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ON ϕ–SOLVABILITY OF A CLASS OF BOUNDARY VALUE PROBLEMS FOR AN OPERATOR-DIFFERENTIAL EQUATION IN HILBERT SPACE

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Abstract. The conditions that provide ϕ– solvability of some nonlocal boundary value problem for third order operator-differential equations in Hilbert space are given.

1. Introduction

In a separable Hilbert space H let us consider the boundary value problem

$$P\left(\frac{d}{dt}\right) u\left(t\right)=\frac{d^3u}{dt^3}+\sum_{j=0}^{2}A_{3-j}\ u^{(j)}\left(t\right)+A^3u(t)=f\left(t\right)\ ,\quad t\in\left(0;\infty\right), \quad (1.1)$$

$$u\left(0\right)=Su, \quad (1.2)$$

where $A,\ A_j\ \left(j=\overline{1,3}\right)$ and S are linear operators, $u\left(t\right)$, $f\left(t\right)$ are vector-functions determined in $R_+=\left(0,\ \infty\right)$ almost everywhere, with the values in H.

Suppose that A is a positive-definite self-adjoint operator in H with domain of definition $D\left(A\right)$. Denote by H_γ $\left(\gamma\geq 0\right)$ a Hilbert space with the scalar product: $\left(x,y\right)_\gamma=\left(A^\gamma x,\ A^\gamma y\right)$. For $\gamma=0$ we assume that $H_0=H$. Let $L_2\left(R_+;H\right)$ be a Hilbert space of all vector-functions $f\left(t\right)$ determined in $R_+=\left(0,\ \infty\right)$ almost everywhere with the values in H with the norm

$$\left\|f\right\|_{L_2(R_+:H)}=\int_0^\infty\left\|f\left(t\right)\right\|^2dt<\infty.$$

Following the monograph [6], let us introduce the Hilbert space

$$\mathrm{W}_2^3\left(R_+:H\right)=\left\{u\left(t\right)\ ;\ u'''\in L_2\left(R_+:H\right),\ A^3u\in\left(R_+:H\right)\ \right\}$$

with the norm

$$\left\|u\right\|_{W_2^3(R_+:H)}=\left(\left\|u'''\right\|^2_{L_2(R_+:H)}+\left\|A^3u\right\|^2_{L_2(R_+:H)}\right)^{1/2}.$$

In what follows, by $L\left(X,\ Y\right)$ we will denote Banach space of all linear bounded operators acting from Hilbert space X to Hilbert space Y.

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Assume that the following conditions are fulfilled:

1) A is a positive-definite self-adjoint operator with completely continuous inverse A^{-1};

2) $S \in L\left(W_2^3\left(R_+ : H\right), H_{5/2}\right)$;

3) The operators $B_j = A_j \cdot A^{-j}$ are completely continuous in H $\left(j = \overline{1,\ 3}\right)$.

Denote

$$P_0 u = P_0\left(d/dt\right)\ u\left(t\right) = \frac{d^3 u}{dt^3} + A^3\ u\ , u \in W_{2,S}^3\left(R_+; H\right),$$

$$P_1 u = \sum_{j=0}^{2} A_{3-j} u^{(j)}\left(t\right)\ ,\quad Pu = \left(P_0 + P_1\right) u\ ;\quad u \in W_{2,S}^3\left(R_+; H\right),$$

where

$$W_{2,\ S}^3\left(R_+ : H\right) = \left\{u\ : u \in W_2^3\left(R_+ : H\right);\ u\left(0\right) = Su\right\}.$$

Definition 1.1. Problem (1.1), (1.2) is said to be ϕ – solvable if the operator P has a finite kernel and co-dimension in the spaces $W_{2,\ S}^3\left(R_+ : H\right)$ and $L_2\left(R_+ : H\right)$ respectively, and the image P is closed in the space $L_2\left(R_+ : H\right)$.

In the present paper we will find conditions on the coefficients of the boundary value problem, that provide ϕ – solvability of problem (1.1), (1.2). Note that for $S = 0$ this problem was studied in [5]. The conditions of regular solvability of problem (1.1), (1.2), i.e. when $Ker\ P = \{0\}$ and $Im\ P = L_2\left(R_+ : H\right)$ were found in the paper [1]-[4], [8], [9].

Let us first study the boundary value problem

$$P_0\left(d/dt\right)\ u\left(t\right) = \frac{d^3 u\left(t\right)}{dt^3} + A^3\ u\left(t\right) = f\left(t\right),\ t \in R_+ = \left(0,\ \infty\right) \qquad (1.3)$$

$$u\left(0\right) = Su. \qquad (1.4)$$

The following theorem holds.

Theorem 1.1. *Let conditions 1) and 2) be fulfilled, and*

$$\kappa = \|S\|_{W_2^3(R_+:H)\to H_{5/2}} < 1\ .$$

Then problem (1.3), (1.4) *has a unique solution* $u\left(t\right) \in W_{2,S}^3\left(R_+ : H\right)$ *for any* $f\left(t\right) \in L_2\left(R_+ : H\right)$, *and* $u\left(t\right)$ *has the following representation*

$$u\left(t\right) = \int_0^\infty Q\left(t-s\right)\ f\left(s\right)\ ds + e^{-tA}\ \left(S\ e^{-tA} - E\right)^{-1} \int_0^\infty Q\left(-s\right)\ f\left(s\right)\ ds$$

$$+ S \int_0^\infty Q\left(t-s\right)\ f\left(s\right)\ ds\ , \qquad (1.5)$$

where

$$Q\left(t-s\right) = \begin{cases} \frac{1}{3}\ e^{A(t-s)}\ A^{-2}, & t > s\ , \\ \left(\frac{1}{6} + \frac{\sqrt{3}}{6}\ i\right) e^{\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\ i\right)\ (t-s)}\ A- & \\ -\left(-\frac{1}{6} + \frac{\sqrt{3}}{6}\ i\right)\ e^{\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\ i\right)\ (t-s)} A^{-2}, & t < s. \end{cases}$$

Proof. Solvability of problem (1.3), (1.4) provided $\kappa < 1$ was proved in the paper [3], the representation (1.5) follows from the formula

$$u(t) = \frac{1}{2\pi} \int_0^{\infty} \left(\int_{-\infty}^{+\infty} \left(i^3\ \xi^3\ E + A^3\right)^{-1} e^{i\ (3-t)\ s} \right) f(s)\ ds + e^{-tA}x, \quad x \in H_{5/2}\ .$$

It is easy to see that

$$u(t) = \int_0^{\infty} Q(t-s)\ f(s)\ ds + e^{-tA}\ x\ , \qquad x \in H_{5/2}\ ,$$

and from condition (1.4) it follows that

$$\int_0^{\infty} Q(-s)\ f(s)\ ds + x = S\ \left(\int_0^{\infty} Q(t-s)\ f(s)\ ds \right) + S\ \left(e^{-tA}\ x\right),$$

since

$$x = \left(S\ e^{-tA} - E\right)^{-1} \left[\int_0^{\infty} Q(-s)\ f(s)\ ds + S \int_0^{\infty} Q(t-s)\ f(s)\ ds \right].$$

Note that here the operator $S\ e^{-tA} - E$ is invertible in $H_{5/2}$ since from the results of the paper [8] it follows that

$$\left\|S\ e^{-tA}\ x\right\|_{5/2} \le \kappa \cdot \left\|e^{-tA}\ x\right\|_{W_2^3(R_+:\ H)} \le \kappa\ \|x\|_{5/2}\,(\kappa < 1)\,.$$

The theorem is proved. □

Now prove a theorem on ϕ – solvability of problem (1.1), (1.2).

Theorem 1.2. *Let conditions 1)-3) be fulfilled, and $\kappa < 1$, on the imaginary axis there exist the resolvent*

$$P^{-1}(\lambda) = \left(\lambda^3\ E + A^3 + \lambda^2\ A_1 + \lambda\ A_2 + A_3\right)^{-1}.$$

Then problem (1.1), (1.2) *is ϕ –solvable.*

Proof. Assuming $\frac{d^3\ u}{dt^3} + A^3 u = v$ in equation (1.1) and using representation (1.5), we get the following integro-differential equation in $L_2(R_+ : H)$ with respect to $v(t)$

$$v(t) + \left(\sum_{j=0}^{2} A_{3-j} \frac{d^j}{dt^j} \right) \int_0^{\infty} Q(t-s)\ v(s)\ ds$$

$$+ \sum_{j=0}^{2} A_{3-j}\ \frac{d^j}{dt^j}\ e^{-tA} \left[\left(Se^{-tA} - E\right)^{-1} \int_0^{\infty} Q(-s)\, v(s)\ ds + S \int_0^{\infty} Q(t-s)\, v(s)\ ds \right].$$

Prove that the second addend is a completely continuous operator in $L_2(R_+ : H)$. Introduce the following operators

$$K_j = A_{3-j} \frac{d^j}{dt^j}\ e^{-tA} \left[\left(Se^{-tA} - E\right)^{-1} \int_0^{\infty} Q(-s)\, v(s)\ ds + S \int_0^{\infty} Q(t-s)\, f(s)\ ds \right]$$

$$= B_{3-j}(-1)^j\, A^3 e^{-tA} \left[\left(Se^{-tA} - E\right)^{-1} \int_0^\infty Q(-s)\, f(s)\, ds + S \int_0^\infty Q(t-s)\, f(s)\, ds \right]$$

$$= B_{3-j} A^3 e^{-tA} \left[\left(Se^{-tA} - E\right)^{-1} \int_0^\infty d_j e^{-\frac{1}{2} sA} f(s)\, ds + S \int_0^\infty Q(t-s)\, v(s)\, ds \right],$$

where $d_j = \frac{1}{3}(-1)^j$.

Show that every addend is a completely continuous operator in $L_2(R_+ : H)$. Denote

$$\tilde{K}_j^{(1)}\, v = A^3 e^{-tA}\left(Se^{-tA} - E\right)^{-1} \int_0^\infty d_j e^{-\frac{1}{2}\ sA}\ v(s)\ ds$$

$$+S \int_0^\infty Q(t-s)\, v(s)\, ds$$

and show that it is bounded in $L_2(R_+ : H)$. Since for any $x \in H_{5/2}$ we have $\left\|A^3 e^{-tA}\ x\right\| \leq \frac{1}{\sqrt{2}}\ \|x\|_{5/2}$ [8], then

$$\left\|\tilde{K}_j^{(1)}\ v\right\|_{L_2(R_+:H)} \leq \left\|\left(Se^{-tA} - E\right)^{-1}\right\|_{H_{5/2}\to H_{5/2}} \left\| \int_0^\infty d_j e^{-\frac{1}{2}\ SA}\ A^{-2} v(s)\ ds \right\|_{5/2}$$

$$+\kappa \left\| \int_0^\infty Q(t-s)\, v(s)\, ds \right\|_{5/2} \leq const\ \|v\|_{L_2(R_+:H)}.$$

Further, denote by L_m an orthoprojector on the first m eigen vectors $\{\varphi_1,\ \ldots,\ \varphi_m\}$ of the operator A responding to eigen values $\lambda_1,\ \ldots,\ \lambda_m$. Then it is known that as $m \to \infty$

$$\|Q_{j,\ m}\| = \|B_{3-j} - B_{3-j}\ L_m\| \to 0\ .$$

On the other hand, the operator

$$B_{3-j}\ L_m\ \tilde{K}_j v = \sum_{k=1}^m \lambda_k^{1/2}\ d_j\ e^{-\frac{1}{2}\ \lambda_k\ s}\left(S\ e^{-tA} - E\right)^{-1} \int_0^\infty e^{-\frac{1}{2}\ sA}\ \left(v(s),\ \varphi_k\right)\ ds$$

$$+ \left(S \int_0^\infty Q(t-s)\ \ v(s),\ \varphi_k \right)\ ds$$

is a finite-dimensional operator in $L_2(R_+; H)$. Then

$$\left\|K_j\ v - B_{3-j}\ L_m\ \tilde{K}_j\ v\right\|_{L_2(R_+:H)} = \left\|B_{3-j}\ \tilde{K}_j\ v - B_{3-j}\ L_m\ \tilde{K}_j\ v\right\|_{L_2(R_+:H)}$$

$$\leq const\ \ \|Q_{m,\ j}\| \cdot \|v\|_{L_2(R_+:H)}\ .$$

Hence it follows that K_j is a uniform limit of finite-dimensional operators, and therefore it is completely continuous in $L_2(R_+ : H)$. Thus, for completing the proof it suffices to prove that the integro-differential equation

$$v(t) + \left(\sum_{j=0}^{2} B_{3-j}\ A^{3-j}\ \frac{d^j}{dt^j}\right) \int_0^\infty Q(t-s)\ v(s)\ ds = f(t), \quad t \in R_+ \tag{1.6}$$

is ϕ- solvable in space $L_2(R_+ : H)$. Denote

$$V(t) = \begin{cases} v(t), & t > 0, \\ v(-t), & t < 0, \end{cases} \qquad F(t) = \begin{cases} f(t), & t > 0 \\ f(-t), & t < 0 \end{cases}$$

and consider in $L_2(R_+ : H) = L_2(R_+ : H) \oplus L_2(R_+ : H)$ the integro-differential equation

$$V(t) = \left(\sum_{j=0}^{2} B_{3-j}\ A^{3-j}\ \frac{d^j}{dt^j}\right) \int_0^\infty Q(t-s)\ V(s)\,ds = F(t), \quad t \in R, \tag{1.7}$$

that is equivalent to the system of equations

$$v(t) + \left(\sum_{j=0}^{2} B_{3-j}\ A^{3-j}\ \frac{d^j}{dt^j}\right) \int_0^\infty Q(t-s)\ v(s)\ ds$$

$$+ \left(\sum_{j=0}^{2} B_{3-j}\ A^{3-j}\ \frac{d^j}{dt^j}\right) \int_0^\infty e^{-(t+s)A}\ v_1(s)\ ds = f(t),$$

$$v_1(t) + \sum_{j=0}^{2} B_{3-j}\ A^{3-j}\ \frac{d^j}{dt^j} \int_0^\infty Q(-t-s)\ v_1(s)\ ds$$

$$+\ell_1 \int_0^\infty e^{-\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\,i\right)\,(t+s)A} f(s)\ ds + \ell_2 \int_0^\infty e^{-\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\,i\right)\,(t+s)A}\ v(s)\ ds = f_1(t),$$

where $\ell_0,\ \ \ell_1,\ \ \ell_2$ are constant numbers. Let us write the system in the operator form

$$\begin{pmatrix} E+L_1 & T_1 \\ T_2 & E+L_2 \end{pmatrix} V = \begin{pmatrix} E+L_1 & 0 \\ 0 & E+L_2 \end{pmatrix} V + \begin{pmatrix} 0 & T_1 \\ T_2 & 0 \end{pmatrix} V = F,$$

where

$$L_1 = \sum_{j=0}^{2} B_{3-j}\ A^{3-j}\ \frac{d^j}{dt^j} \int_0^\infty Q(t-s)\ v(s)\ ds,$$

$$T_1\ v_1 = \alpha_0 \sum_{j=0}^{2} B_{3-j}\ A^{3-j}\ \frac{d^j}{dt^j} \int_0^\infty e^{-(t+s)A}\ v_1(s)\ ds,$$

$$T_1\ v = \ell_1 \int_0^\infty e^{-\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\,i\right)\,(t+s)A}\ v(s)\ ds + \ell_2 \int_0^\infty e^{-\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\,i\right)\,(t+s)A}\ v(s)\ ds,$$

$$L_2\ v_1 = \sum_{j=0}^{2} B_{3-j}\ A^{3-j}\ \frac{d^j}{dt^j}\ \int_0^\infty Q(-t-s)\ v_1(s)\ ds.$$

Show that the equation

$$\begin{pmatrix} E+L_1 & 0 \\ 0 & E+L_2 \end{pmatrix} V = F.$$

is solvable for all $F \in L_2(R\ : H)$. Indeed, after Fourier transformation we have

$$\hat{V}(\xi) = P_0(i\xi)\ P^{-1}(i\xi)\ \hat{F}(\xi), \qquad \xi \in R\ .$$

As A^{-1} is a completely continuous operator, $P^{-1}(\lambda)$ exists for $\lambda = is$, then from the equality

$$\left\|P_0(i\xi)\ P^{-1}(i\xi)\right\|_{H\to H} = \left\|E + \sum_{j=0}^{2} B_{3-j}\ A^{3-j}\ \left((i\xi)_E^3 + A^3\right)^{-1}\right\|_H$$

and from the Keldysh lemma [8] it follows that

$$\sup_{\xi\in R}\left\|P_0(i\xi)\ P^{-1}(i\xi)\right\| \le const$$

and therefore $V(t) \in L_2(R : H)$. For completing the proof it suffices to prove that the operators T_1 and T_2 are completely continuous in $L_2(R_+ : H)$.

As

$$\left\|A^{3-j}\ \frac{d^j}{dt^j}\ e^{-(t+s)\ A}\ A^{-2}\right\|_{H\to H} \le const\ (t+s)^{-1},$$

$$\left\|A^{3-j}\ \frac{d^j}{dt^j}\ e^{-\left(\frac{1}{2}\pm\frac{\sqrt{3}}{2}\ i\right)(t+s)\ A}\ A^{-2}\right\|_{H\to H} \le const\ (t+s)^{-1},$$

then the operators, by the Hilbert inequality

$$R_j\ v = A^{3-j}\ \frac{d^j}{dt^j}\int_0^\infty e^{-(t+s)\ A}\ A^{-2}\ v(s)\ ds\ ,$$

$$R_j^{\pm}\ v = A^{3-j}\ \frac{d^j}{dt^j}\int_0^\infty e^{-\left(\frac{1}{2}\pm\frac{\sqrt{3}}{2}\ i\right)\ (t+s)\ A}\ A^{-2}\ v(s)\ ds$$

are bounded in $L_2(R_+ : H)$.

On the other hand, the operators

$$B_{3-j}\ L_m\ R_j\ v = (-1)^j\ \sum_{k=1}^{m}\int_0^\infty \lambda_k\ e^{-\lambda_k(t+s)}\ (v(s),\ \varphi_k)\ ds\ B_{3-j}\ \varphi_k$$

are also completely continuous in $L_2(R_+ : H)$. Then from the inequality

$$\|B_{3-j}\ R_j\ v - B_{3-j}\ L_m\ R_j\ v\| \le \|Q_{j,\ m}\| \cdot \|R_j\|_{L_2\to L_2} \cdot \|v\|_{L_2(R_+:H)}$$

it follows that $B_{3-j}\ R_j$ is completely continuous in $L_2(R_+ : H)$.

It is similarly proved that the operators $B_{3-j}\ R_j^{\pm}$ are also completely continuous in $L_2(R_+ : H)$.

The theorem is proved. □

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