m,$ $F(x) = 0\}$, where $f_i : X \to R$, $i \in J$, $F : X \to Y$ and $C_1 \subset X$, the functions $f_i$ satisfy $\varphi_i - (\beta, \delta)$ locally Lipschitz condition with the constant $K_i$ at the point $x_0$, $\varphi_i$ and be $\varphi$ continuous positively homogeneous functions, $\beta > 1$, $\mu > 0$ and the mapping $F$ a be strictly differentiable and regular at the point $x_0 \in X$, $f_i(x_0) = 0$ at $i \in J$. Then*

$$K_{1,\mu}(x_0; C, \varphi) \supset \{x \in I_{C_1}(x_0) : \varphi_i(x) < 0,\ i \in J,\ \varphi(x) < 0,\ F'(x_0)x = 0\}.$$

*Proof.* Let $x \in \{x \in I_{C_1}(x_0) : \varphi_i(x) < 0,\ i \in J,\ \varphi(x) < 0,\ F'(x_0)x = 0\}$. If $F'(x_0)x = 0$, then according to Lyusternik's theorem there exists $\exists \lambda_x > 0$, $\exists o_1(x,\lambda) : [0,\lambda_x] \to X$, where $\frac{o_1(x,\lambda)}{\lambda} \to 0$ at $\lambda \downarrow 0$, that $F(x_0 + \lambda x + o_1(x,\lambda)) = 0$ at $\lambda \in [0,\lambda_x]$. Besides let $\lambda_x > 0$ such that $\|\lambda x + o_1(x,\lambda)\| \leq \delta$ at $\lambda \in [0,\lambda_x]$. As the functions $f_i$ satisfy $\varphi_i - (\beta,\delta)$ locally Lipschitz condition with the constant $K_i$ at the point $x_0$, then

$$|f_i(x_0 + y) - f_i(x_0) - \varphi_i(y)| \leq K_i \|y\|^\beta$$

at $y \in \delta B$. Therefore

$$|f_i(x_0 + \lambda x + o_1(x,\lambda)) - f_i(x_0) - \varphi_i(\lambda x + o_1(x,\lambda))| \leq K_i \|\lambda x + o_1(x,\lambda)\|^\beta$$

at $\lambda \in [0,\lambda_x]$. From here we receive that

$$f_i(x_0 + \lambda x + o_1(x,\lambda)) \leq \varphi_i(\lambda x + o_1(x,\lambda)) + K_i \|\lambda x + o_1(x,\lambda)\|^\beta$$

at $\lambda \in [0,\lambda_x]$. As $\varphi_i$ is a positively homogeneous function, then

$$f_i(x_0 + \lambda x + o_1(x,\lambda)) \leq \lambda\varphi_i(x + \frac{o_1(x,\lambda))}{\lambda}) + K_i \|\lambda x + o_1(x,\lambda)\|^\beta . \qquad (3.2)$$

By the condition $\varphi_i$ is a continuous function and $\beta > 1$. Let $\varepsilon > 0$ such that $\varphi_i(x) + \varepsilon < 0$. Then for $\varepsilon > 0$ there exists $\delta_x > 0$ such that $\varphi_i(x + \frac{o_1(x,\lambda))}{\lambda}) \leq \varphi_i(x) + \varepsilon$ at $\lambda \in [0,\delta_x]$. Therefore based on the ratio (3.2) there exists $\bar{\lambda}_x > 0$ such that $f_i(x_0 + \lambda x + o_1(x,\lambda)) \leq 0$ at $\lambda \in [0,\bar{\lambda}_x]$ and $i \in J$.

By the condition $\varphi(x) < 0$, and $\varphi$ is a continuous positively homogeneous function. Let $\varepsilon > 0$ such that $\varphi(x) + \varepsilon < 0$. Then for $\varepsilon > 0$ there exists $\tilde{\lambda}_x > 0$ such that $\varphi(x + \frac{o_1(x,\lambda))}{\lambda}) \leq \varphi(x) + \varepsilon$ at $\lambda \in [0,\tilde{\lambda}_x]$. Therefore $\lambda\varphi(x + \frac{o_1(x,\lambda))}{\lambda}) \leq \lambda(\varphi(x) + \varepsilon) \leq 0$ at $\lambda \in [0,\tilde{\lambda}_x]$.

If $x \in I_{C_1}(x_0)$, then by definition of $I_{C_1}(x_0)$ there exists $\alpha_0 > 0$ such that $x_0 + tx + o_1(x,\lambda) \in C_1$ at $t \in [0,\alpha_0]$. Having put $\lambda_x = \min\{\bar{\lambda}_x, \tilde{\lambda}_x, \alpha_0\}$ we receive that $x \in K_{1,\mu}(x_0; C,\varphi)$. The lemma is proved.

The following lemma 3.4 is similarly proved.

**Lemma 3.4.** *Let $X$ and $Y$ be Banach spaces, $C = \{x \in C_1 : f_i(x) \leq 0,\ i \in J\}$, $f_i : X \to R$, $i \in J$, and $C_1 \subset X$, the functions $f_i$ satisfy $\varphi_i - (\beta,\delta)$ locally Lipschitz condition with the constant $K_i$ at the point $x_0$, $\varphi_i$ and $\varphi$ be continuous positively homogeneous functions of degree $\alpha$, $\beta > \alpha \geq 1$, $\mu > 0$ and $f_i(x_0) = 0$ at $i \in J$. Then*

$$K_{\alpha,\mu}(x_0; C,\varphi) \supset \{x \in K_\alpha(x_0; C_1) : \varphi_i(x) < 0,\ i \in J,\ \varphi(x) < 0\}.$$

**Lemma 3.5.** *Let $X$ and $Y$ be Banach spaces, $C = \{x \in X : f_i(x) \leq 0,\ i \in J,\ F(x) = 0,\ x \in C_1\}$, where $f_i : X \to R$, $i \in J$, $F : X \to Y$, $C_1 \subset X$, the functions $f_i$ satisfy $\varphi_i - (2,\delta)$ locally Lipschitz condition with the constant $K_i$ at the point $x_0$, where $i \in J$, $\varphi_i$ are continuous positively homogeneous functions at $i \in I$, the mapping $F$ satisfy $S - (2,\delta)$ locally Lipschitz condition with the constant $K$ at the point $x_0$, $S : X \to Y$ be continuous positively homogeneous mapping, $x_0 \in C$, $f_i(x_0) = 0$ at $i \in J$, and $\varphi(x) = -\sum_{i=1}^{m} \alpha_i f_i(x_0 + x) - \langle y^*,\ F(x_0 + x)\rangle +$*

$+\sum_{i=0}^{m}\alpha_i\varphi_i(x)+\langle y^*,\, S(x)\rangle$, *where* $\alpha_0\geq 0,\ \alpha_1\geq 0,\ldots,\alpha_m\geq 0$ *and* $y^*\in Y^*$. *Then*

$$K_{1,2}(x_0;C,\varphi)\subset\{x\in K_{C_1}(x_0):\varphi_i(x)\leq 0,\ i\in J,\ \ \alpha_0\varphi_0(x)\ \leq 0,\ \ S(x)=0\}.$$

*Proof.* Let $x\in K_{1,2}(x_0;C,\varphi)$. Then there exists $\exists\lambda_x>0,\ \exists o_1(x,\lambda):[0,\lambda_x]\to X,\ \exists o_2(x,\lambda):[0,\lambda_x]\to R_+$, where $\frac{o_1(x,\lambda)}{\lambda}\to 0$ and $\frac{o_2(x,\lambda)}{\lambda^2}\to 0$ at $\lambda\downarrow 0$, that $f_i(x_0+\lambda x+o_1(x,\lambda))\leq 0$, $i\in J$, $F(x_0+\lambda x+o_1(x,\lambda))=0$, $x_0+\lambda x+o_1(x,\lambda)\in C_1$ and $\varphi(\lambda x+o_1(x,\lambda))\leq o_2(x,\lambda)$ at $\lambda\in[0,\lambda_x]$. Besides let $\lambda_x>0$ such that $\|\lambda x+o_1(x,\lambda)\|\leq\delta$ at $\lambda\in[0,\lambda_x]$. As the function $f_i$ satisfies $\varphi_i-(2,\delta)$ locally Lipschitz condition at the point $x_0$, then

$$|f_i(x_0+y)-f_i(x_0)-\varphi_i(y)|\leq K_i\,\|y\|^2$$

at $y\in\delta B$. Therefore

$$|f_i(x_0+\lambda x+o_1(x,\lambda))-f_i(x_0)-\varphi_i(\lambda x+o_1(x,\lambda))|\leq K_i\,\|\lambda x+o_1(x,\lambda)\|^2$$

at $\lambda\in[0,\lambda_x]$. From here we receive

$$\varphi_i(\lambda x+o_1(x,\lambda))\leq f_i(x_0+\lambda x+o_1(x,\lambda))+K_i\,\|\lambda x+o_1(x,\lambda)\|^2$$

at $\lambda\in[0,\lambda_x]$. As $\varphi_i$ is a continuous positively homogeneous function, then $\lambda\varphi_i(x+\frac{o_1(x,\lambda))}{\lambda})\leq K_i\,\|\lambda x+o_1(x,\lambda)\|^2$ at $\lambda\in[0,\lambda_x]$. From here we receive that $\varphi_i(x)\leq 0$ at $i\in J$. By the condition $\varphi(\lambda x+o_1(x,\lambda))\leq o_2(x,\lambda)$ at $\lambda\in[0,\lambda_x]$. Therefore

$$\varphi(\lambda x+o_1(x,\lambda))=-\sum_{i=1}^{m}\alpha_i f_i(x_0+\lambda x+o_1(x,\lambda))-\langle y^*,\,F(x_0+\lambda x+o_1(x,\lambda))\rangle$$
$$+\sum_{i=0}^{m}\alpha_i\varphi_i(\lambda x+o_1(x,\lambda))+\langle y^*,\,S(\lambda x+o_1(x,\lambda))\rangle\leq o_2(x,\lambda)$$

at $\lambda\in[0,\lambda_x]$. From here we have that

$$\sum_{i=0}^{m}\alpha_i\varphi_i(\lambda x+o_1(x,\lambda))+\langle y^*,\,S(\lambda x+o_1(x,\lambda))\rangle$$
$$\leq\sum_{i=1}^{m}\alpha_i f_i(x_0+\lambda x+o_1(x,\lambda))+\langle y^*,\,F(x_0+\lambda x+o_1(x,\lambda))\rangle+o_2(x,\lambda)$$

at $\lambda\in[0,\lambda_x]$. Therefore

$$f_i(x_0+\lambda x+o_1(x,\lambda))\leq\varphi_i(\lambda x+o_1(x,\lambda))+K_i\,\|\lambda x+o_1(x,\lambda)\|^2,$$
$$\|F(x_0+\lambda x+o_1(x,\lambda))-F(x_0)-S(\lambda x+o_1(x,\lambda))\|\leq L\,\|\lambda x+o_1(x,\lambda)\|^2$$

at $\lambda\in[0,\lambda_x]$. Then we have that $S(x)=0$ and

$$\langle y^*,F(x_0+\lambda x+o_1(x,\lambda))-F(x_0)-S(\lambda x+o_1(x,\lambda))\rangle$$
$$\leq\|y^*\|\,\|F(x_0+\lambda x+o_1(x,\lambda))-F(x_0)-S(\lambda x+o_1(x,\lambda))\|$$
$$\leq L\,\|y^*\|\,\|\lambda x+o_1(x,\lambda)\|^2$$

at $\lambda\in[0,\lambda_x]$. Then we receive that

$$\sum_{i=0}^{m}\alpha_i\varphi_i(\lambda x+o_1(x,\lambda))+\langle y^*,\,S(\lambda x+o_1(x,\lambda))\rangle\leq\sum_{i=1}^{m}\alpha_i(\varphi_i(\lambda x+o_1(x,\lambda))$$
$$+K_i\,\|\lambda x+o_1(x,\lambda)\|^2)+\langle y^*,\,S(\lambda x+o_1(x,\lambda))\rangle+L\,\|y^*\|\,\|\lambda x+o_1(x,\lambda)\|^2$$

at $\lambda\in[0,\lambda_x]$. From here we have that $\alpha_0\varphi_0(x)\leq 0$.

From $x_0 + \lambda x + o_1(x,\lambda) \in C_1$ at $\lambda \in [0, \lambda_x]$ we have that $x \in K_{C_1}(x_0)$. The lemma is proved.

**Lemma 3.6.** *Let $X$ and $Y$ be Banach spaces, $C = \{x \in X : f_i(x) \le 0,\ i \in J,\ F(x) = 0,\ x \in C_1\}$, where $f_i : X \to R$, $i \in J$, $F : X \to Y$, $C_1 \subset X$, the functions $f_i$ satisfy $\varphi_i - (2,\delta)$ locally Lipschitz condition with the constant $K_i$ at the point $x_0$, where $i \in J$, $\varphi_i$ be continuous positively homogeneous functions at $i \in J$ and derivative $F'(z)$ in Frechet's sense exist at $z \in x_0 + 2\delta B$ and there will be $L > 0$ such that $\|F'(u) - F'(v)\| \le L\|u - v\|$ at $u, v \in x_0 + 2\delta B$, $f_i(x_0) = 0$ at $i \in J$, and $\varphi(x) = -\sum\limits_{i=1}^{m} \alpha_i f_i(x_0 + x) - \langle y^*, F(x_0 + x)\rangle$, where $\alpha_1 \ge 0, \dots, \alpha_m \ge 0$ and $y^* \in Y^*$. Then*

$$K_{1,2}(x_0; C, \varphi) \subset \{x \in K_{C_1}(x_0) : \varphi_i(x) \le 0,\ i \in J,\ \sum_{i=1}^{m} \alpha_i\varphi_i(x) \ge 0,\ \ F'(x_0)x = 0\}$$
$$= \{x \in K_{C_1}(x_0) : \varphi_i(x) \le 0,\ \alpha_i\varphi_i(x) = 0,\ i \in J,\ \ F'(x_0)x = 0\}.$$

*Proof.* Let $x \in K_{1,2}(x_0; C, \varphi)$. Then there exists $\exists\lambda_x > 0$, $\exists o_1(x,\lambda) : [0, \lambda_x] \to X$, $\exists o_2(x,\lambda) : [0,\lambda_x] \to R_+$, where $\frac{o_1(x,\lambda)}{\lambda} \to 0$ and $\frac{o_2(x,\lambda)}{\lambda^2} \to 0$ at $\lambda \downarrow 0$, that $f_i(x_0+\lambda x+o_1(x,\lambda)) \le 0$, $i \in J$, $F(x_0+\lambda x+o_1(x,\lambda)) = 0$, $x_0+\lambda x+o_1(x,\lambda) \in C_1$ and $\varphi(\lambda x + o_1(x,\lambda)) \le o_2(x,\lambda)$ at $\lambda \in [0,\lambda_x]$. Besides let $\lambda_x > 0$ such that $\|\lambda x + o_1(x,\lambda)\| \le \delta$ at $\lambda \in [0,\lambda_x]$.

As the function $f_i$ satisfies $\varphi_i - (2,\delta)$ locally Lipschitz condition at the point $x_0$, then

$$|f_i(x_0 + y) - f_i(x_0) - \varphi_i(y)| \le K_i \|y\|^2$$

at $y \in \delta B$. Therefore

$$|f_i(x_0 + \lambda x + o_1(x,\lambda)) - f_i(x_0) - \varphi_i(\lambda x + o_1(x,\lambda))| \le K_i \|\lambda x + o_1(x,\lambda)\|^2$$

at $\lambda \in [0,\lambda_x]$. From here we receive that

$$\varphi_i(\lambda x + o_1(x,\lambda)) \le f_i(x_0 + \lambda x + o_1(x,\lambda)) + K_i \|\lambda x + o_1(x,\lambda)\|^2$$

at $\lambda \in [0,\lambda_x]$. As $\varphi_i$ is a continuous positively homogeneous function, then $\lambda\varphi_i(x + \frac{o_1(x,\lambda))}{\lambda}) \le K_i \|\lambda x + o_1(x,\lambda)\|^2$ at $\lambda \in [0,\lambda_x]$. Then we receive that $\varphi_i(x) \le 0$ at $i \in J$.

By lemma 2.1 we have

$$\left\|F(x_0 + \lambda x + o_1(x,\lambda)) - F(x_0) - F'(x_0)(\lambda x + o_1(x,\lambda))\right\| \le L\|\lambda x + o_1(x,\lambda)\|^2$$

at $\lambda \in [0,\lambda_x]$. Then we have that $F'(x_0)x = 0$ and

$$\left\langle y^*, F(x_0 + \lambda x + o_1(x,\lambda)) - F(x_0) - F'(x_0)(\lambda x + o_1(x,\lambda))\right\rangle$$
$$\le \|y^*\| \left\|F(x_0 + \lambda x + o_1(x,\lambda)) - F(x_0) - F'(x_0)(\lambda x + o_1(x,\lambda))\right\|$$
$$\le L\|y^*\| \|\lambda x + o_1(x,\lambda)\|^2$$

at $\lambda \in [0,\lambda_x]$. Also we have that

$$f_i(x_0 + \lambda x + o_1(x,\lambda)) - f_i(x_0) - \varphi_i(\lambda x + o_1(x,\lambda)) \le K_i \|\lambda x + o_1(x,\lambda)\|^2$$

at $\lambda \in [0,\lambda_x]$. Then we receive that

$$-\sum_{i=1}^{m} \alpha_i f_i(x_0 + \lambda x + o_1(x,\lambda)) - \langle y^*,\ F(x_0 + \lambda x + o_1(x,\lambda))\rangle$$

$$+\sum_{i=1}^{m}\alpha_i\varphi_i(\lambda x+o_1(x,\lambda))+\big\langle y^*,\ F'(x_0)(\lambda x+o_1(x,\lambda))\big\rangle$$

$$\geq -(\sum_{i=1}^{m}\alpha_i K_i+\|y^*\|\,L)\,\|\lambda x+o_1(x,\lambda)\|^2$$

at $\lambda\in[0,\lambda_x]$.

By the condition $\varphi(\lambda x+o_1(x,\lambda))\leq o_2(x,\lambda)$ at $\lambda\in[0,\lambda_x]$. Therefore

$$\varphi(\lambda x+o_1(x,\lambda))=-\sum_{i=1}^{m}\alpha_i f_i(x_0+\lambda x+o_1(x,\lambda))$$

$$-\langle y^*,\ F(x_0+\lambda x+o_1(x,\lambda))\rangle\leq o_2(x,\lambda)$$

at $\lambda\in[0,\lambda_x]$. Therefore

$$\sum_{i=1}^{m}\alpha_i\varphi_i(\lambda x+o_1(x,\lambda))+\big\langle y^*,F'(x_0)(\lambda x+o_1(x,\lambda))\big\rangle$$

$$\leq -(\sum_{i=1}^{m}\alpha_i K_i+\|y^*\|\,L)\,\|\lambda x+o_1(x,\lambda)\|^2-o_2(x,\lambda)$$

at $\lambda\in[0,\lambda_x]$. From here we will receive that $\sum\limits_{i=1}^{m}\alpha_i\varphi_i(x)+\langle y^*,\ F'(x_0)(x)\rangle\geq 0$. As $F'(x_0)x\ =0$, we have that $\sum\limits_{i=1}^{m}\alpha_i\varphi_i(x)\geq 0$.

From $x_0+\lambda x+o_1(x,\lambda)\in C_1$ at $\lambda\in[0,\lambda_x]$ we have that $x\in K_{C_1}(x_0)$. The lemma is proved.

It is possible to receive similar statements in classes of $S-(\alpha,\beta,\nu,\delta,\omega)$ and $S-(o(\beta),\delta)$ locally Lipschitz mappings at the point.

Let's note that the results received in Section 3 are used for receiving relations between the results received in Section 4 and mathematical programming problems.

## 4. The necessary condition of higher order in terms of approximate cone and Lagrange's function

Let $X$ be a Banach space, $C\subset X$, $f:X\to R$, $\varphi:X\to R$, $\varphi_1:X\to R$.

**Theorem 4.1.** *If $X$ is a Banach space, $x_0$ is the minimum point of the function $f$ on the set $C$, there exist $\alpha>0$, $\nu>0$, $\beta\geq\nu$, $\mu>0$, $\sigma>0$, where $\mu\geq\beta-\alpha\nu$, $\sigma\geq\frac{\beta-\alpha\nu}{\alpha}$, finite positively homogeneous function $\varphi_1$ of degree $\mu$, the function $o:R_+\to R_+$, where $\lim\limits_{t\downarrow 0}\frac{o(t)}{t}=0$, the numbers $\delta>0$ and $K$ are such that*

$$|f(x_0+x+y)-f(x_0+x)-\varphi(x+y)+\varphi(x)|\leq K\,\|y\|^\nu\,(\varphi_1(x)+\|y\|^\sigma)+o(\|x\|^\beta)$$

*for $x\in K_{\alpha,\beta}(x_0;C,\varphi)$ $(x\in \mathrm{T}_{\alpha,\beta}(x_0;C,\varphi))$, $\|x\|\leq\delta$, $y\in X$, $\|y\|\leq\|x\|$, $x_0+x+y\in C$, then*

$$f_\varphi^{\{\beta\}-}(x_0;x)=\varliminf_{\lambda\downarrow 0}\tfrac{1}{\lambda^\beta}(f(x_0+\lambda x)-\varphi(\lambda x)-f(x_0))\geq 0 \text{ at } x\in K_{\alpha,\beta}(x_0;C,\varphi),$$

$$f_\varphi^{\{\beta\}+}(x_0;x)=\varlimsup_{\lambda\downarrow 0}\tfrac{1}{\lambda^\beta}(f(x_0+\lambda x)-\varphi(\lambda x)-f(x_0))\geq 0 \text{ at } x\in \mathrm{T}_{\alpha,\beta}(x_0;C,\varphi).$$

**Proof.** If $x \in K_{\alpha,\beta}(x_0; C, \varphi)$, by definition there will be $\lambda_x > 0$, $o_1(x,\lambda) : [0,\lambda_x] \to X$, $o_2(x,\lambda) : [0,\lambda_x] \to R_+$, where $\frac{o_1(x,\lambda)}{\lambda^\alpha} \to 0$ and $\frac{o_2(x,\lambda)}{\lambda^\beta} \to 0$ at $\lambda \downarrow 0$, that $x_0 + \lambda x + o_1(x,\lambda) \in C$ and $\varphi(\lambda x + o_1(x,\lambda)) \le o_2(x,\lambda)$ at $\lambda \in [0,\lambda_x]$. Therefore

$$f_\varphi^{\{\beta\}-}(x_0; x) = \varliminf_{\lambda\downarrow 0} \frac{1}{\lambda^\beta}(f(x_0+\lambda x) - \varphi(\lambda x) - f(x_0))$$

$$\ge \varliminf_{\lambda\downarrow 0} \frac{1}{\lambda^\beta}(f(x_0+\lambda x + o_1(x,\lambda)) - f(x_0) - \varphi(\lambda x + o_1(x,\lambda)) + o_2(x,\lambda))$$

$$+ \varliminf_{\lambda\downarrow 0} \frac{1}{\lambda^\beta}(-f(x_0+\lambda x + o_1(x,\lambda)) + f(x_0+\lambda x) + \varphi(\lambda x + o_1(x,\lambda))$$

$$\varphi(\lambda x) - o_2(x,\lambda) \ge \varliminf_{\lambda\downarrow 0} \frac{1}{\lambda^\beta}(f(x_0+\lambda x + o_1(x,\lambda)) - f(x_0) - \varphi(\lambda x + o_1(x,\lambda))$$

$$-o_2(x,\lambda)) - \varlimsup_{\lambda\downarrow 0} \frac{K}{\lambda^\beta}[\|o_1(x,\lambda)\|^\nu (\lambda^\mu \varphi_1(x) + \|o_1(x,\lambda)\|^\sigma) + o_2(x,\lambda) + o(\|\lambda x\|^\beta)]$$

$$\ge -\varlimsup_{\lambda\downarrow 0} \frac{K}{\lambda^\beta}[\|o_1(x,\lambda)\|^\nu (\lambda^{\beta-\alpha\nu}\lambda^{\mu-\beta+\alpha\nu}\varphi_1(x) + \|o_1(x,\lambda)\|^{\frac{\beta-\alpha\nu}{\alpha}} \|o_1(x,\lambda)\|^{\sigma-\frac{\beta-\alpha\nu}{\alpha}})$$

$$+o_2(x,\lambda) + o(\|\lambda x\|^\beta)] = 0.$$

Let's prove the second part of the theorem. If $x \in \mathrm{T}_{\alpha,\beta}(x_0; C, \varphi)$, then by definition there will be $\exists o_2(x,\lambda) : [0,\lambda_x] \to R_+$, where $\frac{o_2(x,\lambda)}{\lambda^\beta} \to 0$ at $\lambda \downarrow 0$ and $\exists \lambda_i \downarrow 0$, $\exists \upsilon_i \subset X$, where $\frac{1}{\lambda_i^{\alpha-1}}\|\upsilon_i - x\| \to 0$ at $i \to +\infty$, that $x_0 + \lambda_i \upsilon_i \in C$, $\varphi(\lambda_i \upsilon_i) \le o_2(x,\lambda_i)$ for all $i$. Therefore

$$f_\varphi^{\{\beta\}+}(x_0; x) = \varlimsup_{\lambda\downarrow 0} \frac{1}{\lambda^\beta}(f(x_0+\lambda x) - \varphi(\lambda x) - f(x_0))$$

$$\ge \varlimsup_{i\to\infty} \frac{1}{\lambda_i^\beta}(f(x_0+\lambda_i x) - \varphi(\lambda_i x) - f(x_0)) \ge \varlimsup_{i\to\infty} \frac{1}{\lambda_i^\beta}(f(x_0+\lambda_i \upsilon_i) - \varphi(\lambda_i \upsilon_i) - f(x_0)$$

$$+o_2(x,\lambda_i)) + \varliminf_{i\to\infty} \frac{1}{\lambda_i^\beta}(-f(x_0+\lambda_i\upsilon_i) + f(x_0+\lambda_i x) + \varphi(\lambda_i\upsilon_i) - \varphi(\lambda_i x) - o_2(x,\lambda_i))$$

$$\ge \varlimsup_{i\to\infty} \frac{1}{\lambda_i^\beta}(f(x_0+\lambda_i\upsilon_i) - \varphi(\lambda_i\upsilon_i) - f(x_0) + o_2(x,\lambda_i))$$

$$-\varlimsup_{i\to\infty} \frac{1}{\lambda_i^\beta}(f(x_0+\lambda_i x+\lambda_i(\upsilon_i - x)) - f(x_0+\lambda_i x) - \varphi(\lambda_i x+\lambda_i(\upsilon_i-x)) + \varphi(\lambda_i x)) + o_2(x,\lambda_i))$$

$$\ge -\varlimsup_{i\to\infty} \frac{1}{\lambda_i^\beta}(K\|\lambda_i(\upsilon_i - x)\|^\nu (\lambda^\mu \varphi_1(x) + \|\lambda_i(\upsilon_i - x)\|^\sigma) + o_2(x,\lambda_i) + o(\|\lambda_i x\|^\beta) = 0.$$

The theorem is proved.

*Remark 4.1.* From the proof of theorem 4.1 we have that if for any $x \in K_{\alpha,\beta}(x_0; C, \varphi)$ there exists $\exists \lambda_x > 0$, $\exists o_1(x,\lambda) : [0,\lambda_x] \to X$, $\exists o_2(x,\lambda) : [0,\lambda_x] \to R_+$, $\exists o(x,\lambda) : [0,\lambda_x] \to R_+$, where $\frac{o_1(x,\lambda)}{\lambda^\alpha} \to 0$, $\frac{o_2(x,\lambda)}{\lambda^\beta} \to 0$, $\frac{o(x,\lambda)}{\lambda^\beta} \to 0$ at $\lambda \downarrow 0$, that $x_0 + \lambda x + o_1(x,\lambda) \in C$ and $\varphi(\lambda x + o_1(x,\lambda)) \le o_2(x,\lambda)$ as $\lambda \in [0,\lambda_x]$ and

$$|f(x_0+\lambda x + o_1(x,\lambda)) - f(x_0+\lambda x) - \varphi(\lambda x + o_1(x,\lambda)) + \varphi(\lambda x)| \le o(x,\lambda)$$

at $\lambda \in [0, \lambda_x]$ and $x_0$ is the minimum point of the function $f$ on the set $C$, then the statement of theorem 4.1 is also true.

If $\varphi_1(x) = \|x\|^{\mu}$, where $\mu \geq \beta - \alpha\nu$, the following corollary 4.1 follows from theorem 4.1.

**Corollary 4.1.** *If $X$ is a Banach space, $x_0$ is the minimum point of the function $f$ on the set $C$ , there exist $\alpha > 0$, $\nu > 0$, $\mu > 0$, $\beta \geq \nu$, $\sigma > 0$, where $\mu \geq \beta - \alpha\nu$, $\sigma \geq \frac{\beta - \alpha\nu}{\alpha}$, the functions $o : R_+ \to R_+$, where $\lim\limits_{t\downarrow 0} \frac{o(t)}{t} = 0$ and $\varphi : X \to R$, numbers $\delta > 0$ and $K$ are such that*

$$|f(x_0 + x + y) - f(x_0 + x) - \varphi(x + y) + \varphi(x)| \leq K \|y\|^{\nu} (\|x\|^{\mu} + \|y\|^{\sigma}) + o(\|x\|^{\beta})$$

*for $x \in K_{\alpha,\beta}(x_0; C, \varphi)$ ($x \in \mathrm{T}_{\alpha,\beta}(x_0; C, \varphi)$), $\|x\| \leq \delta$, $y \in X$, $\|y\| \leq \|x\|$, $x_0 + x + y \in C$, then*

$$f_{\varphi}^{\{\beta\}-}(x_0; x) = \varliminf_{\lambda\downarrow 0} \frac{1}{\lambda^{\beta}}(f(x_0 + \lambda x) - \varphi(\lambda x) - f(x_0)) \geq 0 \text{ at } x \in K_{\alpha,\beta}(x_0; C, \varphi),$$

$$(f_{\varphi}^{\{\beta\}+}(x_0; x) = \varlimsup_{\lambda\downarrow 0} \frac{1}{\lambda^{\beta}}(f(x_0 + \lambda x) - \varphi(\lambda x) - f(x_0)) \geq 0 \text{ at } x \in \mathrm{T}_{\alpha,\beta}(x_0; C, \varphi)).$$

Let's note that theorem 4.1 and corollary 4.1 are generalizations of theorem 6.1 and corollary 6.1 of [6]. For simplicity further we will consider the use of the first part of theorem 4.1, when $\alpha \geq 1$ and $\mu = \sigma = \beta - \nu$.

Let $X$ and $Y$ be banach spaces, $f_i : X \to R$, $i \in I$, $F : X \to Y$, $C \subset X$.

Let's consider the problem

$$f_0(x) \to \min, \tag{4.1}$$

$P = \{x \in X : f_i(x) \leq 0, \quad i = 1, \ldots, m, \ \ F(x) = 0, \ \ x \in C\}$.

**Theorem 4.2.** *Let $X$ and $Y$ be Banach spaces, $f_i : X \to R$, $i \in I$, and $F : X \to Y$, the functions $f_i$ satisfy $\varphi_i - (1, 2, 1, \delta, o_i(2))$ locally Lipschitz condition with the constant $K_i$ at the point $x_0$, where the functions $\varphi_i$ satisfy the Lipschitz condition with the constant $M_i$ in the set $2\delta B$ and derivative $F'(z)$ in Frechet's sense exists at $z \in x_0 + 2\delta B$ and there will be $L > 0$ such that $\|F'(u) - F'(\upsilon)\| \leq L \|u - \upsilon\|$ at $u, \upsilon \in x_0 + 2\delta B$, $\alpha_0 \geq 0$, $\alpha_1 \geq 0, \ldots, \alpha_m \geq 0$ and $y^* \in Y^*$, and $x_0$ be the minimum point of $f_0$ on the set $P$. Then*

$$f_{\varphi}^{\{2\}-}(x_0; x) = \varliminf_{\lambda\downarrow 0} \frac{1}{\lambda^2}(\alpha_0 f_0(x_0 + \lambda x) - \varphi(\lambda x) - \alpha_0 f_0(x_0)) \geq 0$$

*at $x \in K_{r,2}(x_0; \mathrm{P}, \varphi)$, where $\varphi(x) = -\sum\limits_{i=1}^{m} \alpha_i f_i(x_0 + x) - \langle y^*, F(x_0 + x)\rangle$, $r \geq 2$.*

*Proof.* Using lemma 2.1 by the condition we have that

$$\Bigg|\sum_{i=0}^{m} \alpha_i f_i(x_0 + x + y) + \langle y^*, F(x_0 + x + y)\rangle - \sum_{i=0}^{m} \alpha_i f_i(x_0 + x) - \langle y^*, F(x_0 + x)\rangle$$

$$- \sum_{i=0}^{m} \alpha_i \varphi_i(x + y) - \big\langle y^*, F'(x_0)(x + y)\big\rangle + \sum_{i=0}^{m} \alpha_i \varphi_i(x) + \big\langle y^*, F'(x_0)(x)\big\rangle\Bigg|$$

$$\leq \sum_{i=0}^{m} \alpha_i |f_i(x_0 + x + y) - f_i(x_0 + x) - \varphi_i(x + y) + \varphi_i(x)|$$

$$+\left|\left\langle y^*,\,F(x_0+x+y)-F(x_0+x)-F'(x_0)(x+y)+F'(x_0)(x\right\rangle\right|$$
$$\le (\sum_{i=0}^{m} K_i\alpha_i+\|y^*\|\,L)\,\|y\|\,(\|y\|+\|x\|)+\sum_{i=0}^{m}\alpha_i o_i(\|x\|^2)$$

at $x,y\in\delta B$. Therefore

$$\left|\sum_{i=0}^{m}\alpha_i f_i(x_0+x+y)+\langle y^*,\,F(x_0+x+y)\rangle-\sum_{i=0}^{m}\alpha_i f_i(x_0+x)-\langle y^*,\,F(x_0+x)\rangle\right|$$
$$\le (\sum_{i=0}^{m} K_i\alpha_i+\|y^*\|\,L)\,\|y\|\,(\|y\|+\|x\|)+\sum_{i=0}^{m}\alpha_i o_i(\|x\|^2)$$
$$+\left|\sum_{i=0}^{m}\alpha_i\varphi_i(x+y)+\left\langle y^*,\,F'(x_0)(x+y)\right\rangle-\sum_{i=0}^{m}\alpha_i\varphi_i(x)-\left\langle y^*,\,F'(x_0)(x)\right\rangle\right|$$
$$\le \sum_{i=0}^{m}\alpha_i\,|\varphi_i(x+y)-\varphi_i(x)|+\left|\left\langle y^*,\,F'(x_0)(x+y)-F'(x_0)(x\right\rangle\right|$$
$$+(\sum_{i=0}^{m} K_i\alpha_i+\|y^*\|\,L)\,\|y\|\,(\|y\|+\|x\|)+\sum_{i=0}^{m}\alpha_i o_i(\|x\|^2)$$
$$\le (\sum_{i=0}^{m} K_i\alpha_i+\|y^*\|\,L)\,\|y\|\,(\|y\|+\|x\|)$$
$$+(\sum_{i=0}^{m} M_i\alpha_i+\|y^*\|\,\left\|F'(x_0)\right\|)\,\|y\|+\sum_{i=0}^{m}\alpha_i o_i(\|x\|^2)$$

at $x,y\in\delta B$. If $x\in K_{r,2}(x_0;P,\varphi)$, then by definition there will be $\lambda_x>0$, $o_1(x,\lambda):[0,\lambda_x]\to X$, $o_2(x,\lambda):[0,\lambda_x]\to R_+$, where $\frac{o_1(x,\lambda)}{\lambda^r}\to 0$ and $\frac{o_2(x,\lambda)}{\lambda^2}\to 0$ as $\lambda\downarrow 0$, that $x_0+\lambda x+o_1(x,\lambda)\in P$ and $\varphi(\lambda x+o_1(x,\lambda))\le o_2(x,\lambda)$ at $\lambda\in[0,\lambda_x]$. Let $\lambda_x>0$ such that $\lambda_x\,\|x\|\le\delta$ and $\|o_1(x,\lambda)\|\le\delta$ at $\lambda\in[0,\lambda_x]$. Then

$$\left|\sum_{i=0}^{m}\alpha_i f_i(x_0+\lambda x+o_1(x,\lambda))+\langle y^*,\,F(x_0+\lambda x+o_1(x,\lambda))\rangle\right.$$
$$\left.-\sum_{i=0}^{m}\alpha_i f_i(x_0+\lambda x)-\langle y^*,\,F(x_0+\lambda x)\rangle\right|$$
$$\le (\sum_{i=0}^{m} K_i\alpha_i+\|y^*\|\,L)\,\|o_1(x,\lambda)\|\,(\|o_1(x,\lambda)\|+\|\lambda x\|)$$
$$+(\sum_{i=0}^{m} M_i\alpha_i+\|y^*\|\,\left\|F'(x_0)\right\|)\,\|o_1(x,\lambda)\|+\sum_{i=0}^{m}\alpha_i o_i(\|\lambda x\|^2)$$

at $\lambda\in[0,\lambda_x]$. If $r\ge 2$, from remark 4.1 we have that

$$f_{\varphi}^{\{2\}-}(x_0;x)=\underline{\lim}_{\lambda\downarrow 0}\frac{1}{\lambda^2}(\alpha_0 f_0(x_0+\lambda x)-\varphi(\lambda x)-\alpha_0 f_0(x_0))\ge 0$$

at $x\in K_{r,2}(x_0;P,\varphi)$, where $\varphi(x)=-\sum_{i=1}^{m}\alpha_i f_i(x_0+x)-\langle y^*,\,F(x_0+x)\rangle$. The theorem is proved.

Let us put $L(x,\alpha,y^*)=\alpha_0 f_0(x)+\sum\limits_{i=1}^{m}\alpha_i f_i(x)+\langle y^*,\,F(x)\rangle,\quad \alpha=(\alpha_0,\alpha_1,...,\alpha_m).$

**Corollary 4.2.** *If $\alpha_i f_i(x_0)=0$, $i\in J$, and the condition of theorem 4.2 is satisfied, then*

$$\lim_{\lambda\downarrow 0}\frac{1}{\lambda^2}(L(x_0+\lambda x,\alpha,y^*)-L(x_0,\alpha,y^*))\geq 0$$

*at $x\in K_{r,2}(x_0;P,\varphi)$, where $r\geq 2$.*

*Proof.* If $\alpha_i f_i(x_0)=0$, $i\in J$, under the condition of theorem 4.2 we have that

$$f_\varphi^{\{2\}-}(x_0;x)=\lim_{\lambda\downarrow 0}\frac{1}{\lambda^2}(\alpha_0 f_0(x_0+\lambda x)+\sum_{i=1}^{m}\alpha_i f_i(x_0+\lambda x)$$

$$+\langle y^*,\,F(x_0+\lambda x)\rangle-\alpha_0 f_0(x_0)-\sum_{i=1}^{m}\alpha_i f_i(x_0)-\langle y^*,\,F(x_0)\rangle)$$

$$=\lim_{\lambda\downarrow 0}\frac{1}{\lambda^2}(L(x_0+\lambda x,\alpha,y^*)-L(x_0,\alpha,y^*))\geq 0$$

at $x\in K_{r,2}(x_0;P,\varphi)$. The corollary is proved.

Let the condition of theorem 4.2 be satisfied and $x_0$ be the local minimum point of $f_0$ on the set $P$, $\alpha_0\geq 0$, $\alpha_1\geq 0,\ldots,\alpha_m\geq 0$ and $y^*\in Y^*$. As $K_2(x_0;P,\varphi)\subset K_{r,2}(x_0;P,\varphi)$, from theorem 2 we have that

$$f_\varphi^{\{2\}-}(x_0;x)=\lim_{\lambda\downarrow 0}\frac{1}{\lambda^2}(\alpha_0 f_0(x_0+\lambda x)-\varphi(\lambda x)-\alpha_0 f_0(x_0))\geq 0$$

at $x\in K_2(x_0;P,\varphi)$, where $\varphi(x)=-\sum_{i=1}^{m}\alpha_i f_i(x_0+x)-\langle y^*,\,F(x_0+x)\rangle$.

**Theorem 4.3.** *Let $X$ and $Y$ be Banach spaces, $f_i:X\to R$, $i\in I$, and $F:X\to Y$, the functions $f_i$ satisfy $\varphi_i-(1,2,1,\delta,o_i(2))$ locally Lipschitz condition with the constant $K_i$ at the point $x_0$ at $i\in I$, derivative $F'(z)$ in Frechet's sense exist at $z\in x_0+2\delta B$ and there will be $L>0$ such that $\|F'(u)-F'(\upsilon)\|\leq L\,\|u-\upsilon\|$ at $u,\upsilon\in x_0+2\delta B$, $\alpha_0\geq 0$, $\alpha_1\geq 0,\ldots,\alpha_m\geq 0$ and $y^*\in Y^*$, and $x_0$ be the minimum point of the function $f_0$ on the set $P$. Then*
$f_\varphi^{\{2\}-}(x_0;x)=\lim\limits_{\lambda\downarrow 0}\frac{1}{\lambda^2}(\alpha_0 f_0(x_0+\lambda x)-\varphi(\lambda x)-\alpha_0 f_0(x_0))\geq 0$

*at $x\in K_{r,2}(x_0;P,\varphi)$, where $\varphi(x)=-\sum\limits_{i=1}^{m}\alpha_i f_i(x_0+x)-\langle y^*,\,F(x_0+x)\rangle+$
$+\sum\limits_{i=0}^{m}\alpha_i\varphi_i(x)+\langle y^*,\,F'(x_0)x\rangle$, $r\geq 1$.*

*Proof* Using lemma 2.1 by the condition we have that

$$\Bigg|\sum_{i=0}^{m}\alpha_i f_i(x_0+x+y)+\langle y^*,\,F(x_0+x+y)\rangle-\sum_{i=0}^{m}\alpha_i f_i(x_0+x)-\langle y^*,\,F(x_0+x)\rangle$$

$$-\sum_{i=0}^{m}\alpha_i\varphi_i(x+y)-\big\langle y^*,\,F'(x_0)(x+y)\big\rangle+\sum_{i=0}^{m}\alpha_i\varphi_i(x)+\big\langle y^*,\,F'(x_0)x\big\rangle\Bigg|$$

$$\leq\sum_{i=0}^{m}\alpha_i\,|f_i(x_0+x+y)-f_i(x_0+x)-\varphi_i(x+y)+\varphi_i(x)|$$

$$+\left|\left\langle y^*,\, F(x_0+x+y)-F(x_0+x)-F'(x_0)(x+y)+F'(x_0)x\right\rangle\right|$$
$$\le (\sum_{i=0}^{m} K_i\alpha_i + \|y^*\|\, L)\,\|y\|\,(\|y\|+\|x\|)+\sum_{i=0}^{m}\alpha_i o_i(\|x\|^2)$$

at $x, y \in \delta B$. Therefore

$$\left|\alpha_0 f_0(x_0+x+y)-\alpha_0 f_0(x_0+x)+\sum_{i=1}^{m}\alpha_i f_i(x_0+x+y)\right.$$
$$+\langle y^*,\, F(x_0+x+y)\rangle-\sum_{i=1}^{m}\alpha_i f_i(x_0+x)-\langle y^*,\, F(x_0+x)\rangle$$
$$\left.-\sum_{i=0}^{m}\alpha_i\varphi_i(x+y)-\left\langle y^*,\, F'(x_0)(x+y)\right\rangle+\sum_{i=0}^{m}\alpha_i\varphi_i(x)+\left\langle y^*,\, F'(x_0)x\right\rangle\right|$$
$$\le (\sum_{i=0}^{m} K_i\alpha_i + \|y^*\|\, L)\,\|y\|\,(\|y\|+\|x\|)+\sum_{i=0}^{m}\alpha_i o_i(\|x\|^2)$$

at $x, y \in \delta B$, i.e. $\alpha_0 f_0$ is $-\sum\limits_{i=1}^{m}\alpha_i f_i(x_0+x)-\langle y^*,\, F(x_0+x)\rangle+\sum\limits_{i=0}^{m}\alpha_i\varphi_i(x)+$ $+\langle y^*, F'(x_0)x\rangle$ $-(1,2,1,\delta,\sum\limits_{i=0}^{m}\alpha_i o_i(2))$ locally Lipschitz function with the constant $(\sum\limits_{i=0}^{m} K_i\alpha_i+\|y^*\|\, L)$ at the point $x_0$.

Then from the corollary 4.1 we have

$$f_{\varphi}^{\{2\}-}(x_0;x)=\varliminf_{\lambda\downarrow 0}\frac{1}{\lambda^2}(\alpha_0 f_0(x_0+\lambda x)-\varphi(\lambda x)-\alpha_0 f_0(x_0))\ge 0$$

at $x \in K_{r,2}(x_0;P,\varphi)$, where $\varphi(x)=-\sum\limits_{i=1}^{m}\alpha_i f_i(x_0+x)-\langle y^*,\, F(x_0+x)\rangle+$ $+\sum\limits_{i=0}^{m}\alpha_i\varphi_i(x)+\langle y^*,\, F'(x_0)x\rangle$ . The theorem is proved.

Let the condition of theorem 4.3 be satisfied and $x_0$ be the local minimum point of function $f_0$ on the set $P$, $\alpha_0\ge 0$, $\alpha_1\ge 0,\dots,\alpha_m\ge 0$ and $y^*\in Y^*$. As $K_2(x_0;P,\varphi)\subset K_{r,2}(x_0;P,\varphi)$, from theorem 4.3 we have that

$$f_{\varphi}^{\{2\}-}(x_0;x)=\varliminf_{\lambda\downarrow 0}\frac{1}{\lambda^2}(\alpha_0 f_0(x_0+\lambda x)-\varphi(\lambda x)-\alpha_0 f_0(x_0))\ge 0$$

at $x\in K_2(x_0;C,\varphi)$, where $\varphi(x)=-\sum\limits_{i=1}^{m}\alpha_i f_i(x_0+x)-\langle y^*,\, F(x_0+x)\rangle+$ $+\sum\limits_{i=0}^{m}\alpha_i\varphi_i(x)+\langle y^*,\, F'(x_0)x\rangle$.

If $\sum\limits_{i=1}^{m}\alpha_i f_i(x_0)+\langle y^*,\, F(x_0)\rangle=0$, having put that $L(x,\alpha,y^*)=\sum\limits_{i=0}^{m}\alpha_i f_i(x)+$ $+\langle y^*,\, F(x)\rangle$ , $\alpha=(\alpha_0,\,\alpha_1,\dots,\alpha_m)$, $y^*\in Y^*$ from theorem 4.3 we have that

$$L_{q_{\alpha,y^*}}^{\{2\}-}(x_0,\alpha,y^*;x)=\varliminf_{\lambda\downarrow 0}\frac{1}{\lambda^2}(L(x_0+\lambda x,\alpha,y^*)$$

$$-\sum_{i=0}^{m}\alpha_i\varphi_i(\lambda x)-\left\langle y^*,\,F'(x_0)(\lambda x)\right\rangle - L(x_0,\alpha,y^*))\geq 0$$

at $x\in K_{r,2}(x_0;P,q_{\alpha,y^*})$, where

$$q_{\alpha,y^*}(x)=-\sum_{i=1}^{m}\alpha_i f_i(x_0+x)-\langle y^*,\,F(x_0+x)\rangle+\sum_{i=0}^{m}\alpha_i\varphi_i(x)+\left\langle y^*,\,F'(x_0)(x)\right\rangle.$$

If the function $x\to L^{\{2\}-}_{q_{\alpha,y^*}}(x_0,\alpha,y^*;x)$ is continuous and

$$clK_{r,2}(x_0;P,\varphi)\supset\{x\in K_C(x_0):\varphi_i(x)\leq 0,\ i\in I,\ \alpha_0\,\varphi_0(x)\leq 0,\ \ F'(x_0)x=0\},$$

then we have from here that $L^{\{2\}-}_{q_{\alpha,y^*}}(x_0,\alpha,y^*;z)\geq 0$ at

$$z\in\{x\in K_C(x_0):\varphi_i(x)\leq 0,\ i\in I,\ \alpha_0\varphi_0(x)\leq 0,\ \ F'(x_0)x=0\}.$$

Let $C\subset X$, $x_0\in C$. Let's put

$$T_C\,(x_0)=\{v\in X:\,\forall h_n>0,\ h_n\to 0,\forall x_n\to x_0,\exists u_n\to v,\ x_n+h_nu_n\in C\},$$

$$T\,(x_0;C)=\{v\in X:\,\exists h_n>0,\ h_n\to 0,\ \exists u_n\to v,\ x_0+h_nu_n\in C\},$$

where $T_C\,(x_0)$ is the tangent cone to $C$ at the point $x_0$, and $T(x_0;C)$ is the contingent cone to $C$ at the point $x_0$ (see [2]).

If $C\subset X$ is a convex set, then $T_C\,(x_0)=T\,(x_0;C)=cl\bigcup_{\lambda>0}\frac{C-x_0}{\lambda}$ (see [2]).

If $C\subset X$ is a convex set and $x_0\in C$, we will designate $\tilde{C}=intC\bigcup\{x_0\}$, $S_{\tilde{C}}(x_0)=\bigcup_{\lambda>0}\frac{\tilde{C}-x_0}{\lambda}$. Let's note that if $intC\neq\emptyset$ and $C$ is a convex set, then $T_C(x_0)=clS_{\tilde{C}}(x_0)$.

**Theorem 4.4.** *Let $X$ and $Y$ be Banach spaces, $x_0$ be the local minimum point in the problem (4.1), $\beta>1$, the function $f_i$ satisfy $\varphi_i-(\beta,\delta)$ locally semi-Lipschitz condition with the constant $K$ at the point $x_0$, where $i\in I$, $\varphi_i:X\to R$ sublinear continuous functions at $i\in I$, $f_j(x_0)=0$ at $j\in J$, the operator $F:X\to Y$ be strictly differentiable at the point $x_0$ and $F'(x_0)X=Y$, $C$ be a convex set, $intC\neq\emptyset$, then there exist simultaneously non zero $\alpha_0\geq 0,\ \alpha_1\geq 0,\ldots,\alpha_m\geq 0$ and $y^*\in Y^*$ such that $\sum_{i=0}^{m}\alpha_i\varphi_i(x)+\langle y^*,F'(x_0)x\rangle\geq 0$ at $x\in T_C(x_0)$.*

*Proof.* Let's denote $\Lambda=F'(x_0)$. Let's show that system

$$\varphi_0(x)<0,\varphi_1(x)<0,\ldots,\varphi_m(x)<0,\Lambda x=0$$

has no solution on $S_{\tilde{C}}(x_0)$. Let's assume the contrary. Let there exist $\bar{x}\in S_{\tilde{C}}(x_0)$ such that $\varphi_i(\bar{x})<0$ at $i\in I$ and $\Lambda\bar{x}=0$. As $\Lambda\bar{x}=F'(x_0)\bar{x}=0$ and $F'(x_0)X=Y$, according to Lyusternik's theorem there exists $\varepsilon>0$ and the mapping $r:[0,\varepsilon]\to X$ such that $\frac{r(t)}{t}\to 0$ as $t\downarrow 0$ and $F(x_0+t\bar{x}+r(t))=0$ at $t\in[0,\varepsilon]$. By the condition the function $f_i$ satisfies $\varphi_i-(\beta,\delta)$ locally semi-Lipschitz condition with the constant K at the point $x_0$, where $i\in I$, $\beta>1$. Then we have that

$$f_i(x_0+t\bar{x}+r(t))-f_i(x_0)-\varphi_i(t\bar{x}+r(t))\leq K\left\|t\bar{x}+r(t)\right\|^{\beta}$$

at $t\in[0,\varepsilon]$, $\|t\bar{x}+r(t)\|\leq\delta$, $i\in I$. As $\varphi_i$ is a continuous function, there exists $0<\delta_0\leq\frac{1}{2}\delta$ such that $\left|\varphi_i(\bar{x}+\frac{r(t)}{t})-\varphi_i(\bar{x})\right|\leq\frac{1}{2}\left|\varphi_i(\bar{x})\right|$ at $\left\|\frac{r(t)}{t}\right\|\leq\delta_0$ and $i\in I$. Then we get that $\varphi_i(\bar{x}+\frac{r(t)}{t})\leq\frac{1}{2}\varphi_i(\bar{x})$ at $\left\|\frac{r(t)}{t}\right\|\leq\delta_0$ and $i\in I$. As $\frac{r(t)}{t}\to 0$

at $t \downarrow 0$, then there exists $\lambda$, where $0 < \lambda < 1$, it that $\left\| \frac{r(t)}{t} \right\| \le \delta_0$ at $t \in (0, \lambda]$. Then $\|t\bar{x} + r(t)\| \le \delta$ at $t \in [0, \lambda_1]$, where $\lambda_1 = \min\{\lambda, \frac{1}{2\|\bar{x}\|}\delta, \varepsilon\}$. Therefore we have that

$$f_i(x_0 + t\bar{x} + r(t)) - f_i(x_0) \le 0,5\ t\varphi_i(\bar{x}) + Kt^\beta \left\| \bar{x} + \frac{r(t)}{t} \right\|^\beta$$
$$\le 0,5\ t\varphi_i(\bar{x}) + Kt^\beta(\|\bar{x}\| + \delta_0)^\beta$$

at $t \in [0, \lambda_1]$ and $i \in I$. From here we have that

$f_0(x_0 + t\bar{x} + r(t)) - f_0(x_0) < 0, f_j(x_0 + t\bar{x} + r(t)) < 0, j \in J, F(x_0 + t\bar{x} + r(t)) = 0$

at rather small $t > 0$. If $\bar{x} \in S_{\tilde{C}}(x_0) = \bigcup_{\lambda>0} \frac{\tilde{C} - x_0}{\lambda}$, then there exists $\lambda_0 > 0$ such that $\bar{x} \in \frac{\tilde{C} - x_0}{\lambda_0}$, i.e. $x_0 + \lambda_0\bar{x} \in \tilde{C}$. Therefore $x_0 + \lambda_0\bar{x} \in int\ C$. Then there exists $\nu_0 > 0$, where $\lambda_0 > \nu_0$, such that $x_0 + \lambda_0\bar{x} + (\lambda_0\frac{r(t)}{t}) \in int\ C$ at $t \in [0, \nu_0]$. Therefore

$$x_0 + t(\bar{x} + \frac{r(t)}{t}) = (1 - \frac{t}{\lambda_0})x_0 + \frac{t}{\lambda_0}(x_0 + \lambda_0\bar{x} + \lambda_0(\frac{r(t)}{t})) \in C,$$

i.e. we have that $x_0 + t\bar{x} + r(t) \in C$ at $t \in [0, \nu_0]$.

As $x_0$ is the local minimum point in problem (4.1), we get contradiction. Therefore the system $\varphi_0(x) < 0$, $\varphi_1(x) < 0$, $\ldots$, $\varphi_m(x) < 0$, $\Lambda x = 0$ has no solution on $S_{\tilde{C}}(x_0)$. According to theorem 5.5.3[9] there exists simultaneously non-zero $\alpha_0 \ge 0,\ \alpha_1 \ge 0, \ldots, \alpha_m \ge 0$ and $y^* \in Y^*$ such that $\sum_{i=0}^{m} \alpha_i\varphi_i(x) + \langle y^*, F'(x_0)x\rangle \ge 0$ at $x \in S_{\tilde{C}}(x_0)$. As $\varphi_i : X \to R$ are continuous functions at $i \in I$ and $\Lambda = F'(x_0)$ is a linear continuous operator, then $\sum_{i=0}^{m} \alpha_i\varphi_i(x) + \langle y^*, F'(x_0)x\rangle \ge 0$ at $x \in clS_{\tilde{C}}(x_0) = T_C(x_0)$. The theorem is proved.

From theorem 4.4 we have that the zero point minimizes the convex function $\sum_{i=0}^{m} \alpha_i\varphi_i(x) + \langle y^*, F'(x_0)x\rangle + \delta_{T_C(x_0)}(x)$ in $X$. As $\partial\delta_{T_C(x_0)}(0) = N_C(x_0)$, we will get that

$$0 \in \partial(\sum_{i=0}^{m} \alpha_i\varphi_i(x) + \langle y^*, F'(x_0)x\rangle + \delta_{T_C(x_0)}(x))_{x=0}$$
$$= \sum_{i=0}^{m} \alpha_i\partial\varphi_i(0) + F'(x_0)^*y^* + N_C(x_0).$$

From the proof of theorem 4.4 we have that $\max_{0\le i\le m} \varphi_i(x) \ge 0$ at $x \in T_C(x_0) \bigcap Ker\ F'(x_0)$.

Let's note that in theorem 4.4 we can replace the condition: function $f_i$ satisfies $\varphi_i - (\beta, \delta)$ locally semi-Lipschitz condition with the constant $K$ as the point $x_0$, where $\beta > 1$, by: the function $f_i$ satisfies $\varphi_i - (o(1), \delta)$ locally semi-Lipschitz condition as the point $x_0$.

We denote the set of all hypertangents to the set $C$ at the point $x_0 \in C$ by $I_C(x_0)$ (see [3]). If $\bar{x} \in I_C(x_0)$ and $r : R_+ \to X$, where $\frac{r(t)}{t} \to 0$ at $t \downarrow 0$, by definition of $I_C(x_0)$ there exists $\alpha_0 > 0$ such that $x_0 + t\bar{x} + r(t) \in C$ at $t \in [0, \alpha_0]$. Therefore from the proof of theorem 4.4 we have that if $C \subset X$ is any set and there exists a hypertangent vector to the set $C$ at the point $x_0 \in C$, theorem 4.4 remains true if in theorem 4.4 we replace the cone $S_{\tilde{C}}(x_0)$ by $intT_C(x_0)$, where $T_C(x_0)$ is Clarke's tangent cone to the set $C$ at the point $x_0$ (see [3]), i.e. there exists simultaneously non zero $\alpha_0 \geq 0,\ \alpha_1 \geq 0, \ldots, \alpha_m \geq 0$ and $y^* \in Y^*$ such that $\sum\limits_{i=0}^{m} \alpha_i \varphi_i(x) + \langle y^*, F'(x_0)x\rangle \geq 0$ at $x \in intT_C(x_0)$. As $\varphi_i : X \to R$ are sublinear continuous functions at $i \in I$, we will get that $\sum\limits_{i=0}^{m} \alpha_i \varphi_i(x) + \langle y^*, F'(x_0)x\rangle \geq 0$ at $x \in T_C(x_0)$.

Theorem 4.4 remains true, if $F$ is an affine continuous operator, $C \subset X$ is a convex set.

Let's note that theorem 4.3 is true for all $\alpha_0 \geq 0$, $\alpha_1 \geq 0, \ldots, \alpha_m \geq 0$ and $y^* \in Y^*$, but in theorem 4.4 in which the necessary condition of the first order is received, $\alpha_0 \geq 0$, $\alpha_1 \geq 0, \ldots, \alpha_m \geq 0$ and $y^* \in Y^*$ is not arbitrary.

Let's denote $V_{\alpha,y^*}(x) = \sum\limits_{i=0}^{m} \alpha_i \varphi_i(x) + \langle y^*, F'(x_0)x\rangle$, $\alpha = (\alpha_0, \alpha_1, \ldots \alpha_m)$,

$$\Omega = \{(\alpha, y^*) :\ \alpha_i \geq 0,\ y^* \in Y^*,\ \sum_{i=0}^{m} \alpha_i + \|y^*\| = 1,\ V_{\alpha,y^*}(x) \geq 0 \text{ at } x \in T_C(x_0)\}.$$

**Corollary 4.3.** *If $\alpha_i f_i(x_0) = 0$, $i \in J$, and the condition of theorems 4.3 and 4.4 is satisfied, then*

$$\sup\{L^{\{2\}-}_{q_{\alpha,y^*}}(x_0, \alpha, y^*; x) :\ (\alpha, y^*) \in \Omega\} \geq 0 \ \text{ at } x \in \bigcup\nolimits_{(\alpha,y^*)\in\Omega} K_{r,2}(x_0; P, q_{\alpha,y^*}),$$
$$\sup\{L^{\{2\}+}_{q_{\alpha,y^*}}(x_0, \alpha, y^*; x) :\ (\alpha, y^*) \in \Omega\} \geq 0 \ \text{ at } x \in \bigcup\nolimits_{(\alpha,y^*)\in\Omega} T_{r,2}(x_0; P, q_{\alpha,y^*}).$$

Using Section 3 the set $\bigcup_{(\alpha,y^*)\in\Omega} K_{r,2}(x_0; P, q_{\alpha,y^*})$ may be substitute of by a simpler set (see lemma 3.3 and lemma 3.4).

**Theorem 4.5.** *If $X$ and $Y$ are Banach spaces, $f_i : X \to R$, $i \in I$, and $F : X \to Y$, the functions $f_i$ satisfy $\varphi_i-(1, \beta, \nu, \delta, o_i(\beta))$ locally Lipschitz condition with the constant $K_i$ at the point $x_0$ at $i \in I$, the mapping $F(x)$ satisfies $S-(1, \beta, \nu, \delta, o(\beta))$ locally Lipschitz condition with the constant $K$ at the point $x_0$, $\alpha_0 \geq 0$, $\alpha_1 \geq 0, \ldots, \alpha_m \geq 0$ and $y^* \in Y^*$, $r \geq 1$, $\beta \geq \nu > 0$ and $x_0$ is the minimum point in problem (4.1), then*

$$f^{\{\beta\}-}_{\varphi}(x_0; x) = \varliminf_{\lambda\downarrow 0} \tfrac{1}{\lambda^\beta}(\alpha_0 f_0(x_0+\lambda x) - \varphi(\lambda x) - \alpha_0 f_0(x_0)) \geq 0 \ \text{ at } x \in K_{r,\beta}(x_0; P, \varphi),$$
$$f^{\{\beta\}+}_{\varphi}(x_0; x) = \varlimsup_{\lambda\downarrow 0} \tfrac{1}{\lambda^\beta}(\alpha_0 f_0(x_0+\lambda x) - \varphi(\lambda x) - \alpha_0 f_0(x_0)) \geq 0 \ \text{ at } x \in T_{r,\beta}(x_0; P, \varphi),$$

*where* $\varphi(x) = -\sum\limits_{i=1}^{m} \alpha_i f_i(x_0 + x) - \langle y^*,\ F(x_0 + x)\rangle + \sum\limits_{i=0}^{m} \alpha_i \varphi_i(x) + \langle y^*,\ S(x)\rangle$.

*Proof.* By the condition we have that

$$\Bigg|\sum_{i=0}^{m} \alpha_i f_i(x_0 + x + y) + \langle y^*,\ F(x_0 + x + y)\rangle - \sum_{i=0}^{m} \alpha_i f_i(x_0 + x)$$
$$- \langle y^*,\ F(x_0 + x)\rangle - \sum_{i=0}^{m} \alpha_i \varphi_i(x + y) - \langle y^*,\ S(x + y)\rangle + \sum_{i=0}^{m} \alpha_i \varphi_i(x) + \langle y^*,\ S(x)\rangle\Bigg|$$
$$\leq \sum_{i=0}^{m} \alpha_i \left|f_i(x_0 + x + y) - f_i(x_0 + x) - \varphi_i(x + y) + \varphi_i(x)\right|$$
$$+ \left|\langle y^*,\ F(x_0 + x + y) - F(x_0 + x) - S(x + y) + S(x)\rangle\right|$$

$$\leq (\sum_{i=0}^{m} K_i\alpha_i + \|y^*\| K) \|y\|^{\nu} (\|y\|^{\beta-\nu} + \|x\|^{\beta-\nu}) + \sum_{i=0}^{m} \alpha_i o_i(\|x\|^{\beta}) + \|y^*\| o(\|x\|^{\beta})$$

at $x, y \in \delta B$. From here we have that $\alpha_0 f_0$ satisfies

$$-\sum_{i=1}^{m} \alpha_i f_i(x_0+x) - \langle y^*, F(x_0+x)\rangle + \sum_{i=0}^{m} \alpha_i \varphi_i(x)$$

$$+ \langle y^*, S(x)\rangle - (1, \beta, \nu, \delta, \sum_{i=0}^{m} \alpha_i o_i(\beta) + \|y^*\| o(\beta))$$

locally Lipschitz condition with the constant $(\sum_{i=0}^{m} K_i\alpha_i + \|y^*\| K)$ at the point $x_0$. Validity of theorem 4.5 follows from corollary 4.1. The theorem is proved.

Let's denote

$$\varphi_{\alpha,y^*}(x) = -\sum_{i=1}^{m} \alpha_i f_i(x_0+x) - \langle y^*, F(x_0+x)\rangle + \sum_{i=0}^{m} \alpha_i \varphi_i(x) + \langle y^*, S(x)\rangle ,$$

$$L^{\{\beta\}-}_{\varphi_{\alpha,y^*}}(x_0, \alpha, y^*; x) = \varliminf_{\lambda\downarrow 0} \frac{1}{\lambda^{\beta}} (L(x_0+\lambda x, \alpha, y^*)$$

$$-\sum_{i=0}^{m} \alpha_i \varphi_i(\lambda x) - \langle y^*, S(\lambda x)\rangle L(x_0, \alpha, y^*)),$$

$$L^{\{\beta\}+}_{\varphi_{\alpha,y^*}}(x_0, \alpha, y^*; x) = \varlimsup_{\lambda\downarrow 0} \frac{1}{\lambda^{\beta}} (L(x_0+\lambda x, \alpha, y^*)$$

$$-\sum_{i=0}^{m} \alpha_i \varphi_i(\lambda x) - \langle y^*, S(\lambda x)\rangle - L(x_0, \alpha, y^*)).$$

**Corollary 4.4.** *If $\alpha_i f_i(x_0) = 0$, $i \in J$, and the condition of theorem 4.5 is satisfied, then*
*$L^{\{\beta\}-}_{\varphi_{\alpha,y^*}}(x_0, \alpha, y^*; x) \geq 0$ at $x \in K_{r,\beta}(x_0; P, \varphi_{\alpha,y^*})$;*
*$L^{\{\beta\}+}_{\varphi_{\alpha,y^*}}(x_0, \alpha, y^*; x) \geq 0$ at $x \in T_{r,\beta}(x_0; P, \varphi_{\alpha,y^*})$, where $r \geq 1$, $\beta \geq \nu > 0$.*

*Proof.* If $\alpha_i f_i(x_0) = 0$, $i \in J$, under the condition of theorem 4.5 we have that

$$L^{\{\beta\}-}_{\varphi_{\alpha,y^*}}(x_0, \alpha, y^*; x) = \varliminf_{\lambda\downarrow 0} \frac{1}{\lambda^{\beta}} (\sum_{i=0}^{m} \alpha_i f_i(x_0+\lambda x) + \langle y^*, F(x_0+\lambda x)\rangle - \sum_{i=0}^{m} \alpha_i \varphi_i(\lambda x)$$

$$- \langle y^*, S(\lambda x)\rangle - \sum_{i=0}^{m} \alpha_i f_i(x_0) - \langle y^*, F(x_0)\rangle) = \varliminf_{\lambda\downarrow 0} \frac{1}{\lambda^{\beta}} (L(x_0+\lambda x, \alpha, y^*)$$

$$-\sum_{i=0}^{m} \alpha_i \varphi_i(\lambda x) - \langle y^*, S(\lambda x)\rangle - L(x_0, \alpha, y^*)) \geq 0$$

at $x \in K_{r,\beta}(x_0; P, \varphi_{\alpha,y^*})$.

The second case is proved similarly. The corollary is proved.

Let's note that all statements of Section 4 remain true if $x_0$ is the local minimum point.

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