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CUBATURE FORMULA FOR A CLASS OF SURFACE INTEGRALS GENERATED BY WEAKLY-SINGULAR INTEGRALS

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Abstract. In the paper, a cubature formula is constructed for a class of surface integrals generated by weakly-singular integrals.

1. Statement of the problem

Let us consider the surface integrals

$$(Af)\ (x) = \int_S \frac{K(x,\, y)}{|x-y|^n} f(y) dS_y, \quad x \in S,$$

and

$$(Bf)\ (x) = \int_S \frac{F(x,\, y)}{|x-y|^m} f(y) dS_y, \quad x \in S,$$

where $S \subset R^3$ is the Lyapunov surface with the exponent $0 < \alpha \leq 1$, $f(x)$ is a continuous function on S, m and n are natural numbers, $K(x,y)$ and $F(x,y)$ are continuous functions on $S \times S$, and there exist numbers $\lambda\,,\, \mu \in (0,2)$ such that

$$|K(x,y)| \leq M\, |x-y|^{n-\lambda}, \quad \forall x, y \in S,$$

and

$$|F(x,y)| \leq M\ |x-y|^{m-\mu} \quad , \quad \forall x\,,\, y \in S.$$

Here and in the sequel M denotes positive constants different and various inequalities. It is known (see [1,2,5] that numerous theoretical and applied problems of mathematics, physics and mechanics are reduced to different classes of boundary integral equations (BIE) dependent on one integral

$$((AB)\, f)\ (x) = \int_S \frac{K(x,\, y)}{|x-y|^n} \left(\int_S \frac{F(y,\, t)}{|y-t|^m} f(t) dS_t \right) dS_y. \tag{1.1}$$

Since BIE are solved exactly only in very rare cases, development of approximate methods for solving BIE with appropriate theoretical ground takes central stage, and for that we should construct a cubature formula for the integral $((AB)\, f)\ (x)$. This paper is devoted to this problem.

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2. Cubature formulas for the integral (1.1)

We start this section with a partition of S into elementary domains $S = \bigcup_{l=1}^{N} S_l$:

(1) for $\forall\, l \in \{1,\, 2,\, ...,\, N\}$ the domain S_l is closed and the set $\overset{0}{S_l}$ of its points internal with respect to S is not empty, and $mes\, \overset{0}{S_l} = mes S_l$ and $\overset{0}{S_l} \bigcap \overset{0}{S_j} = \varnothing$ for $j \in \{1,\, 2,\, ...N\}\,,\ j \neq l$;

(2) for $\forall\, l \in \{1,\, 2,\, ...,\, N\}$ the domain S_l is a connected piece of the surface S with continuous boundary;

(3) for $\forall\, l \in \{1,\, 2,\, ...,\, N\}$ there exists the so called support point $x_l \in S_l$ such that:

(3.1) $r_l(N) \sim R_l(N)$ ($r_l(N) \sim R_l(N) \ \Leftrightarrow\ C_1 \leq r_l\,(N)\,/R_l\,(N) \leq C_2$, C_1 and C_2 are positive constants independent of N), where $r_l(N) = \min\limits_{x \in \partial S_l} |x - x_l|$ and $R_l(N) = \max\limits_{x \in \partial S_l} |x - x_l|$;

(3.2) $R_l(N) \leq d/2$, where d is the radius of the standard sphere (see [6]);

(3.3) for $\forall\, j \in \{1, 2, ..., N\} : r_j(N) \sim r_l(N)$.

Obviously, $\lim\limits_{N\to\infty} R(N) = \lim\limits_{N\to\infty} r(N) = 0$ and $r(N) \sim R(N)$, where $R(N) = \max\limits_{l=\overline{1,\,N}} R_l(N)$, $r(N) = \min\limits_{l=\overline{1,\,N}} r_l(N)$.

Such a partition, as a partition of a unit sphere into elementary parts, was earlier cited in [4].

For the function $\varphi\,(x) \in C\,(S)$ (by $C\,(S)$ the space of continuous functions on S with the norm $\|f\|_\infty = \max\limits_{x\in S} |f\,(x)|$ is denoted) we introduce the continuity modulus of the form $\omega\,(\varphi,\, \delta) = \delta \sup\limits_{\tau \geq \delta} \frac{\overline{\omega}\,(\varphi,\, \tau)}{\tau}$, $\ \delta > 0$, where $\bar{\omega}(\varphi,\, \tau) = \underset{\substack{|x-y|\leq\tau \\ x,y\in S}}{max} |\varphi(x) - \varphi(y)|\,.$

Let

$$a_{lj} = \begin{cases} 0 & \text{for} \quad l = j, \\ \frac{K(x_l, x_j)}{|x_l - x_j|^n} mes S_j & \text{for} \quad l \neq j, \end{cases}$$

and

$$b_{lj} = \begin{cases} 0 & \text{for} \quad l = j, \\ \frac{F(x_l, x_j)}{|x_l - x_j|^m} mes S_j & \text{for} \quad l \neq j. \end{cases}$$

Theorem 2.1. *Let there exist natural numbers ℓ and k such that*

$$\left|K\left(x, y'\right) - K\left(x, y''\right)\right| \leq M \sum_{j=1}^{\ell} \left|y' - y''\right|^{\alpha_j} \left|x - y'\right|^{\beta_j} \left|x - y''\right|^{\gamma_j}, \quad \forall x, y', y'' \in S,$$

and

$$\left|F\left(x, y'\right) - F\left(x, y''\right)\right| \leq M \sum_{j=1}^{k} \left|y' - y''\right|^{a_j} \left|x - y'\right|^{b_j} \left|x - y''\right|^{c_j}, \ \forall x, y', y'' \in S,$$

where $0<\alpha_j\le 1$, $\beta_j\ge 0$, $\gamma_j\ge 0$, $0<a_j\le 1$, $b_j\ge 0$, $c_j\ge 0$, $\alpha_j+\beta_j+\gamma_j>n-2$, $j=\overline{1,\ell}$, *and* $a_j+b_j+c_j>m-2$, $\quad j=\overline{1,k}$. *Then the expression*

$$((AB)\,f)^N(x_l)=\sum_{j=1}^{N}\left(\sum_{p=1}^{N}a_{lp}\,b_{pj}\right)\,f(x_j)$$

at the points x_l, $\;l=\overline{1,\,N}$ *is the cubature formula for the integral* $((AB)\,f)\,(x)$, *and the following estimation is valid:*

$$\max_{l=\overline{1,\,N}}\left|((AB)\,f)\,\,(x_l)-((AB)\,f)^N(x_l)\right|$$

$$\le M\,\left[(1+\|B\|)\,\|f\|_\infty\,(R(N))^{\varepsilon}\,\,|\ln R(N)|+\omega\,(f,R(N))\,+\omega\,(Bf,R(N))\right],$$

where $\alpha=\min\limits_{j=\overline{1,\ell}},\ \alpha_j,\ \beta=\min\limits_{j=\overline{1,\ell}}\{\alpha_j+\beta_j+\gamma_j\}-\alpha$,
$\gamma=\min\{\alpha,2-\lambda,\alpha+\beta+2-n\}$, $a=\min\limits_{j=\overline{1,k}}a_j$, $b=\min\limits_{j=\overline{1,k}}\{a_j+b_j+c_j\}-a$,
$c=\min\{a,2-\mu,a+b+2-m\}$, $\varepsilon=\min\{\gamma,c\}$.

Proof. In the paper [3] it is proved that the expressions

$$(Af)^N(x_l)=\sum_{j=1}^{N}a_{lj}\,f(x_j)$$

and

$$(Bf)^N(x_l)=\sum_{j=1}^{N}b_{lj}\,f(x_j)$$

at the points x_l, $\;l=\overline{1,\,N}$ are cubature formulas for the integrals $(Af)\,(x)$ and $(Bf)\,(x)$, respectively,

$$\max_{l=\overline{1,\,N}}\left|(Af)\,(x_l)-(Af)^N(x_l)\right|$$

$$\le M\,\left[\,\|f\|_\infty\,(R(N))^{\gamma}\,\,|\ln R(N)|+\omega\,(f,R(N))\,\right]$$

and

$$\max_{l=\overline{1,\,N}}\left|(Bf)\,(x_l)-(Bf)^N(x_l)\right|$$

$$\le M\,\left[\,\|f\|_\infty\,(R(N))^{c}\,\,|\ln R(N)|+\omega\,(f,R(N))\,\right].$$

Then taking into account the equalities

$$\sum_{j=1}^{N}\left(\sum_{p=1}^{N}a_{lp}\,b_{pj}\right)\,f(x_j)=\sum_{j=1}^{N}\left(a_{lj}\sum_{p=1}^{N}b_{jp}f(x_p)\right),\quad l=\overline{1\,,\,N},$$

we have:

$$\left|((AB)\,f)\,\,(x_l)-((AB)\,f)^N(x_l)\,\right|\le\left|(A\,(Bf))\,(x_l)-\sum_{j=1}^{N}a_{lj}\,\,(Bf)\,(x_j)\right|$$

$$+\left|\sum_{j=1}^{N}a_{lj}\,\left((Bf)\,(x_j)-(Bf)^N(x_j)\right)\,\right|$$

$$\leq M \left[\|B\| \, \|f\|_\infty \left(R\left(N\right)\right)^\gamma \, \left|\ln R\left(N\right)\right| + \omega\left(Bf, R\left(N\right)\right) \right]$$

$$+ M \left(\max_{l=\overline{1,N}} \sum_{j=1}^{N} |\, a_{lj}| \right) \left[\|f\|_\infty \left(R\left(N\right)\right)^c \, \left|\ln R\left(N\right)\,\right| + \omega\left(f, R\left(N\right)\right) \right]. \qquad (2.1)$$

Taking into account the equality

$$\int_S \frac{|K(x_l,\, y)|}{|x_l - y|^n}\, dS_y - \sum_{j=1}^{N} |\, a_{lj}| \;=\; \int_{S_l} \frac{|K(x_l,\, y)|}{|x_l - y|^n}\, dS_y$$

$$+ \sum_{\substack{j=1 \\ j\neq l}}^{N} \int_{S_j} \frac{|K\left(x_l, y\right)| - |K\left(x_l, x_j\right)|}{|x_l - y|^n}\, dS_y$$

$$+ \sum_{\substack{j=1 \\ j\neq l}}^{N}{}' \int_{S_j} \left(\frac{1}{|x_l - y|^n} - \frac{1}{|x_l - x_j|^n} \right) |K\left(x_l, x_j\right)|\, dS_y,$$

it is easy to show that

$$\left| \int_S \frac{|K(x_l,\, y)|}{|x_l - y|^n}\, dS_y - \sum_{j=1}^{N} |\, a_{lj}| \right| \leq M \, \left(R\left(N\right)\right)^\gamma \, \left|\ln R\left(N\right)\right|,$$

and so,

$$\sum_{j=1}^{N} |\, a_{lj}| \;\leq \max_{x\in S} \int_S \frac{|K(x,\, y)|}{|x - y|^n}\, dS_y + M \, \left(R\left(N\right)\right)^\gamma \, \left|\ln R\left(N\right)\right| \leq M. \qquad (2.2)$$

Then, taking into account inequality (2.2) in (2.1), we get the proof of the theorem. □

3. Example

Let us consider an external problem with impedance condition for the Helmholtz equation: to find the function u twice continuously-differentiable on $R^3 \backslash \bar{D}$ and continuous on S and possessing a normal derivative in the sense of uniform convergence, satisfying the Helmholtz equation $\Delta u + k^2 u = 0$ in $R^3 \backslash \bar{D}$, the Sommerfeld radiation condition

$$\left(\frac{x}{|x|},\, grad u\left(x\right) \right) - i\, k\, u\left(x\right) = o\left(\frac{1}{|x|} \right), \;\; |x| \to \infty,$$

uniform in all directions of $x/\,|x|$ and the boundary condition

$$\frac{\partial u\left(x\right)}{\partial \vec{n}\left(x\right)} + f\left(x\right)\, u\left(x\right) = g\, \left(x\right) \;\; \text{on} \;\; S,$$

where $D \subset R^3$ is a bounded domain with the boundary $S \in \Lambda_\alpha$ (here Λ_α denotes a class of Lyapunov surfaces with the exponent $0 < \alpha \leq 1$), k is a wave number, moreover $Im\, k \geq 0$, $\vec{n}\left(x\right)$ is a unit external normal at the point $x \in S$, while f and g are the given continuous functions on S, and

$$Im\, \left(\bar{k}\, f\right) \geq 0 \;\;\; \text{on} \;\; S.$$

Let $v\ (x,\varphi)$ be an acoustic simple layer potential, $w\ (x,\varphi)$ be an acoustic double layer potential, $v_0\ (x,\varphi)$ be a simple layer potential for the Laplace equation, i.e.

$$v\ (x,\varphi) = \int_S \Phi_k\,(x,y)\ \varphi\,(y)\ dS_y,$$

$$w\ (x,\varphi) = \int_S \frac{\partial \Phi_k\,(x,y)}{\partial \vec{n}\ (y)}\,\varphi\,(y)\ dS_y,$$

$$v_0\ (x,\varphi) = v\ (x,\varphi)\,|_{k=0}\ = \int_S \Phi_0\,(x,y)\ \varphi\,(y)\ dS_y,$$

where $\Phi_k\,(x,y) = e^{i\,k\,|x-y|}/\ (4\pi\ |x-y|)$ is the fundamental solution of the Helmholtz equation. In the paper [2] it is shown that the function

$$u\,(x) = v\ (x,\varphi) + i\eta\, w\ (x,v_0)\,, x \in R^3\backslash\bar{D},$$

(where η is a real number, moreover if $Im\,k > 0$, then $\eta = 0$, and if $Im\,k = 0$, then $\eta \neq 0$) is the solution of the external problem with impedance boundary condition for the Helmholtz equation if the density $\varphi \in C\,(S)$ is the solution of uniquely solvable integral equation

$$\varphi + A\,\varphi = \psi,$$

where $\psi = -4\ (2+i\eta)^{-1}\,g$,

$$A = -2\ (2+i\eta)^{-1}\,(2K + 2i\eta\,(T+G) + f\ (2L + 2i\eta\,F + i\eta\,L_0))\,,$$

$$(L\varphi)\,(x) = \int_S \Phi_k\,(x,y)\ \varphi\,(y)\ dS_y,$$

$$(L_0\varphi)\,(x) = \int_S \Phi_0\,(x,y)\ \varphi\,(y)\ dS_y,$$

$$(K\varphi)\,(x) = \int_S \frac{\partial \Phi_k\,(x,y)}{\partial \vec{n}\ (x)}\,\varphi\,(y)\ dS_y,$$

$$(F\varphi)\,(x) = \int_S \frac{\partial \Phi_k\,(x,y)}{\partial \vec{n}\ (y)}\left(\int_S \Phi_0\,(y,t)\,\varphi\,(t)\ dS_t\right) dS_y,$$

$$(G\varphi)\,(x) = \int_S \frac{\partial \Phi_0\,(x,y)}{\partial \vec{n}\ (x)}\left(\int_S \frac{\partial \Phi_0\,(y,t)}{\partial \vec{n}\,(y)}\varphi\,(t)\ dS_t\right) dS_y,$$

$$(T\varphi)\,(x) = \int_S \frac{\partial}{\partial \vec{n}\ (x)}\left(\frac{\partial\ (\Phi_k\,(x,y) - \Phi_0\,(x,y))}{\partial \vec{n}\ (y)}\right)$$

$$\times\left(\int_S \Phi_0\,(y,t)\,\varphi\,(t)\ dS_t\right) dS_y, x \in S.$$

Using theorem 2.1 we construct a cubature formula for the integrals $(F\varphi)\,(x)$, $(G\varphi)\,(x)$ and $(T\varphi)\,(x)$. It is easy to show that

$$\frac{\partial \Phi_k\,(x,y)}{\partial \vec{n}\ (y)} = \frac{K_1\,(x,y)}{|x-y|^3},$$

$$\frac{\partial \Phi_0\,(x,y)}{\partial \vec{n}\ (x)} = \frac{K_2\,(x,y)}{|x-y|^3}$$

and

$$\frac{\partial}{\partial \vec{n}\ (x)}\left(\frac{\partial\ (\Phi_k\,(x,y) - \Phi_0\,(x,y))}{\partial \vec{n}\ (y)}\right) = \frac{K_3\,(x,y)}{|x-y|^5},$$

where

$$K_1(x,y) = -\frac{1}{4\pi}\left(\overrightarrow{xy}, \vec{n}(y)\right)\,(1 - i\,k\,|x-y|)\,e^{ik|x-y|},$$

$$K_2(x,y) = \frac{1}{4\pi}\left(\overrightarrow{xy}, \vec{n}(x)\right)$$

and

$$K_3(x,\,y) = \frac{1}{4\pi}\left(\overrightarrow{yx}, \vec{n}(x)\right)\left(\overrightarrow{xy}, \vec{n}(y)\right)$$
$$\times\left(\left(3 - 3\,i\,k\,|x-y| - k^2\,|x-y|^2\right)e^{ik|x-y|} - 3\right)$$
$$+\left(\vec{n}(y), \vec{n}(x)\right)\left((1 - i\,k\,|x-y|)\,e^{ik|x-y|} - 1\right)|x-y|^2.$$

Therefore,

$$|\Phi_0(x,y)| \le \frac{M}{|x-y|},\quad \left|\frac{\partial\Phi_k(x,y)}{\partial\vec{n}\,(y)}\right| \le \frac{M}{|x-y|^{2-\alpha}},$$

$$\left|\frac{\partial\Phi_0(x,y)}{\partial\vec{n}\,(x)}\right| \le \frac{M}{|x-y|^{2-\alpha}},$$

$$\left|\frac{\partial}{\partial\vec{n}\,(x)}\left(\frac{\partial\,(\Phi_k(x,y) - \Phi_0(x,y))}{\partial\vec{n}\,(y)}\right)\right| \le \frac{M}{|x-y|},$$

and in the paper [3] it is shown that

$$\left|K_1(x,y') - K_1(x,y'')\right| \le M\,\left|y' - y''\right|^{\alpha}\,\left|x - y'\right|,$$

$$\left|K_2(x,y') - K_2(x,y'')\right| \le M\,\left|y' - y''\right|\,\left|x - y'\right|^{\alpha},$$

$$\left|K_3(x,y') - K_3(x,y'')\right| \le M\,\left|y' - y''\right|^{\alpha}\,\left|x - y'\right|^{4-\alpha}.$$

Let

$$c_{lj} = |sgn\,(l-j)|\,\frac{\partial\Phi_k(x_l,x_j)}{\partial\vec{n}\,(x_j)} mesS_j,$$

$$d_{lj} = |sgn\,(l-j)|\,\Phi_0(x_l,x_j)\,mesS_j,$$

$$g_{lj} = |sgn\,(l-j)|\,\frac{\partial\Phi_0(x_l,x_j)}{\partial\vec{n}\,(x_l)} mesS_j$$

and

$$t_{lj} = |sgn\,(l-j)|\,\frac{\partial}{\partial\vec{n}\,(x_l)}\left(\frac{\partial\,(\Phi_k(x_l,x_j) - \Phi_0(x_l,x_j))}{\partial\vec{n}\,(x_j)}\right) mesS_j.$$

Then applying theorem 2.1, we get that the expressions

$$(F\varphi)^N(x_l) = \sum_{j=1}^{N}\left(\sum_{p=1}^{N} c_{lp}\,d_{pj}\right)\varphi(x_j),$$

$$(G\varphi)^N(x_l) = \sum_{j=1}^{N}\left(\sum_{p=1}^{N} g_{lp}\,g_{pj}\right)\varphi(x_j)$$

and

$$(T\varphi)^N(x_l) = \sum_{j=1}^{N}\left(\sum_{p=1}^{N} t_{lp}\,d_{pj}\right)\varphi(x_j)$$

at the points $x_l,\ l=\overline{1,\ N(h)}$ are the cubature formulas for the integrals $(F\varphi)(x)$, $(G\varphi)(x)$ and $(T\varphi)(x)$, respectively, and

$$\max_{l=\overline{1,N}}\left|(F\varphi)(x_l)-(F\varphi)^N(x_l)\right|$$
$$\leq M\ \left(\|\varphi\|_\infty (R(N))^\alpha\ |\ln R(N)|+\omega(\varphi,R(N))\right),$$
$$\max_{l=\overline{1,N}}\left|(G\varphi)(x_l)-(G\varphi)^N(x_l)\right|$$
$$\leq M\ \left(\|\varphi\|_\infty (R(N))^\alpha\ |\ln R(N)|+\omega(\varphi,R(N))\right)$$

and

$$\max_{l=\overline{1,N}}\left|(T\varphi)(x_l)-(T\varphi)^N(x_l)\right|$$
$$\leq M\ \left(\|\varphi\|_\infty (R(N))^\alpha\ |\ln R(N)|+\omega(\varphi,R(N))\right).$$

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