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ON THE TIETZE EXTENSION THEOREM IN SOFT TOPOLOGICAL SPACES

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Abstract. The theory of soft set was introduced by Molodtsov in 1999 as a new mathematical tool for deal with uncertainties. The idea of soft topological space was first given by Shabir and Naz. The purpose of this paper is to give Uryshon's Lemma and prove the Tietze Extension Theorem using this lemma in the so-called "soft topological space".

1. Introduction

Many practical problems in economics, engineering, environment, social science, medical science etc. cannot be dealt with by classical methods, because classical methods have inherent difficulties. The reason for these difficulties may be due to the inadequacy of the theories of parametrization tools. Several theories exist, for example, fuzzy set theory [19], intuitionistic fuzzy set theory [4], rough set theory [14], i.e., which can be considered as mathematical tools for dealing with uncertainties. Each of these theories has its inherent difficulties as what were pointed out by Molodtsov in [13]. Molodtsov [13] initiated a completely new approach for modeling uncertainties and applied successfully in directions such as smoothness of functions, game theory, operations research, Riemann-integration, Perron integration, and so on. Maji et al. [10], [11] defined operations on soft set and gave the first practical application of soft sets in decision making problems. The algebraic structure of set theories dealing with uncertainties is an important problem. Many researchers have contributed towards the algebraic structure of soft set theory. Aktas and Cagman [2] defined soft groups and derived their basic properties. U.Acar et al. [1] introduced initial concepts of soft rings. F.Feng et al. [7] defined soft semirings and several related notions to establish a connection between soft sets and semirings. Qiu- Mei Sun et al. [18] defined soft modules and investigated their basic properties. M.Shabir et al. [15] studied soft ideals over a semigroup which characterized generalized fuzzy ideals and fuzzy ideals with thresholds of a semigroup. C.Gunduz (Aras) and S.Bayramov [8], [9] introduced fuzzy soft modules and intuitionistic fuzzy soft modules and investigated some basic properties.

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Shabir and Naz [16] firstly introduced the notion of soft topological space which are defined over an initial universe with a fixed set of parameters and showed that a soft topological space gives a parameterized family of topological spaces. Theoretical studies of soft topological spaces have also been by some authors in [3], [6], [12], [17], [20]. In these studies the concept of soft point is given almost similarly. When the definition of soft point is given in this form, some results of classical topology are not valid in soft topological spaces.

In the present study, we first give some basic ideas about soft sets and the results already studied. In addition to these, we give the concept of soft completely regular space and we investigate its some important properties. Later we give Urysohn's lemma. Finally we prove Tietze extension theorem using Urysohn's lemma.

2. Preliminaries

In this section we will introduce necessary definitions and theorems for soft sets. Molodtsov [13] defined the soft set in the following way. Let X be an initial universe set and E be a set of parameters. Let $P(X)$ denotes the power set of X and $A \subset E$.

Definition 2.1. ([13]) A pair (F, A) is called a soft set over X, where F is a mapping given by $F : A \to P(X)$.

Definition 2.2. ([11]) For two soft sets (F, A) and (G, B) over X, (F, A) is called a soft subset of (G, B) if

(i) $A \subset B$ and

(ii) $\forall e \in A$, $F(e)$ and $G(e)$ are identical approximations.

This relationship is denoted by $(F, A) \widetilde{\subseteq} (G, B)$. Similarly, (F, A) is said to be a soft super set of (G, B), if (G, B) is a soft subset of (F, A). This relationship is denoted by $(F, A) \widetilde{\supseteq} (G, B)$.

Definition 2.3. ([11]) Two soft sets (F, A) and (G, B) over X are said to be soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 2.4. ([11]) The soft intersection of two soft sets (F, A) and (G, B) over X is the soft set (H, C), where $C = A \cap B$ and $\forall e \in C$, $H(e) = F(e) \cap G(e)$. This is denoted by $(F, A) \widetilde{\cap} (G, B) = (H, C)$.

Definition 2.5. ([11])The soft union of two soft sets (F, A) and (G, B) over X is the soft set, where $C = A \cup B$ and $\forall e \in C$,

$$H(e) = \begin{cases} F(e), & \text{if } e \epsilon A - B \\ G(e), & \text{if } e \epsilon B - A \\ F(e) \cup G(e), & \text{if } e \epsilon A \cap B. \end{cases}$$

This relationship is denoted by $(F, A) \widetilde{\cup} (G, B) = (H, C)$.

Definition 2.6. ([11]) A soft set (F, A) over X is said to be a null soft set, denoted by Φ, if for all $e \in A$, $F(e) = \varnothing$(NULL set).

Definition 2.7. ([16]) The complement of a soft set (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, A)$, where $F^c : A \to P(X)$ is a mapping given by $F^c(e) = X \setminus F(e)$ for all $e \in A$.

Definition 2.8. ([16]) Let (F,E) be a soft set over X and $x \in X$. We say that $x \in (F,E)$ read as belongs to the soft set (F,E), whenever $x \in F(e)$ for all $e \in E$.

Note that for any $x \in X$, $x \notin (F,E)$, if $x \notin F(e)$ for some $e \in E$.

Definition 2.9. ([16]) Let Y be a non-empty subset of X, then $\widetilde{Y}$ denotes the soft set (Y,E) over X for which $Y(e) = Y$, for all $e \in E$.

In particular, (X,E) will be denoted by $\widetilde{X}$.

Definition 2.10. ([16]) Let (F,E) be a soft set over X and Y be a non-empty subset of X. Then the soft subset of (F,E) over Y denoted by $\left({}^{Y}F,E\right)$, is defined as follows ${}^{Y}F(e) = Y \cap F(e)$ for all $e \in E$.

In other words $\left({}^{Y}F,E\right) = \widetilde{Y} \cap (F,E)$.

Definition 2.11. ([16]) Let τ be the collection of soft sets over X, then τ is said to be a soft topology on X if

(1) Φ, $\widetilde{X}$ belong to τ
(2) the union of any number of soft sets in τ belongs to τ
(3) the intersection of any two soft sets in τ belongs to τ.

The triplet (X,τ,E) is called a soft topological space over X.

Proposition 2.1. *([16]) Let (X,τ,E) be a soft topological space over X. Then the collection $\tau_e = \{F(e) : (F,E) \in \tau\}$, for each $e \in E$, defines a topology on X.*

Definition 2.12. ([16]) Let (X,τ,E) be a soft topological space over X and (F,E) be a soft set over X . Then the soft closure of (F,E), denoted by $\overline{(F,E)}$, is the intersection of all soft closed super sets of (F,E). Clearly $\overline{(F,E)}$ is the smallest soft closed set over X which contains (F,E).

Definition 2.13. ([5]) Let (F,E) be a soft set over X. The soft set (F,E) is called a soft point, denoted by x_e, if for the element $e \in E$, $F(e) = \{x\}$ and $F(e') = \varnothing$ for all $e' \in E - \{e\}$.

Definition 2.14. ([5]) Two soft points x_e and $y_{e'}$ over a common universe X, we say that the soft points are different points if $x \neq y$ or $e \neq e'$.

Definition 2.15. ([16]) Let (X,τ,E) be a soft topological space over X, (F,E) and (G,E) be two soft closed sets over X such that $(F,E) \cap (G,E) = \Phi$. If there exist soft open sets (F_1,E) and (F_2,E) such that $(F,E) \subset (F_1,E), (G,E) \subset (F_2,E)$ and $(F_1,E) \cap (F_2,E) = \Phi$, then (X,τ,E) is called a soft T_4- space.

3. Tietze Extension Theorem

Let (X,τ) be a topological space. If we consider $\tau = \{U\}$ and $E = \{*\}$, we can define soft set (F_U,E) such that $F_U(*) = U$, for all $U \in \tau$. Then the family $\widetilde{\tau} = \{(F_U,E) : U \in \tau\}$ is a soft topology on X and the soft topological space $(X,\widetilde{\tau},E)$ is the topological space (X,τ). Hence every topological space is considered as a soft topological space depend on single parameter. Here we think (I,τ_I) as a soft topological space with single parameter $\{*\}$.

Definition 3.1. Let (X,τ,E) be a soft topological space over X , (F,E) be a soft closed set and $x_e \notin (F,E)$. If there exists a soft continuous mapping

$$(f,\varphi):(X,\tau,E)\to(I,\tau_I)$$

such that

$$\begin{aligned}(f,\varphi)(x_e) &= f(x)_{\varphi(e)} = 0_* \text{ and}\\ (f,\varphi)(y_{e'}) &= f(y)_{\varphi(e\prime)} = 1_* \text{ for all } y_{e'}\in(F,E),\end{aligned}$$

then (X,τ,E) is called a soft $T_{3\frac{1}{2}}-$ space or completely regular space.

Remark 3.1. It is clear that

$$\varphi:E\to\{*\}$$

is a constant mapping.

Theorem 3.1. *Let (X,τ,E) be a soft topological space over X . If (X,τ,E) is a soft $T_{3\frac{1}{2}}-$ space, then (X,τ_e) is a $T_{3\frac{1}{2}}-$ space for all $e\in E$.*

Proof. Let $B\subset X$ be a closed set in τ_e and $x\notin B$. We consider the point x as a soft point x_e. From the definition of τ_e, there exists a soft closed set (F,E) such that $F(e)=B$ and $x_e\notin(F,E)$. Since (X,τ,E) is a soft $T_{3\frac{1}{2}}-$space, there exists a soft continuous mapping

$$(f,\varphi):(X,\tau,E)\to(I,\tau_I)$$

such that

$$\begin{aligned}(f,\varphi)(x_e) &= f(x)_{\varphi(e)} = 0_{\varphi(e)} = 0_* \text{ and}\\ (f,\varphi)(y_{e'}) &= f(y)_{\varphi(e\prime)} = 1_{\varphi(e\prime)} = 1_*.\end{aligned}$$

Then

$$f:(X,\tau_e)\to(I,\tau_I)$$

is a continuous mapping, for all $e\in E$ and the conditions $f(x)=0$ and $f(y)=1$ are satisfied, for $y\in B$. Thus (X,τ_e) is a $T_{3\frac{1}{2}}-$ space, for all $e\in E$. □

Theorem 3.2. *Let (X,τ,E) be a soft topological space over X and $x_e\in(X,\tau,E)$. Then (X,τ,E) is a soft $T_{3\frac{1}{2}}-$ space if and only if there exists a soft continuous mapping $(f,\varphi):(X,\tau,E)\to(I,\tau_I)$ such that $(f,\varphi)(y_{e'})=1_*$ for all $y_{e'}\in(V,E)^c$ and $(f,\varphi)(x_e)=0_*$, where (V,E) is a soft neighborhood of x_e and (V,E) belongs to soft subbase of x_e.*

Proof. $\Longrightarrow$ It is clear.

$\Longleftarrow$ Let $x_e\in(X,\tau,E)$ and (F,E) be a soft closed set and $x_e\notin(F,E)$. $(F,E)^c$ is a soft open set and $x_e\in(F,E)^c$. From the definition of soft subbase, there exist soft open sets $(V_1,E),(V_2,E),...,(V_n,E)$ such that

$$x_e\in\bigcap_{i=1}^{n}(V_i,E)\subset(F,E)^c .$$

Here (V_i,E) belong to soft subbase of x_e, $1\leq i\leq n$. From the definition of theorem, there exists a soft continuous mapping

$$(f_i,\varphi_i):(X,\tau,E)\to(I,\tau_I)$$

such that $(f_i,\varphi_i)(x_e) = 0_*$, $(f_i,\varphi_i)((V_i,E)^c) = 1_*$. Then we take $f = \max\{f_1,f_2,...,f_n\}$. Since $\varphi = \varphi_i, 1 \leq i \leq n$, then $(f,\varphi)(x_e) = 0_*$ and $(f,\varphi)(y_{e'}) = 1_*$ are satisfied. □

Theorem 3.3. *Let (X,τ,E) be a soft topological space over X and $Y \subset X$. If (X,τ,E) is a soft $T_{3\frac{1}{2}}-$ space, then (Y,τ_Y,E) is a soft $T_{3\frac{1}{2}}-$ space.*

Proof. Let (X,τ,E) be a soft $T_{3\frac{1}{2}}-$ space and (Y,τ_Y,E) be a soft subspace. Let $y_e \in (Y,\tau_Y,E)$ and (F_1,E) be a soft closed set in (Y,τ_Y,E) such that $y_e \notin (F_1,E)$. Since (F_1,E) is a soft closed set in $\widetilde{Y}$, $(F_1,E) = \widetilde{Y} \cap (F,E)$, for some closed set (F,E) in (X,τ,E). It is clear that $y_e \notin (F,E)$. Since (X,τ,E) is a soft $T_{3\frac{1}{2}}-$ space, there exists a soft continuous mapping

$$(f,\varphi) : (X,\tau,E) \to (I,\tau_I)$$

such that $(f,\varphi)(y_e) = 0_*$ and $(f,\varphi)(x_{e'}) = 1_*$, for all $x_{e'} \in (F,E)$. Then the soft mapping

$$(f,\varphi)\mid_{(Y,\tau_Y,E)} : (Y,\tau_Y,E) \to (I,\tau_I)$$

provides conditions of the theorem. So (Y,τ_Y,E) is a soft $T_{3\frac{1}{2}}-$ space. □

Theorem 3.4. *Let $\{(X_s,\tau_s,E_s)\}_{s\in S}$ be a family of soft $T_{3\frac{1}{2}}-$ spaces. Then the topological product of (X_s,τ_s,E_s) is a soft $T_{3\frac{1}{2}}-$space, for $s \in S$.*

Proof. Let $\{(X_s,\tau_s,E_s)\}_{s\in S}$ be a family of soft $T_{3\frac{1}{2}}-$ spaces. We want to show that soft space $(\Pi X_s,\Pi\tau_s,\Pi E_s)$ is a soft $T_{3\frac{1}{2}}-$ space. Let $\{x^s_{e_s}\} \in (\Pi X_s,\Pi\tau_s,\Pi E_s)$ be a soft point and $(V,\Pi E_s) = (p_{s_0},q_{s_0})^{-1}(W_{s_0},E_{s_0})$ be a soft neighborhood of $\{x^s_{e_s}\}$. Here $(W_{s_0},E_{s_0}) \in \tau_{s_0}$ and $p_{s_0} : \Pi X_s \to X_{s_0}$, $q_{s_0} : \Pi E_s \to E_{s_0}$ are soft projections mappings. Since $(X_{s_0},\tau_{s_0},E_{s_0})$ is a soft $T_{3\frac{1}{2}}-$ space, from the Theorem 3.2., there exists a soft continuous mapping

$$(f_{s_0},\varphi_{s_0}) : (X_{s_0},\tau_{s_0},E_{s_0}) \to (I,\tau_I)$$

such that $(f_{s_0},\varphi_{s_0})\left(x^{s_0}_{e_{s_0}}\right) = 0_*$ and $(f_{s_0},\varphi_{s_0})((W_{s_0},E_{s_0}))^c = 1_*$, for the soft point $x^{s_0}_{e_{s_0}} \in (W_{s_0},E_{s_0}) \in \tau_{s_0}$. Then the soft continuous mapping

$$(f,\varphi) = (f_{s_0},\varphi_{s_0}) \circ (p_{s_0},q_{s_0}) : (\Pi X_s,\Pi\tau_s,\Pi E_s) \to (I,\tau_I)$$

is obtained and the following conditions

$$(f,\varphi)\left(\{x^s_{e_s}\}\right) = 0_* \text{ and } (f,\varphi)\left(\{y^s_{e_s}\}\right) = 1_*, \text{ for all } \{y^s_{e_s}\} \in (V,\Pi E_s)^c$$

are satisfied. □

Theorem 3.5. *Let (X,τ,E) be a soft topological space over X. Then (X,τ,E) is a soft T_4-space if and only if for each soft closed set (F,E) and soft open set (G,E) with $(F,E) \subset (G,E)$, there exists soft open set (D,E) such that*

$$(F,E) \subset (D,E) \subset \overline{(D,E)} \subset (G,E).$$

Proof. Let (X,τ,E) be a soft T_4-space, (F,E) be a soft closed set and $(F,E) \subset (G,E)$, $(G,E) \in \tau$. Then $(G,E)^c$ is a soft closed set and $(F,E) \cap (G,E)^c = \Phi$. Since (X,τ,E) is a soft T_4- space, there exist soft open sets (D_1,E) and (D_2,E)

such that $(F,E) \subset (D_1,E)$, $(G,E)^c \subset (D_2,E)$ and $(D_1,E) \cap (D_2,E) = \Phi$. This implies that

$$(F,E) \subset (D_1,E) \subset (D_2,E)^c \subset (G,E).$$

$(D_2,E)^c$ is a soft closed set and $\overline{(D_1,E)} \subset (D_2,E)^c$ is satisfied. Thus

$$(F,E) \subset (D_1,E) \subset \overline{(D_1,E)} \subset (G,E)$$

is obtained. Conversely, let (F_1,E), (F_2,E) be two soft closed sets and $(F_1,E) \cap (F_2,E) = \Phi$. Then $(F_1,E) \subset (F_2,E)^c$. From the condition of theorem, there exists a soft open set (D,E) such that

$$(F_1,E) \subset (D,E) \subset \overline{(D,E)} \subset (F_2,E)^c.$$

So, (D,E), $\overline{(D,E)}^c$ are soft open sets and $(F_1,E) \subset (D,E)$, $(F_2,E) \subset \overline{(D,E)}^c$ and $(D,E) \cap \overline{(D,E)}^c = \Phi$ are obtained. Hence (X,τ,E) is a soft T_4- space.

In the following theorems, we use the concept of a soft point as given in [5]. □

Theorem 3.6. *Let (X,τ,E) be a soft topological space X . If (X,τ,E) is a soft compact T_2- space, then (X,τ,E) is a soft T_4- space.*

Proof. Let (X,τ,E) be a soft T_2- space and and (F_1,E) and (F_2,E) be two soft closed sets and $(F_1,E) \cap (F_2,E) = \Phi$. For the arbitrary soft points $(x_e,E) \in (F_1,E)$ and $(y_{e'},E) \in (F_2,E), (x_e,E) \neq (y_{e'},E)$. Since (X,τ,E) is a soft T_2- space, there exist soft open sets $\left(G_{1_{x_e,y_{e'}}},E\right)$ and $\left(G_{2_{y_{e'},x_e}},E\right)$ such that $(x_e,E) \in \left(G_{1_{x_e,y_{e'}}},E\right)$, $(y_{e'},E) \in \left(G_{2_{y_{e'},x_e}},E\right)$ and $\left(G_{1_{x_e,y_{e'}}},E\right) \cap \left(G_{2_{y_{e'},x_e}},E\right) = \Phi$.

Let (x_e,E) be fixed point and $(y_{e'},E) \in (F_2,E)$ be an arbitrary point. Then $\left\{\left(G_{2_{x_e,y_{e'}}},E\right)\right\}_{(y_{e'},E)\in(F_2,E)}$ is a family of soft open sets and is an soft open cover of (F_2,E). Since (F_2,E) is a soft compact set, there exists finite subfamily of this cover such that $(F_2,E) \subset \bigcup_{i=1}^{n}\left(G_{2_{y_{ie_i},x_e}},E\right)$. Thus $\bigcap_{i=1}^{n}\left(G_{1_{x_e,y_{ie_i}}},E\right)$ is a soft open set and $(x_e,E) \in \bigcap_{i=1}^{n}\left(G_{1_{x_e,y_{ie_i}}},E\right)$. If we take as $(G_{x_e,F_2},E) = \bigcap_{i=1}^{n}\left(G_{1_{x_e,y_{ie_i}}},E\right)$ and $(D_{F_2,x_e},E) = \bigcup_{i=1}^{n}\left(G_{2_{y_{ie_i},x_e}},E\right)$, then $(G_{x_e,F_2},E) \cap (D_{F_2,x_e},E) = \Phi$ is satisfied. Hence there exist soft open sets $(G_{x_e,F_2},E), (D_{F_2,x_e},E)$ in τ such that $(x_e,E) \in (G_{x_e,F_2},E)$, $(F_2,E) \subset (D_{F_2,x_e},E)$. Then the family of soft open sets $\{(G_{x_e,F_2},E)\}_{(x_e,E)\in(F_1,E)}$ is a soft open cover of (F_1,E). Since (F_1,E) is a soft compact set, there exists finite subfamily of this cover such that $(F_1,E) \subset \bigcup_{i=1}^{n}(G_{x_i,e_i},E)$, $(F_2,E) \subset \bigcap_{i=1}^{n}\left(D_{F_2,x_{ie_i}},E\right)$ and $\bigcup_{i=1}^{n}(G_{x_i,e_i},E) \cap \bigcap_{i=1}^{n}\left(D_{F_2,x_{ie_i}},E\right) = \Phi$. This means that (X,τ,E) is a soft compact T_4- space.

Note that, this theorem is not valid if we consider soft point as in [16]. □

Theorem 3.7. *Let (X,τ,E) be a soft topological space over X. If (X,τ,E) is a soft T_4-space and $\widetilde{Y}$ is a soft closed set of X, then (Y,τ_Y,E) is a soft T_4-space.*

Proof. Let (X,τ,E) be a soft T_4-space and $\widetilde{Y}$ be a soft closed set of X. Let (F_1,E) and (F_2,E) be two soft closed sets over $\widetilde{Y}$ such that $(F_1,E) \cap (F_2,E) = \Phi$. Since $\widetilde{Y}$ is a soft closed set, (F_1,E) and (F_2,E) are soft closed sets in X. Since

(X,τ,E) is a soft T_4- space, there exist soft open sets (G_1,E) and (G_2,E) such that $(F_1,E)\subset(G_1,E)$, $(F_2,E)\subset(G_2,E)$ and $(G_1,E)\cap(G_2,E)=\Phi$. Then $(F_1,E)=(G_1,E)\cap\widetilde{Y}$, $(F_2,E)=(G_2,E)\cap\widetilde{Y}$ and $\left((G_1,E)\cap\widetilde{Y}\right)\cap\left((G_2,E)\cap\widetilde{Y}\right)=\Phi$. This implies that (Y,τ_Y,E) is a soft T_4- space. □

Theorem 3.8. *(Urysohn's Lemma) Let (X,τ,E) be a soft topological space over X. (X,τ,E) is a soft T_4- space if and only if there exists a soft continuous mapping*

$$(f,\varphi):(X,\tau,E)\to(I,\tau_I)$$

such that $(f,\varphi)(F,E)=0_*$, $(f,\varphi)(G,E)=1_*$, for every disjoint soft closed sets (F,E) and (G,E).

Proof. We consider soft real numbers set $\{r_*\}_{r\in Q\cap I}$. If $r<r'$, then $r_*<r'_*$. Thus the set $\{r_*\}$ is totally ordered set. Now we arrange the set $\{r_*\}$ as a sequence such that $r_*^1=0_*, r_*^2=1_*, r_*^3, r_*^4$,...

We define soft open sets (V_{r_*},E) for each r_* as provide conditions below

1) If $r_*<r'_*$, then $\overline{(V_{r_*},E)}\subset\left(V_{r'_*},E\right)$

2) $(F,E)\subset(V_{0_*},E)$ and $(G,E)\subset(V_{1_*},E)^c$.

We now define the soft open sets (V_{r_*},E) by inductive method. Since (X,τ,E) is a soft T_4- space, there exist soft open sets $(U,E),(V,E)\in\tau$ such that

$$(F,E)\subset(U,E)\,,\ \ (G,E)\subset(V,E)\ \text{ and }\ (U,E)\cap(V,E)=\Phi$$

for disjoint soft closed sets (F,E) and (G,E). Let $(V_{0_*},E)=(U,E)$ and $(V_{1_*},E)=(G,E)^c$. The soft set (V_{0_*},E) and (V_{1_*},E) provide conditions 1) and 2). Let consider soft sets $\left(V_{r_*^i},E\right)$ that providing condition 1) for $i\le n$. Now we define soft set $\left(V_{r_*^{n+1}},E\right)$. We choose soft numbers r_*^l,r_*^m within $r_1,r_2,...,r_n$ so that r_*^l and r_*^m are chosen the closest to r_*^{n+1}. It is obvious that $r_*^l<r_*^{n+1}<r_*^m$. By using inductive method, we obtain soft open sets $\left(V_{r_*^l},E\right)$, $\left(V_{r_*^m},E\right)$ such that

$$\overline{\left(V_{r_*^l},E\right)}\subset\left(V_{r_*^m},E\right)\ \text{ for } r_*^l<r_*^m.$$

Then there exist soft open sets $(U,E),(V,E)\in\tau$ such that

$$\overline{\left(V_{r_*^l},E\right)}\subset(U,E)\,,\ \ \left(V_{r_*^m},E\right)^c\subset(V,E)\ \text{ and }\ (U,E)\cap(V,E)=\Phi$$

for disjoint soft closed sets $\overline{\left(V_{r_*^l},E\right)}$ and $\left(V_{r_*^m},E\right)^c$. Let $\left(V_{r_*^{n+1}},E\right)=(U,E)$. Hence the soft open sets $\left(V_{r_*^n},E\right)$ that providing conditions 1) and 2) are defined for all r_*^n. We now wish to define the following mappings

$$f:X\to I \text{ and } \varphi:E\to\{*\}\,.$$

Here $\varphi(e)=*$ for all $e\in E$ and

$$(f,\varphi)(x_e)=\begin{cases}\inf\limits_r\{r_*:x_e\in(V_{r_*},E)\}\,, & \text{if } x_e\in(V_{1_*},E)\\ 1_*, & \text{if } x_e\in(V_{1_*},E)^c\,.\end{cases}$$

It is obvious that $(f,\varphi)(F,E)=0_*$ and $(f,\varphi)(G,E)=1_*$. □

Finally, we show that (f,φ) is a soft continuous mapping. Since

$$\begin{aligned}(f,\varphi)^{-1}([0,a),*) &= \{x_e \in (X,\tau,E):\ (f,\varphi)(x_e) < a_*\} \\ &= \bigcup \{(V_{r_*},E) : r_* < a_*\}\end{aligned}$$

and

$$\begin{aligned}(f,\varphi)^{-1}((b,1],*) &= \{x_e \in (X,\tau,E):\ (f,\varphi)(x_e) > b_*\} \\ &= \bigcup \left\{\overline{(V_{r_*},E)^c} : r_* > b_*\right\}\end{aligned}$$

are soft open sets, and (f,φ) is a soft continuous mapping.

Conversely, let (F,E) and (G,E) be disjoint soft closed sets. From the condition of the theorem, there exists a soft continuous mapping

$$(f,\varphi):(X,\tau,E)\to(I,\tau_I)$$

such that $(f,\varphi)(F,E)=0_*, (f,\varphi)(G,E)=1_*$. Then $(U,E)=(f,\varphi)^{-1}\left(\left[0,\frac{1}{2}\right),*\right)$, $(V,E)=(f,\varphi)^{-1}\left(\left(\frac{1}{2},1\right],*\right)$ are disjoint soft open sets and $(F,E)\subset(U,E)$,$(G,E)\subset(V,E)$. So (X,τ,E) is a soft T_4- space.

Corollary 3.1. *Let (X,τ,E) be a soft T_4-space, then it is also soft $T_{3\frac{1}{2}}-$ space.*

Proof. Let $x_e \in (X,\tau,E)$ and (F,E) be a soft closed set such that $x_e \notin (F,E)$. Since the soft point $\{x_e\}$ is a soft closed set in soft T_4- space, from the Urysohn Lemma, there exists a soft continuous mapping

$$(f,\varphi):(X,\tau,E)\to(I,\tau_I)$$

such that $(f,\varphi)(x_e)=0_*, (f,\varphi)(F,E)=1_*$. So (X,τ,E) is a soft $T_{3\frac{1}{2}}-$ space. □

Theorem 3.9. *Let (X,τ,E) be a soft local compact T_2- space. Then (X,τ,E) is a soft $T_{3\frac{1}{2}}-$ space.*

Proof. Let (X,τ,E) be a soft local compact T_2- space, $x_e \in (X,\tau,E)$ be a soft point and (F,E) be a soft closed set and $x_e \notin (F,E)$. Since (X,τ,E) is a soft local compact space, there exists a soft compact neighborhood $\overline{(U,E)}$ of x_e. Then the soft set

$$(F_0,E)=\left(\overline{(U,E)}\setminus(U,E)\right)\cup\left(\overline{(U,E)}\cap(F,E)\right)$$

is a soft closed subset of $\overline{(U,E)}^c$.

It is clear that $x_e \in \overline{(U,E)}$ and $x_e \notin (F_0,E)$. The soft set $\overline{(U,E)}$ is a soft T_4- space. So it is also soft $T_{3\frac{1}{2}}-$ space. For $x_e \notin (F_0,E)$, there exists a soft continuous mapping

$$(f_1,\varphi):\overline{(U,E)}\to(I,*)$$

such that

$$(f_1,\varphi)(x_e)=0_* \text{ and } (f_1,\varphi)(F_0,E)=1_*.$$

Now we define the following soft mapping in the soft set $\widetilde{X}\setminus(U,E)$ by

$$(f_2,\varphi):\ \widetilde{X}\setminus(U,E)\to(I,*)$$

such that $(f_2,\varphi)(y_e)=1_*$, for all $y_e\in\widetilde{X}\setminus(U,E)$. Hence we give the soft mapping

$$(f,\varphi):(X,\tau,E)\to(I,*)$$

by

$$(f,\varphi)(x_e)=\begin{cases}(f_1,\varphi)(x_e), & \text{if } x_e\in\overline{(U,E)}\\ (f_2,\varphi)(x_e), & \text{if } x_e\notin(U,E).\end{cases}$$

It is clear that the soft mapping (f,φ) is a soft continuous mapping and $(f,\varphi)(x_e)=0_*$ and $(f,\varphi)(F,E)=1_*$ is obtained. Thus (X,τ,E) is a soft $T_{3\frac{1}{2}}-$ space. □

Theorem 3.10. *(Tietze's Extension Theorem) Let (X,τ,E) be a soft topological space. (X,τ,E) is a soft T_4- space if and only if there exists a soft continuous mapping*

$$(F,\varphi):(X,\tau,E)\to(R,\tau_R)$$

such that $(F,\varphi)\mid_{(B,E)}=(f,\varphi)$, for each soft closed set (B,E) and soft continuous mapping $(f,\varphi):\left((B,E),\tau_{(B,E)},E\right)\to(R,\tau_R)$.

Proof. We firstly prove this theorem for soft continuous mapping $(f,\varphi):\left((B,E),\tau_{(B,E)},E\right)\to([-1,1],*)$. Let us consider disjoint soft closed subsets

$$\begin{aligned}(A_1,E)&=(f,\varphi)^{-1}\left(\left[\frac{1}{3},1\right],*\right)\subset(B,E)\\(B_1,E)&=(f,\varphi)^{-1}\left(\left[-1,-\frac{1}{3}\right],*\right)\subset(B,E).\end{aligned}$$

From the Uryshon's Lemma, there exists a soft continuous mapping

$$(f_1,\varphi):(X,\tau,E)\to\left(\left[-\frac{1}{3},\frac{1}{3}\right],*\right)$$

such that $(f_1,\varphi)(A_1,E)=\left(\frac{1}{3}\right)_*,(f_1,\varphi)(B_1,E)=-\left(\frac{1}{3}\right)_*$. Since $|f(x)-f_1(x)|\leq\frac{2}{3}$, then $(f-f_1,\varphi)(B,E)=\left(\left[-\frac{2}{3},\frac{2}{3}\right],*\right)$. If we take $(f-f_1,\varphi)$ instead of (f,φ), there exists a soft continuous mapping

$$(f_2,\varphi):(X,\tau,E)\to\left(\left[-\frac{2}{9},\frac{2}{9}\right],*\right)$$

such that $(f_2,\varphi)(A_2,E)=\left(\frac{2}{9}\right)_*,(f_2,\varphi)(B_2,E)=-\left(\frac{2}{9}\right)_*$, for soft disjoint closed sets

$$\begin{aligned}(A_2,E)&=(f-f_1,\varphi)^{-1}\left(\left[\frac{2}{9},\frac{2}{3}\right],*\right)\\(B_2,E)&=(f-f_1,\varphi)^{-1}\left(\left[-\frac{2}{3},\frac{-2}{9}\right],*\right).\end{aligned}$$

It is obvious that

$$|(f-f_1)-f_2|\leq\left(\frac{2}{3}\right)^2.$$

Thus we obtain soft continuous mappings (f_k,φ) on (B,E) such that $\left|f-\sum_{k=1}^{n}f_k\right|\leq\left(\frac{2}{3}\right)^n$, and $|f_k(x)|\leq\left(\frac{2}{3}\right)^{k-1}\frac{1}{3}$ is satisfied. Since the series $\sum_{k=1}^{\infty}f_k(x)$ is a uniform convergence, (F,φ) is a soft continuous for the mapping $F(x)=\sum_{k=1}^{\infty}f_k(x)$ and $(F,\varphi)\mid_{(B,E)}=(f,\varphi)$ is obtained.

Now we prove the theorem for the soft mapping

$$(f,\varphi):(B,E)\to((-1,1),*).$$

Suppose that

$$(f,\varphi):(B,E)\to((-1,1),*)\subset([-1,1],*).$$

Then there exists a soft extension mapping

$$(F^*,\varphi):(X,\tau,E)\to([-1,1],*)$$

of the soft mapping (f,φ). Define

$$(A_0,E)=\left\{x_e\in(X,\tau,E):(F^*,\varphi)(x_e)=\bar{+}1_*\right\}.$$

From the Uryshon Lemma, there exists a soft continuous mapping

$$(g,\varphi):(X,\tau,E)\to(I,*)$$

such that $(g,\varphi)(A_0,E)=0_*,(g,\varphi)(B,E)=1_*$, for disjoint soft closed sets $(B,E),(A_0,E)$. Then $(F,\varphi)(x_e)=(g,\varphi)(x_e)\circ(F^*,\varphi)(x_e)$ and the soft mapping (F,φ) is an extension of (f,φ).

Conversely let (A,E) and (B,E) be disjoint soft closed sets. We define the following soft mapping

$$(f,\varphi):(A,E)\,\widetilde{\cup}\,(B,E)\to(I,*)$$

such that $(f,\varphi)(A,E)=0_*,(f,\varphi)(B,E)=1_*$. It is clear that (f,φ) is a soft continuous mapping and

$$(F,\varphi):(X,\tau,E)\to(I,*)$$

is an extension of (f,φ). Also $(F,\varphi)(A,E)=0_*,(F,\varphi)(B,E)=1_*$ is satisfied. So the proof is completed from Uryshon Lemma. □

Remark 3.2. Under the conditions of the Tietze theorem, let us take the mapping $f:B(e)\to(R,\tau)$ for each $e\in E$. Since (X,τ_e) is a normal topological space, there exists an extension mapping $F_e:(X,\tau_e)\to(R,\tau)$ of f. But the family $\{F_e:(X,\tau_e)\to(R,\tau)\}_{e\in E}$ doesn't determine any extension on (X,τ,E). Conversely, if there exists an extension (F,φ) of the soft mapping $(f,\varphi):(B,E)\to(R,\tau)$ then $F_e:(X,\tau_e)\to(R,\tau)$ is an extension of the continuous mapping $f:B(e)\to(R,\tau)$, for each $e\in E$.

Conclusion. We have introduced soft $T_{3\frac{1}{2}}-$space based on the soft point in [5] and investigated its important properties. Then we have given Uryshon's Lemma and proved Tietze Extension Theorem using Uryshon's Lemma.

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