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TRAVELLING-WAVE SOLUTIONS FOR KLEIN-GORDON AND HELMHOLTZ EQUATIONS ON CANTOR SETS

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Abstract. In the present paper, a class of the partial differential equations (PDEs) on Cantor sets is investigated for the first time. The travelling-wave solutions for the Klein-Gordon and Helmholtz equations on Cantor sets are graphically discussed. The travelling-wave transformation technology is accurate and efficient for finding the exact solutions of the PDEs in mathematical physics.

1. Introduction

Local fractional calculus (LFC) has been successful as one of important mathematical tools to describe the complexities and non-differentiability of the physical phenomena in nature (see [1,4,6,7] and the cited references therein). For example, the fractal Burgers' equation (FBE) in a class of fluid flows was proposed in [9]. The fractional Korteweg-de Vries equation (FKdV) on non-differentiable shallow water surface was developed in [8]. The fractal heat-transfer equations (FHE) were reported in [3,10]. The fractional Tricomi equation (FTE) in the transonic flow was demonstrated in [5].

The travelling-wave transformation of the non-differentiable type was for the first time proposed to find the exact solutions for the FKdV (see, for example, [8]). However, the travelling-wave solutions for the linear partial differential equations (PDEs) have not reported. Motivated by the idea, this study aims to find travelling-wave solutions for the linear Klein-Gordon equation (KGE) and Helmholtz equation (HE) within local fractional derivative (LFD).

The structures of this paper are arranged as follows. In Section 2, the concept and properties of the LFD are given. In Section 3, the travelling-wave transformation of non-differentiable type is proposed. In Section 4, the travelling-wave solutions for a class of local fractional PDEs are presented. Finally, the conclusion is outlined in Section 5.

2. Theory of LFD

Let $C_\vartheta(a,\, b)$ be a set of the non-differentiable functions (see, for example, [6,7]).

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Let $\Lambda_\vartheta(x) \in C_\vartheta(a, b)$. The LFD of $\Lambda_\vartheta(x)$ of order ϑ $(0 < \vartheta < 1)$ at the point $x = x_0$ is defined by (see, for example, [6,7]):

$$D^{(\vartheta)}\Lambda_\vartheta(x_0) = \frac{d^\vartheta \Lambda_\vartheta(x_0)}{dx^\vartheta} = \lim_{x\to x_0} \frac{\Delta^\vartheta(\Lambda_\vartheta(x) - \Lambda_\vartheta(x_0))}{(x - x_0)^\vartheta}, \tag{2.1}$$

where

$$\Delta^\vartheta(\Lambda_\vartheta(x) - \Lambda_\vartheta(x_0)) \cong \Gamma(1+\vartheta)[\Lambda_\vartheta(x) - \Lambda_\vartheta(x_0)]. \tag{2.2}$$

Let $\Lambda_\vartheta(x_0), \Pi_\vartheta(x_0) \in C_\vartheta(a, b)$. The properties of the LFD are listed as follows (see, for example, [7]) :

(a) $D^{(\vartheta)}[\Lambda_\vartheta(x_0) \pm \Pi_\vartheta(x_0)] = D^{(\vartheta)}\Lambda_\vartheta(x_0) \pm D^{(\vartheta)}\Pi_\vartheta(x_0)$;

(b) $D^{(\vartheta)}[\Lambda_\vartheta(x_0)\Pi_\vartheta(x_0)] = \Pi_\vartheta(x_0)D^{(\vartheta)}\Lambda_\vartheta(x_0) + \Lambda_\vartheta(x_0)D^{(\vartheta)}\Pi_\vartheta(x_0)$;

(c) $D^{(\vartheta)}[\Lambda_\vartheta(x_0)/\Pi_\vartheta(x_0)] = \left\{\Pi_\vartheta(x_0)D^{(\vartheta)}\Lambda_\vartheta(x_0) + \Lambda_\vartheta(x_0)D^{(\vartheta)}\Pi_\vartheta(x_0)\right\}/\Pi_\vartheta^2(x_0)$, provided that $\Pi_\vartheta(x_0) \neq 0$.

The special functions on Cantor sets are as (see, for example, [7]):

$$M_\vartheta\left(x^\vartheta\right) = \sum_{n=0}^{\infty} \frac{x^{n\vartheta}}{\Gamma(1+n\vartheta)}, \tag{2.3}$$

$$\sin_\vartheta\left(x^\vartheta\right) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)\vartheta}}{\Gamma(1+(2n+1)\vartheta)} \tag{2.4}$$

$$\cos_\vartheta\left(x^\vartheta\right) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n\vartheta}}{\Gamma(1+2n\vartheta)}. \tag{2.5}$$

Useful formulas of the LFD of non-differentiable functions (see, e.g., [7]) are listed in Table 1.

Table 1. The basic formulas of the LFD

Non-differentiable functions	**LFDs**
$M_\vartheta\left(x^\vartheta\right)$	$M_\vartheta\left(x^\vartheta\right)$
$\sin_\vartheta\left(x^\vartheta\right)$	$\cos_\vartheta\left(x^\vartheta\right)$
$\cos_\vartheta\left(x^\vartheta\right)$	$-\sin_\vartheta\left(x^\vartheta\right)$

3. Travelling-wave transformation technology applied

In this section, we present the travelling-wave transformation technology for finding the linear local fractional PDEs.

Let us consider the local fractional PDE in the form:

$$\Omega\left(\frac{\partial^{2\vartheta}\mathrm{T}(x,t)}{\partial x^{2\vartheta}}, \frac{\partial^{2\vartheta}\mathrm{T}(x,t)}{\partial t^{2\vartheta}}, \frac{\partial^{\vartheta}\mathrm{T}(x,t)}{\partial x^{\vartheta}}, \frac{\partial^{\vartheta}\mathrm{T}(x,t)}{\partial t^{\vartheta}}, \mathrm{T}(x,t)\right) = 0, \tag{3.1}$$

where both $\partial^{2\vartheta}\mathrm{T}(x,t)/\partial x^{2\vartheta}$ and $\partial^{2\vartheta}\mathrm{T}(x,t)/\partial t^{2\vartheta}$ are the local fractional partial derivatives (LFPDs) of 2ϑ order with respect to x and t, respectively, and both $\partial^{\vartheta}\mathrm{T}(x,t)/\partial x^{\vartheta}$ and $\partial^{\vartheta}\mathrm{T}(x,t)/\partial t^{\vartheta}$ are the LFPDs of ϑ order with respect to x and t, respectively.

Traveling wave transformation of non-differentiability is defined by:

$$\psi^\vartheta = x^\vartheta - \mu^\vartheta t^\vartheta, \tag{3.2}$$

where

$$\lim_{\vartheta\to 1}\psi = x - \mu t. \tag{3.3}$$

From Eq.(3.2) and Eq.(3.3), we have the following:

$$\mathrm{T}(x,t) = \mathrm{T}(\psi). \tag{3.4}$$

Making use of the chain rule of the LFD, we have

$$\frac{\partial^{\vartheta}\mathrm{T}(x,t)}{\partial t^{\vartheta}} = \frac{\partial^{\vartheta}\mathrm{T}(\psi)}{\partial \psi^{\vartheta}}\left(\frac{\partial\psi}{\partial t}\right)^{\vartheta} = -\mu^{\vartheta}\frac{\partial^{\vartheta}\mathrm{T}(\psi)}{\partial \psi^{\vartheta}}, \tag{3.5}$$

$$\frac{\partial^{\vartheta}\mathrm{T}(x,t)}{\partial x^{\vartheta}} = \frac{\partial^{\vartheta}\mathrm{T}(\psi)}{\partial \psi^{\vartheta}}, \tag{3.6}$$

$$\frac{\partial^{2\vartheta}\mathrm{T}(x,t)}{\partial x^{2\vartheta}} = \frac{\partial^{2\vartheta}\mathrm{T}(\psi)}{\partial \psi^{2\vartheta}}. \tag{3.7}$$

Thus, with the aid of Eq.(3.5), (3.6) and (3.7), Eq.(3.1) can be written as

$$\Omega\left(\frac{d^{2\vartheta}\mathrm{T}(\psi)}{d\psi^{2\vartheta}}, \frac{d^{\vartheta}\mathrm{T}(\psi)}{d\psi^{\vartheta}}, \mathrm{T}(\psi)\right) = 0, \tag{3.8}$$

where $d^{2\vartheta}\mathrm{T}(\psi)/d\psi^{2\vartheta}$ and $d^{\vartheta}\mathrm{T}(\psi)/d\psi^{\vartheta}$ are the LFDs of the orders 2ϑ and ϑ with respect to ψ, respectively.

In this case, we easily present solution of Eq.(3.8). Thus, we easily obtain the exact travelling-wave solutions of Eq.(3.1).

4. Exact solutions for local fractional PDEs

In this section, two examples for solving the local fractional PDEs are discussed. Let us consider the local fractional KGE on Cantor sets (see[7])

$$\frac{\partial^{2\vartheta}\mathrm{T}(\zeta,t)}{\partial \zeta^{2\vartheta}} - \frac{\partial^{2\vartheta}\mathrm{T}(\zeta,t)}{\partial t^{2\vartheta}} = \mathrm{T}(\zeta,t). \tag{4.1}$$

With the help of the travelling-wave transformation given as

$$\psi^{\vartheta} = \zeta^{\vartheta} - \mu^{\vartheta}t^{\vartheta}, \tag{4.2}$$

we have

$$\frac{\partial^{\vartheta}\mathrm{T}(\zeta,t)}{\partial t^{\vartheta}} = \frac{\partial^{\vartheta}\mathrm{T}(\psi)}{\partial \psi^{\vartheta}}\left(\frac{\partial\psi}{\partial t}\right)^{\vartheta} = -\mu^{\vartheta}\frac{\partial^{\vartheta}\mathrm{T}(\psi)}{\partial \psi^{\vartheta}}, \tag{4.3}$$

$$\frac{\partial^{2\vartheta}\mathrm{T}(x,t)}{\partial t^{2\vartheta}} = \mu^{2\vartheta}\frac{\partial^{\vartheta}\mathrm{T}(\psi)}{\partial \psi^{2\vartheta}}, \tag{4.4}$$

$$\frac{\partial^{2\vartheta}\mathrm{T}(x,t)}{\partial x^{2\vartheta}} = \frac{\partial^{2\vartheta}\mathrm{T}(\psi)}{\partial \psi^{2\vartheta}}, \tag{4.5}$$

such that

$$\left(1-\mu^{2\vartheta}\right)\frac{d^{2\vartheta}\mathrm{T}(\psi)}{d\psi^{2\vartheta}} = \mathrm{T}(\psi). \tag{4.6}$$

The non-differentiable solution of Eq.(4.6) can be written as (see [6]):

$$\mathrm{T}(\psi) = \phi_1 \mathrm{M}_{\vartheta}\left(\sqrt{(1-\mu^{2\vartheta})}\psi^{\vartheta}\right) + \phi_2 \mathrm{M}_{\vartheta}\left(-\sqrt{(1-\mu^{2\vartheta})}\psi^{\vartheta}\right), \tag{4.7}$$

where ϕ_1 and ϕ_2 are two coefficients.

From Eq.(4.6), the exact travelling-wave solution of Eq.(4.1) takes the form:

$$\mathrm{T}(\zeta,t)=\phi_1\mathrm{M}_\vartheta\left(\sqrt{(1-\mu^{2\vartheta})}\left(\zeta^\vartheta-\mu^\vartheta t^\vartheta\right)\right)+\phi_2\mathrm{M}_\vartheta\left(-\sqrt{(1-\mu^{2\vartheta})}\left(\zeta^\vartheta-\mu^\vartheta t^\vartheta\right)\right), \tag{4.8}$$

where ϕ_1 and ϕ_2 are two coefficients. The chart of Eq.(4.8) for $\phi_1=0$, $\phi_2=1$ and $\mu^\vartheta=0.5$ is illustrated in Figure 1; The chart of Eq.(4.8) for $\phi_1=1$, $\phi_2=0$ and $\mu^\vartheta=0.5$ is illustrated in Figure 2; The chart of Eq.(4.8) for $\phi_1=1$, $\phi_2=1$ and $\mu^\vartheta=0.5$ is illustrated in Figure 3.

As the second example, we consider the local fractional HE on Cantor sets (see[7]):

$$\frac{\partial^{2\vartheta}\mathrm{T}(x,y)}{\partial x^{2\vartheta}}+\frac{\partial^{2\vartheta}\mathrm{T}(x,y)}{\partial y^{2\vartheta}}+\mathrm{T}(x,y)=0. \tag{4.9}$$

Making use of the travelling-wave transformation:

$$\psi^\vartheta=x^\vartheta-\mu^\vartheta y^\vartheta, \tag{4.10}$$

we have

$$\frac{\partial^{\vartheta}\mathrm{T}(x,y)}{\partial y^{\vartheta}}=\frac{\partial^{\vartheta}\mathrm{T}(\psi)}{\partial \psi^{\vartheta}}\left(\frac{\partial\psi}{\partial y}\right)^\vartheta=-\mu^\vartheta\frac{\partial^{\vartheta}\mathrm{T}(\psi)}{\partial \psi^{\vartheta}}, \tag{4.11}$$

$$\frac{\partial^{2\vartheta}\mathrm{T}(x,y)}{\partial y^{\vartheta}}=\mu^{2\vartheta}\frac{\partial^{2\vartheta}\mathrm{T}(\psi)}{\partial \psi^{2\vartheta}}, \tag{4.12}$$

$$\frac{\partial^{2\vartheta}\mathrm{T}(x,t)}{\partial x^{2\vartheta}}=\frac{\partial^{2\vartheta}\mathrm{T}(\psi)}{\partial \psi^{2\vartheta}}, \tag{4.13}$$

such that

$$\left(1+\mu^{2\vartheta}\right)\frac{d^{2\vartheta}\mathrm{T}(\psi)}{d\psi^{2\vartheta}}=-\mathrm{T}(\psi). \tag{4.14}$$

The non-differentiable solution of Eq.(4.14) is given as (see [6]):

$$\mathrm{T}(\psi)=\phi_1\sin_\vartheta\left(-\sqrt{(1+\mu^{2\vartheta})}\psi^\vartheta\right)+\phi_2\cos_\vartheta\left(-\sqrt{(1+\mu^{2\vartheta})}\psi^\vartheta\right), \tag{4.15}$$

where ϕ_1 and ϕ_2 are two coefficients.

From Eq.(4.15) , the exact travelling-wave solution of Eq.(4.9) reads

$$\mathrm{T}(\zeta,t)=\phi_1\sin_\vartheta\left(-\sqrt{(1+\mu^{2\vartheta})}\left(\zeta^\vartheta-\mu^\vartheta t^\vartheta\right)\right)+\phi_2\cos_\vartheta\left(-\sqrt{(1+\mu^{2\vartheta})}\left(\zeta^\vartheta-\mu^\vartheta t^\vartheta\right)\right) \tag{4.16}$$

where ϕ_1 and ϕ_2 are two coefficients. The plot of Eq.(4.16) for $\phi_1=0$, $\phi_2=1$ and $\mu^\vartheta=0.5$ is shown in Figure 4; The plot of Eq.(4.16) for $\phi_1=1$, $\phi_2=0$ and $\mu^\vartheta=0.5$ is shown in Figure 5; The plot of Eq.(4.16) for $\phi_1=1$, $\phi_2=1$ and $\mu^\vartheta=0.5$ is shown in Figure 6.

5. Conclusion

In this work we considered the KGE and HE on Cantor sets within local fractional derivative. With the help of the travelling-wave transformation of non-differentiable type, the exact solutions of them were graphically discussed in detail. The presented method for the obtained results is accurate and efficient for us to find the exact solutions for the local fractional PDEs in mathematical physics.

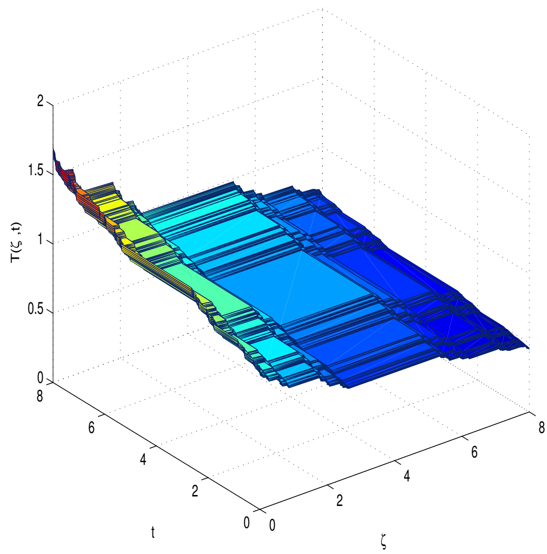

FIGURE 1. Plot of the exact solution (4.8) for $\phi_1 = 0$, $\phi_2 = 1$ and $\mu^{\vartheta} = 0.5$.

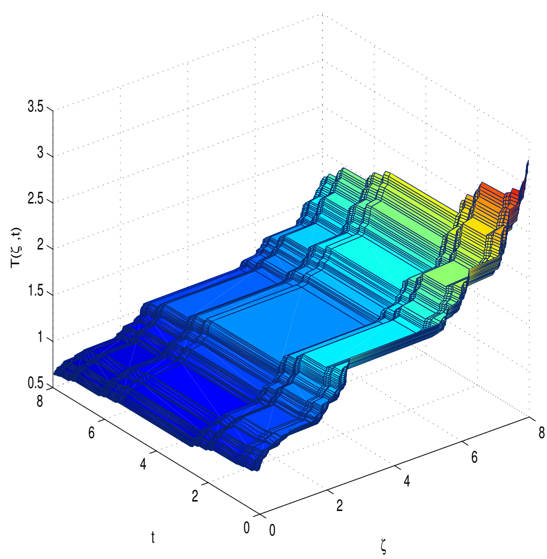

FIGURE 2. Plot of the exact solution (4.8) for $\phi_1 = 1$, $\phi_2 = 0$ and $\mu^{\vartheta} = 0.5$.

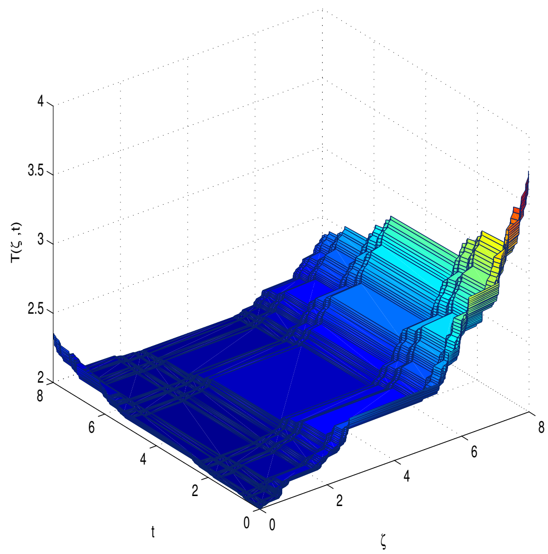

FIGURE 3. Plot of the exact solution (4.8) for $\phi_1 = 1$, $\phi_2 = 1$ and $\mu^{\vartheta} = 0.5$.

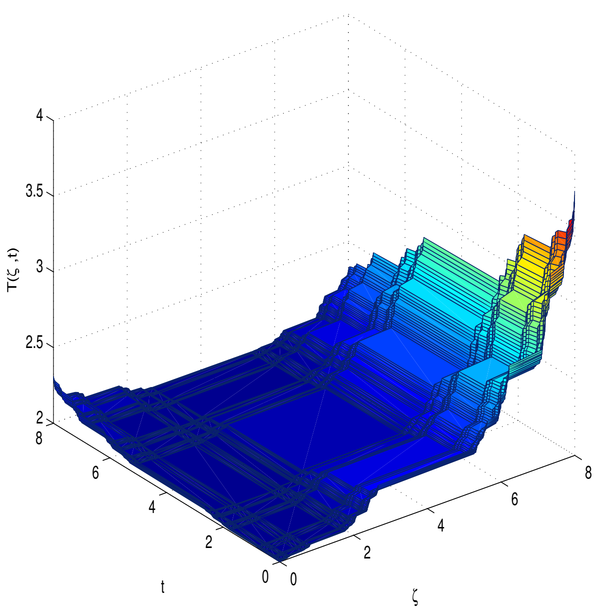

FIGURE 4. Plot of the exact solution (4.16) for $\phi_1 = 0$, $\phi_2 = 1$ and $\mu^{\vartheta} = 0.5$.

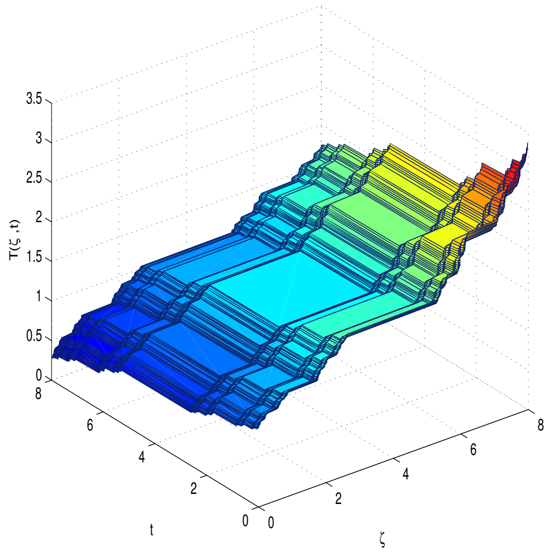

FIGURE 5. Plot of the exact solution (4.16) for $\phi_1 = 1$, $\phi_2 = 0$ and $\mu^\vartheta = 0.5$.

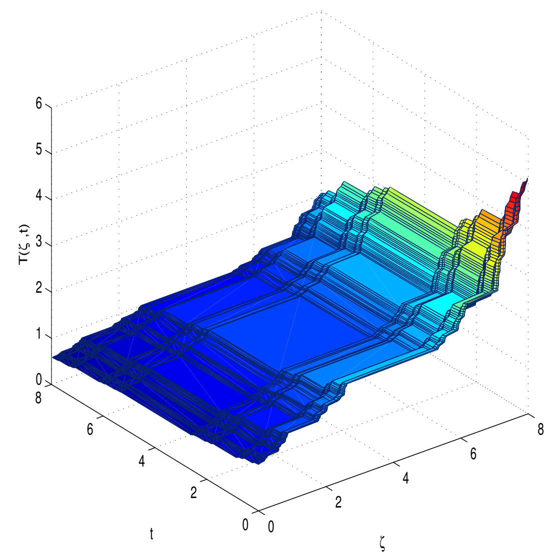

FIGURE 6. Plot of the exact solution (4.16) for $\phi_1 = 1$, $\phi_2 = 1$ and $\mu^\vartheta = 0.5$.

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