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# ON DISCRETENESS OF NEGATIVE PART OF SPECTRUM AND ESTIMATES FOR THE NUMBER OF EIGEN VALUES OF SECOND ORDER EQUATION WITH OPERATOR COEFFICIENTS ON THE SEMI-AXIS

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**Abstract**. In the paper we consider a second order operator-differential operator on the semi-axis. Discreteness of the negative spectrum is studied, lower and upper estimates of the number of eigen values of the given problem, less than the given number $-\varepsilon \ (\varepsilon > 0)$, are obtained.

## 1. Introduction

Spectral theory of differential operators is one of the major fields of contemporary mathematics. Extensive spectral theory of such operators was constructed by the efforts of many mathematicians. But matter with differential equations with operator coefficients acting in infinite-dimensional spaces is different. Numerous studies both of home and foreign mathematicians were devoted to spectral theory of such equations. Detailed bibliography on this theme may be found in review papers of V.I. Gorbachuk and M.L. Gorbachuk [10], [11]. M.Sh. Birman and M.Z. Solomiak [6] and also in the book of A.G. Kostyuchenko and I.S. Sargsyan [15]. The present paper is devoted to investigation of negative spectrum and obtaining the estimation for the number of negative eigen values of a second order operator differential equation given on the semi-axis. Asymptotic formulas for negative eigen values of scalar differential equations were obtained in the paper of Rosenfeld [17], B.Ya. Skachek [18, 19], G.I. Rosenbloom [16], A.N. Kochubey [13], while for operator-differential equations, a negative spectrum was studied in the papers of M.G. Gasymov, V.V. Zhikov, B.M. Levitan [8], D.R. Yafayev [21], A.A. Adigezelov [2], J.A. Zeynalov [22], A.B. Bayramov [5] and others. Asymptotic formula for the number of eigen values of abstract differential operators when the operators have pure discrete spectra, were obtained in the papers of A.G. Kostychenko and B.M. Levitan [14], E. Abdukadyrov [1], M.Bayramoglu [4], H.I. Aslanov [3] and others.

Let $H$ be a separable Hilbert space. In the Hilbert space $H_1 = L_2(H; [0,\infty))$ we consider the operator $L$ generated by the expression

$$l(y) = -\left(p(x)\,y'\right)' - Q(x)\,y \tag{1.1}$$

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and the boundary condition

$$y'(0) = 0. \tag{1.2}$$

It is supposed that the function $p(x)$ and the operator function $Q(x)$ satisfy the conditions:

1) $p(x)$ is a scalar continuous function with bounded derivatives. The function $p(x)$ does not decrease and there exist positive constants $c_1$ and $c_2$ such that the inequalities

$$c_1 \leq p(x) \leq c_2$$

are fulfilled.

2) For every $x \in [0, \infty)$ the operator function $Q(x)$ is a completely continuous, positive, monotonically decreasing operator, $\|Q(x)\|_H$ is a continuous function and $\lim\limits_{x \to \infty} \|Q(x)\|_H = 0$.

## 2. Discreteness of negative spectrum

It holds the following theorem.

**Theorem 2.1.** *Subject to conditions 1), 2), the operator $L$ is a lower bounded operator and the negative part of the spectrum is discrete.*

*Proof.* Denote by $L_{01}$ a self-adjoint operator in space $H_1 = L_2(H : [0, \infty))$, generated by the expression

$$l_0(y) = -y'' - c_1^{-1} Q(x)\, y$$

and the boundary condition $y'(0) = 0$.

From the conditions imposed on the function $p(x)$ it follows that $p'(x)\, y'(x) \in L_2(H; [0, \infty))$. Then from the equality $(p(x)\, y'(x))' = p'(x)\, y'(x) + p(x)\, y''(x)$ we get $p(x)\, y''(x) \in L_2(H; [0, \infty))$. Since $p(x) \geq c_1$, then we get

$$\int_0^\infty \|y''(x)\|_H^2\, dx \leq c_1^{-1} \int_0^\infty \|p(x)\, y''(x)\|_H^2\, dx < \infty$$

i.e.

$$y''(x) \in L_2(H; [0, \infty)).$$

Thus, we get that if $y(x) \in D(L)$, then $y(x) \in D(L_{01})$. In the same way we can show that the inverse is also true, i.e. if $y(x) \in D(L_{01})$ then $y(x) \in D(L)$. Hence it follows $D(L_{01}) = D(L)$.

For all $y(x) \in D(L_{01})$ the following inequality is fulfilled

$$\int_0^\infty \|y'(x)\|^2\, dx = -\int_0^\infty \left(y''(x)\,,\, y(x)\right)_H dx.$$

Then we have:

$$(L_0, y, y) = \int_0^\infty \left(-y''(x) - c_1^{-1} Q(x)\, y(x)\,,\, y(x)\right)_H dx$$

$$= -\int_0^\infty \left(y''(x)\,, y(x)\right)_H dx - c_1^{-1} \int_0^\infty (Q(x)\, y(x)\,, y(x))_H\, dx$$

$$\geq -c_1^{-1} \cdot c \int_0^\infty \|y(x)\|^2\, dx = -c_1^{-1} \cdot c\, (y, y)_{H_1}.$$

This shows lower boundedness of the operator $L_{01}$. From the inequality $L \geq c_1 \cdot L_{0\,1}$ it follows that the operator $L$ is a lower bounded operator as well. Therefore, the negative part of the spectrum of the operator $L_{01}$ is discrete [see [21]]. In other words, the negative part of the spectrum of the operator $L_{01}$ consists of eigen values. Every eigen value has a finite multiplicity. The set of eigen values may have a unique limit point at zero. Since $L \geq c_1 L_1$, the negative part of the spectrum of the operator $L$ is also discrete [see [19]].

Denote by $\alpha_1(x) \geq \alpha_2(x) \geq ... \geq \alpha_j(x) ...$ eigen values of the operator $Q(x)$ in space $H$. As $Q(x) > 0$ for all $x \in [0, \infty)$, we get $\alpha_j(x) > 0$ $(j = 1, 2, ...)$. It is known [see [20]] that $\alpha_1(x) = \sup\limits_{\|f\|=1} (Q(x)\, f,\, f)$ and $\|Q(x)\| = \sup\limits_{\|f\|=1} |(Q(x)\, f,\, f)|$. Hence we get that $\alpha_1(x) = \|Q(x)\|$ is a continuous function in the interval $[0, \infty)$. Since the operator function $Q(x)$ is monotonically decreasing, therefore the functions $\alpha_1(x)$, $\alpha_2(x) ... \alpha(x) ...$ are also monotonically decreasing functions in the interval $[0, \infty)$.

By the condition, $\lim\limits_{x\to\infty} \alpha_1(x) = 0$. The interval $(0, \alpha, (0))$ is the image of the function $\alpha_1(x)$. Therefore in the interval $(0, \alpha, (0))$ the function $\alpha_1(x)$ has a continuous inverse. We denote this function by $\psi_1(x)$. Take any point $\varepsilon \in (0, \alpha, (0))$. Let $L_{0,2}$ be an operator in space $\tilde{H}_{02} = L_2(H; (\psi_1(\varepsilon), \infty))$ generated by expression (1.1) and the boundary condition $y'(\psi_1(\varepsilon)) = 0$. The following theorem holds. □

**Theorem 2.2.** *Subject to conditions 1), 2) for any $y(x) \in D(L_{02})$ the following inequality is fulfilled*

$$(L_{02}y, y)_{\tilde{H}_{02}} \geq -\varepsilon\,(y, y)_{\tilde{H}_{0,2}}.$$

The proof of this theorem is completely similar to the proof of theorem 2.1.

## 3. Estimation of the number of negative eigen values of the operator $L$

Denote by $L_1$ and $L_2$ the operators in space $\tilde{H}_{01} = L_2(H; [0, \psi_1(\varepsilon)])$, generated by expression (1.1) and the boundary conditions

$$y(0) = y(\psi_1(\varepsilon)) = 0$$

$$y'(0) = y'(\psi_1(\varepsilon)) = 0.$$

Let $0 = x_0 < x_1 < ... < x_m = \psi_1(\varepsilon)$ be the dividing point of the interval $[0, \psi_1(\varepsilon)]$. Denote by $L_{(1)\,i}$ and $L_{(2)\,i}$ the operators in space $L_2(H; [x_{i-1}, x_i])$, generated by expressions (1.1) and the boundary conditions

$$y(x_{i-1}) = y(x_i) = 0$$

$$y'(x_{i-1}) = y'(x_i) = 0.$$

Denote by $N(\varepsilon)$, $N_1(\varepsilon)$ and $N_2(\varepsilon)$ the numbers of negative eigen values of the operator $L$, $L_1$, $L_2$, less than $-\varepsilon$ $(\varepsilon > 0)$. Denote the negative values of the operator $L$ by $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq ...$, and appropriate orthonormed eigen functions by $u_1(x)$, $u_2(x)$, ..., $u_n(x)$.... Denote by $L_\varepsilon$, $L_{1\varepsilon}, L_{2\varepsilon}$ the following operators:

$L_\varepsilon = L + \varepsilon E$, $L_{1\,\varepsilon} = L_1 + \varepsilon E$, $L_{2\varepsilon} = L_2 + \varepsilon E$ where $E$ is a unit operator in the space $H_1$.

Obviously, the number of the negative eigen values of operators $L_\varepsilon$, $L_{1\,\varepsilon}$, $L_{2\,\varepsilon}$ equals $N(\varepsilon)$ , $N_1(\varepsilon)$ and $N_2(\varepsilon)$, respectively.

Let $\mu_1^{(1)} \le \mu_2^{(1)} \le ... \le \mu_{N_1(\varepsilon)}^{(1)}$ and $\mu_1^{(2)} \le \mu_2^{(2)} \le ... \le \mu_{N_2(\varepsilon)}^{(2)}$ be negative eigen values, while $\varphi_1(x), \varphi_2(x) ..., \varphi_{N_1(\varepsilon)}^{(x)}$ and $\theta_1(x), \theta_2(x), ..., \theta_{N_2(\varepsilon)}(x)$ appropriate orthonormed eigen functions of the operators $L_{1\,\varepsilon}$ and $L_{2\,\varepsilon}$, respectively. The following lemma holds:

**Lemma 3.1.** *Subject to condition 1), 2) the following inequality holds* $N(\varepsilon) \ge N_1(\varepsilon)$.

*Proof.* Assume that the statement of the lemma is invalid, i.e. $N(\varepsilon) < N_1(\varepsilon)$.

In this case there exists a nonzero combination $\varphi = \sum_{i=1}^{N_1(\varepsilon)} c_i\varphi_i(x)$ such that

$$(u_i, \varphi) = \int_0^{\psi_1(\varepsilon)} (u_i(x), \varphi(x))\,dx = 0,\ i = 1, 2, ...N(\varepsilon).$$

We have

$$(L_{1\varepsilon}\varphi, \varphi) = \left(L_{1\varepsilon}\left(\sum_{i=1}^{N_1(\varepsilon)} c_i\varphi_i\right), \sum_{i=1}^{N_1(\varepsilon)} c_i\varphi_i\right) = \sum_{i=1}^{N_1(\varepsilon)} \mu_i^1 |c_i|^2 = \alpha < 0. \tag{3.1}$$

Similar to one in I.M. Glazman's monograph [9] one can construct a vector-function $\tilde{\varphi} = \tilde{\varphi}(x)$ satisfying the following conditions.

(1) The function $\tilde{\varphi}''(x)$ is continuous on the interval $[0, \psi_1(\varepsilon)]$,
(2) Outside of some interval $[a, b] \subset (0, \psi_1(\varepsilon))$ $\tilde{\varphi}(x)$ equals zero.
(3) $\left|(L_{1\varepsilon}\tilde{\varphi}, \tilde{\varphi})_{\tilde{H}_{01}} - (L_{1\varepsilon}\varphi, \varphi)_{\tilde{H}_{01}}\right| < \frac{\alpha}{2}$
(4) $(u_i, \tilde{\varphi})_{\tilde{H}_{01}} = 0$ $i = 1, 2, ..., N(\varepsilon)$

In what follows, as

$$\inf (L_\varepsilon y, y) = \mu_{N(\varepsilon)+1}$$
$$y \in D(L_\varepsilon)$$
$$y \perp u_i,\ i = 1, 2, ..., N(\varepsilon) \tag{3.2}$$

then

$$\left(L_{1\varepsilon}\left(\frac{\tilde{\varphi}}{\|\tilde{\varphi}\|}\right), \frac{\tilde{\varphi}}{\|\tilde{\varphi}\|}\right)_{\tilde{H}_{01}} = \left(L_\varepsilon\left(\frac{\tilde{\varphi}}{\|\tilde{\varphi}\|}\right), \frac{\tilde{\varphi}}{\|\tilde{\varphi}\|}\right)_{H_1} \ge \mu_{N(\varepsilon)+1} \ge 0.$$

From (3.1) and from the last relation we get:

$$(L_{1\varepsilon}\tilde{\varphi}, \tilde{\varphi})_{\tilde{H}_{01}} - (L_{1\varepsilon}\varphi, \varphi)_{\tilde{H}_{01}} = (L_{1\varepsilon}\tilde{\varphi}, \tilde{\varphi}) - \alpha \ge -\alpha. \tag{3.3}$$

The obtained relation (3.3) contradicts condition 3). We get that our assumption is invalid. This shows the validity of the statement of the lemma.

In the same way we can prove that the inequality $N(\varepsilon) \le N_2(\varepsilon)$ is valid.

Assuming that $\varepsilon$ is a rather small positive number, we partition $[0, \psi_1(\varepsilon)]$ into intervals whose lengths are equal to

$$\delta = \frac{\psi_1(\varepsilon)}{\left[\psi_1^k(\varepsilon)\right] + 1}. \tag{3.4}$$

Here $k$ is a positive number less than unit, $\left[\psi_1^k(\varepsilon)\right]$ denotes the entire part of the number $\psi_1^k(\varepsilon)$.

Let $0 < x_0 < x_1 < x_2 < ... < x_m = \psi_1(\varepsilon)$ be the dividing points of the interval $[0, \psi_1(\varepsilon)]$. In space $L_2[H; [x_{i-1}, x_i]]$ we determine the operators $L_{i\,(1)}$ and $L_{i\,(2)}$ generated by the differential expression

$$l_{i\,(1)}y = -p(x_i)\, y''(x) - Q\,(x_i)\, y(x),$$

and the boundary condition $y(x_{i-1}) = y(x_i) = 0$, and also by the differential expression

$$l_{i\,(2)}y = -p(x_{i-1})\, y''(x) - Q\,(x_{i-1})\, y(x)$$

and the boundary condition $y'(x_{i-1}) = y'(x_i) = 0$.

The following lemma holds. □

**Lemma 3.2.** *Subject to conditions 1), 2) the inequalities $L_{(1)\,i} < L_{i(1)}$ and $L_{(2)\,i} < L_{i(2)}$ are valid.*

*Proof.* Let $y(x) \in D\left(L_{i_{(1)}}\right)$. Then the functions $y(x)$ and $y'(x)$ are absolutely continuous functions in the norm of space $H$. Therefore $y''(x) \in L_2[H; [x_{i-1}, x_i]]$.

We have

$$\int_{x_{i-1}}^{x_i} \left\| \left(p(x)\, y'(x)\right)' \right\|_H^2 dx = \int_{x_{i-1}}^{x_i} \left\| p'(x)\, y'(x) + p(x)\, y''(x) \right\|_H^2 dx$$

$$\leq 2 \int_{x_{i-1}}^{x_i} \left| p'(x) \right| \cdot \left\| y'(x) \right\|_H^2 dx + \int_{x_{i-1}}^{x_i} \left| p(x) \right| \cdot \left\| y''(x) \right\|_H^2 dx. \tag{3.5}$$

By the condition, the function $p'(x)$ is continuous, therefore

$$\left| p'(x) \right| < c_1\ ,\ \ c_1 > 0. \tag{3.6}$$

The function $y'(x)$ is continuous in the norm of space $H$, therefore $\|y'(x)\|$ is continuous on the interval $[x_{i-1}, x_i]$.

Then we get

$$\left\| y'(x) \right\|_H^2 < c_2, \quad c_2 > 0. \tag{3.7}$$

From inequalities (3.5), (3.6) and (3.7) it follows

$$\int_{x_{i-1}}^{x_i} \left\| \left(p(x)\, y'\right) \right\|_H^2 dx \leq c_1 + c_2 \int_{x_{i-1}}^{x_i} \left\| y'(x) \right\|_H^2 dx \leq c_3. \tag{3.8}$$

This shows that if $y(x) \in D(L_{i_(1)})$ , then $y(x) \in L_{1(i)}$, i.e.

$$D\left(L_{i_{(1)}}\right) \subset D\left(L_{1_{(i)}}\right). \tag{3.9}$$

For all $y(x) \in D\left(L_{i_{(1)}}\right)$ , $y(x) \neq 0$ we have:

$$\left(L_{(1)_i} y, y\right)_{L_2[H;[x_{i-1},x_1]]} = \int_{x_{i-1}}^{x_i} \left[ -\left(p(x)\, y'(x)\right)_H^1 - Q(x)\, y(x)\, , y(x) \right]_H dx$$

$$= \int_{x_{i-1}}^{x_i} p(x) \left\| y'(x) \right\|_H^2 dx - \int_{x_{i-1}}^{x_i} \left(Q(x)\, y(x)\, , y(x)\right)_H dx. \tag{3.10}$$

As the function $p(x)$ is monotonically decreasing, and the operator function $Q(x)$ is monotonically decreasing, then from (3.10) we have

$$\left(L_{(1)i}y, y\right)_{L_2[H;[x_{i-1},x_1]]} < \int\limits_{x_{i-1}}^{x_i} p(x_i)\left\|y'(x)\right\|_H^2 dx - \int\limits_{x_{i-1}}^{x_i}\left(Q(x_i)\,y(x), y(x)\right)_H dx$$

$$= \int\limits_{x_{i-1}}^{x_i}\left(-p(x_i)\,y''(x) - Q(x_i)\,y(x), y(x)\right)_H dx = \left(L_{i(1)}y, y\right)_{L_2[H;[x_{i-1},x_1]]}. \tag{3.11}$$

From (3.9) and (3.11) we get $L_{(1)\,i} < L_{i\,(1)}$. In the similar way it is shown that $L_{(2)\,i} < L_{i\,(2)}$. The lemma is proved.

Take $\varepsilon \in (0, \alpha_i(0))$. Introduce the following sets:

$$E_{j,\varepsilon} = \{x\,|x \in [0,\infty)\ ;\ \alpha_j(x) \geq \varepsilon\}\ ,\ \tilde{\psi}_j(\varepsilon) = \sup E_{j,\varepsilon}. \tag{3.12}$$

Denote by $n_{1(i)}$ and $n_{i(1)}$ the number of negative eigen numbers of the operators $L_{1(i)}$ and $L_{i(1)}$, less than the number $\varepsilon$.

The following theorem holds. □

**Theorem 3.1.** *If the coefficients $p(x)$ and $Q(x)$ satisfy conditions 1), 2), then for rather small values of the number $\varepsilon$, it holds the inequality*

$$n_{(1)\,i} > \sum_j\left[\frac{1}{\pi}\int_{x_i}^{\varphi_{ij(\varepsilon)}}\sqrt{\frac{\alpha_i(x)-\varepsilon}{p(x)}}\,dx - 1\right]. \tag{3.13}$$

*Here* $\varphi_{i\,j}(\varepsilon) = \min\left\{x_{i+1}, \tilde{\psi}_j(\varepsilon)\right\}$ $i = 1, 2, \ldots.$

*Proof.* The eigen values of the operator $L_{i(1)}$ have the form

$$p(x_i)\left(\frac{n\pi}{x_i - x_{i-1}}\right)^2 - \alpha_j(x_i)\ ,\ \begin{matrix} n = 0,1,2,... \\ j = 1,2,3,... \end{matrix}.$$

Therefore $n_{i(1)}$ is the number of the pair of the form $(n, j)$, satisfying the inequality

$$p(x_i)\left(\frac{n\pi}{x_i - x_{i-1}}\right)^2 - \alpha_j(x_i) < -\varepsilon. \tag{3.14}$$

From in equality (3.14) we get

$$n < \frac{\delta}{\pi}\sqrt{\frac{\alpha_i(x_i)-\varepsilon}{p(x_i)}} \qquad (\delta = x_i - x_{i-1}). \tag{3.15}$$

For fixed values of $j$, the number of natural numbers $n$ satisfying inequality (3.14) is determined as follows:

$$n_{i(1)\,j} = \begin{cases} \frac{\delta}{\pi}\sqrt{\frac{\alpha_j(x_i)-\varepsilon}{p(x_i)}} - 1; & \frac{\delta}{\pi}\sqrt{\frac{\alpha_j(x_i)-\varepsilon}{p(x_i)}} \in \Lambda \\ \frac{\delta}{\pi}\sqrt{\frac{\alpha_j(x_i)-\varepsilon}{p(x_i)}}\ , & \frac{\delta}{\pi}\sqrt{\frac{\alpha_j(x_i)-\varepsilon}{p(x_i)}} \notin \Lambda \end{cases}. \tag{3.16}$$

As $n_{i(1)}=\sum_{\substack{j\\ (\alpha_j(x_i))>\varepsilon}} n_{i(1)\,j}$, from inequality (3.15) it follows

$$n_{i(1)} = \sum_{\substack{j\\ \alpha_{j(x_i)}>\varepsilon}} \left[\frac{\delta}{\pi}\sqrt{\frac{\alpha_j(x_i)-\varepsilon}{p(x_i)}}-1\right] \quad (i=1,2,...m-1)\,. \tag{3.17}$$

By the condition, the functions $\alpha_j(x)$ $(j=1,2,...)$ are decreasing, while the function $p(x)$ is non-decreasing, we get that $\alpha_j(x_{i+1})\geq\varepsilon$ or $x_{i+1}\leq\psi_j(\varepsilon)$. Then for $j\in\Lambda$ it holds

$$\delta\sqrt{\frac{\alpha_j(x_i)-\varepsilon}{p(x_i)}}=\int_{x_i}^{x_{i+1}}\sqrt{\frac{\alpha_j(x_i)-\varepsilon}{p(x_i)}}\,dx>\int_{x_i}^{x_{i+1}}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\,dx. \tag{3.18}$$

For all $j\in\Lambda$ satisfying the condition

$$\alpha_j(x_{i+1})<\varepsilon<\alpha_j(x_i)$$

or the condition $x_i\leq\psi_j(\varepsilon)\leq x_{i+1}$ we have

$$\delta\sqrt{\frac{\alpha_j(x_i)-\varepsilon}{p(x_i)}}\geq\int_{x_i}^{\varphi_j(\varepsilon)}\sqrt{\frac{\alpha_j(x_i)-\varepsilon}{p(x_i)}}\,dx>\int_{x_i}^{\varphi_j(\varepsilon)}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\,dx.$$

Fro $j\in\Lambda$ satisfying the condition $\alpha_j(x_i)>\varepsilon$ we have

$$\delta\sqrt{\frac{\alpha_j(x_i)-\varepsilon}{p(x_i)}}>\int_{x_i}^{\varphi_{i,j}(\varepsilon)}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\,dx. \tag{3.19}$$

Here $\varphi_{i,j}(\varepsilon)=\min\left\{x_{i+1},\tilde{\psi}_j(\varepsilon)\right\}$.

From (3.16) and (3.19) we get

$$n_{i(1)}>\sum_{\substack{j\\ (\alpha_j(x_i)>\varepsilon)}}\left[\frac{1}{\pi}\int_{x_1}^{\varphi_{i,j}(\varepsilon)}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\,dx-1\right]. \tag{3.20}$$

From lemma 3.1 it follows the validity of the inequality

$$n_{(1)\,i}\geq n_{i(1)}. \tag{3.21}$$

From (3.20) and (3.21) we finally get:

$$n_{(1)\,i}>\sum_{\substack{j\\ \alpha_j(x_i>\varepsilon)}}\left[\frac{1}{\pi}\int_{x_1}^{\varphi_{i,j}(\varepsilon)}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\,dx-1\right].$$

The theorem is proved. □

**Theorem 3.2.** *Subject to conditions 1), 2) for the number of eigenvalues, the following estimation is valid*

$$N(\varepsilon)>\frac{1}{\pi}\sum_{j=1}^{l_\varepsilon}\int_{0_1}^{\varphi_j(\varepsilon)}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\,dx-const\cdot l_3\cdot\int_0^{\delta}\sqrt{\alpha_1(x)}\,dx-const\cdot l_3\cdot\psi_1^k(\varepsilon)\,.$$

*Here* $l_3=\sum_{\alpha_j(0)\geq\varepsilon}1\,.$

*Proof.* By R. Courant's variational principle [see [7], p. 380] and theorem 3.1 we have:

$$N(\varepsilon) \geq \sum_{i=1}^{M-1} n_{(1)\,i} > \sum_{i=1}^{M-1} \sum_{j} \left[ \frac{1}{\pi} \int_{x_1}^{\varphi_{i,j}(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\, dx - 1 \right]$$

$$= \sum_{\substack{i\geq 1 \\ \alpha_j(x_i)>\varepsilon j}} \sum_{\alpha_j(x_j)>\varepsilon} \left[ \frac{1}{\pi} \int_{x_1}^{\varphi_{i,j}(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\, dx - 1 \right]. \tag{3.22}$$

If we use expression (3.13) for $\varphi_{i,j}(\varepsilon)$, then we get

$$\sum_{\alpha_j(x_i)>\varepsilon} \sum_{\substack{i\geq 1 \\ \alpha_j(x_i)>\varepsilon}} \int_{x_1}^{\varphi_{i,j}(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\, dx$$

$$= \sum_{\alpha_j(x_i)>\varepsilon} \left[ \int_{x_1}^{x_2} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\, dx + \int_{x_2}^{x_3} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\, dx + \ldots \right.$$

$$\left. + \int_{x_{i_0}}^{\tilde{\psi}_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\, dx \right] = \sum_{\substack{j \\ \alpha_{j(x_i)>\varepsilon}}} \int_{x_1}^{\tilde{\psi}_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\, dx$$

$$= \sum_{\psi_j(\varepsilon)>x_1} \int_{x_1}^{\tilde{\psi}_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\, dx = \sum_{\psi_j(\varepsilon)>x_1} \int_{x_1}^{\tilde{\psi}_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\, dx$$

$$= \sum_{\psi_j(\varepsilon)>x_1} \int_{0}^{\tilde{\psi}_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\, dx - \sum_{\psi_j(\varepsilon)>x_1} \int_{0}^{x_1} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\, dx$$

$$= \sum_{j=1}^{l_3} \int_{0}^{\tilde{\psi}_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\, dx - \sum_{\psi_j(\varepsilon)>x_1} \int_{0}^{\tilde{\psi}_{j(\varepsilon)}} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\, dx$$

$$- \sum_{\psi_j(\varepsilon)>x} \int_{0}^{x_1} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\, dx = \sum_{j=1}^{l_\varepsilon} \int_{0}^{\tilde{\psi}_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\, dx$$

$$- \sum_{i} \int_{0}^{\varphi_0,j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}} dx$$

$$\geq \sum_{j=1}^{l_3} \int_{0}^{\tilde{\psi}_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\, dx - \sum_{\alpha_j(0)\geq\varepsilon} \int_{0}^{x_1} \sqrt{\frac{\alpha_1(x)-\varepsilon}{p(x)}}\, dx$$

$$\geq \sum_{j=1}^{l_3} \int_{0}^{\tilde{\psi}_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\, dx - const \cdot \int_{0}^{\delta} \sqrt{\alpha_1(x)-\varepsilon}\, dx \cdot \sum_{\alpha_j(0)\geq\varepsilon} 1$$

$$= \sum_{j=1}^{l_3} \int_{0}^{\tilde{\psi}_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\, dx - const \cdot l_\varepsilon \cdot \int_{0}^{\delta} \sqrt{\alpha(x)-\varepsilon}\, dx. \tag{3.23}$$

Here $i_0$ is a natural number for which the inequality $\tilde{\psi}_j(\varepsilon) \le x_{i_0+1}$ is fulfilled.

For the number $j$ for which $\alpha_j(x_i) > \varepsilon$ it holds

$$\sum_{\substack{i\ge 1\\ \alpha_j(x_i)>\varepsilon_i}} 1 = \max\{i : \alpha_j(x_i) > \varepsilon\}. \tag{3.24}$$

Denote $\max\{i : \alpha_j(x_i) > \varepsilon\} = m(j,\varepsilon)$.

Since $\alpha_j\left(x_{m_{(j,\varepsilon)}}\right) > \varepsilon$ we have $x_{m(j,\varepsilon)} \le \widetilde{\psi}_j(\varepsilon)$. On the other hand, $m(j,\varepsilon) = \frac{x_{m_{(j,\varepsilon)}}}{\delta}$

Therefore $\sum_{\substack{i\ge 1\\ \alpha_j(x_i)>\varepsilon_i}} 1 \le \frac{\psi_j(\varepsilon)}{\delta}$.

Using this inequality we get

$$\sum_{\substack{i\ge 1\\ \alpha_j(x_i)>\varepsilon_i}} \sum_{\substack{i\ge 1\\ \alpha_j(x_i)>\varepsilon_i}} < \sum_{\substack{j\\ \alpha_j(x)>\varepsilon}} \frac{\tilde{\psi}_j(\varepsilon)}{\delta} < \sum_{\alpha_j(0)>\varepsilon} \frac{\tilde{\psi}_j(\varepsilon)}{\delta} < \frac{\psi_1(\varepsilon)}{\delta} \sum_{\alpha_j(0)>\varepsilon} 1 = \delta^{-1}\psi_1(\varepsilon)\, l_3. \tag{3.25}$$

From (3.22), (3.23) and (3.25) we get

$$N(\varepsilon) > \frac{1}{\pi}\sum_{j=1}^{l_\varepsilon}\int_{0_1}^{\tilde{\varphi}_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\, dx - const\cdot l_3\cdot \int_0^\delta \sqrt{\alpha_1(x)-\varepsilon}\, dx - \delta^{-1}\cdot\psi_1(\varepsilon)\, l_3. \tag{3.26}$$

From lemma 3.1 and relations (3.4) and (3.25) we finally get:

$$N(\varepsilon) > \frac{1}{\pi}\sum_{j=1}^{l_\varepsilon}\int_{0_1}^{\tilde{\varphi}_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\, dx - const\cdot l_3\cdot \int_0^\delta \sqrt{\alpha_1(x)}\, dx - const\cdot l_3\cdot\psi_1^k(\varepsilon). \tag{3.27}$$

Theorem 3.2 is proved. □

Denote by $n_{(2)\,i}$ and $n_{i\,(2)}$ the number of negative eigen values of the operators $L_{(2)\,i}$ and $L_{i\,(2)}$, less than the number $\varepsilon$.

The following theorem holds.

**Theorem 3.3.** *If the coefficients of the differential operator $L$ satisfies conditions 1), 2), then for small values of $\varepsilon$ it holds*

$$n_{(2)\,i} < \sum_{\substack{j\\ (\alpha_j(x_{i-1}))>\varepsilon}} \left[\frac{1}{\pi}\int_{x_{i-2}}^{x_{i-1}} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\, dx + 1\right].$$

*Proof.* The eigen values of the operator $L_{i\,(2)}$ have the form

$$p(x_{i-1})\left(\frac{(n-1)\pi}{x_i - x_{i-1}}\right)^2 - \alpha_j(x_i) \quad (n = 1,2,... \quad j = 1,2,...).$$

Therefore, $n_{i\,(2)}$ is the number of the pairs $(n,\, j)$ $(n, j \in \Lambda)$ satisfying the inequality

$$p(x_{i-1})\frac{(n-1)^2\pi^2}{(x_i - x_{i-1})^2} - \alpha_j(x_{i-1}) < -\varepsilon \tag{3.28}$$

and for it the following estimation is fulfilled

$$n_{i\,(2)} < \sum_{\substack{j \\ (\alpha_j(x_{i-1}))>\varepsilon}} \left[\frac{\delta}{\pi}\sqrt{\frac{\alpha_j\,(x_{i-1})-\varepsilon}{p\,(x_{i-1})}}+1\right]. \tag{3.29}$$

As the functions $\alpha_j\,(x)$ $(j=1,2,...)$ are monotonically decreasing, while the function $p\,(x)$ is a non-decreasing function, we get

$$\delta\sqrt{\frac{\alpha_j\,(x_{i-1})-\varepsilon}{p\,(x_{i-1})}} = \int_{x_{i-2}}^{x_{i-1}}\sqrt{\frac{\alpha_j\,(x_{i-1})-\varepsilon}{p\,(x_{i-1})}}\,dx$$
$$< \int_{x_{i-2}}^{x_{i-1}}\sqrt{\frac{\alpha_j\,(x)-\varepsilon}{p\,(x)}} \quad (i=2,3,...)\,. \tag{3.30}$$

From (3.29) and (3.30) we easily get the estimation

$$n_{i\,(2)} < \sum_{\substack{j \\ (\alpha_j(x_{i-1}))>\varepsilon}} \left[\frac{1}{\pi}\int_{x_{i-2}}^{i-1}\sqrt{\frac{\alpha_j\,(x)-\varepsilon}{p\,(x)}}\,dx+1\right]. \tag{3.31}$$

From lemma 3.2 it follows

$$n_{(2)\,i} \le n_{i(2)} \quad (i=1,2,...)\,. \tag{3.32}$$

As a result, from (3.31) and (3.32) we get the estimation

$$n_{2\,(i)} < \sum_{\substack{j \\ (\alpha_j(x_{i-1}))>\varepsilon}} \left[\frac{1}{\pi}\int_{x_{i-2}}^{x_{i-1}}\sqrt{\frac{\alpha_j\,(x)-\varepsilon}{p\,(x)}}\,dx+1\right] \quad (i=2,3,...)\,. \tag{3.33}$$

Theorem 3.3 is proved. □

**Theorem 3.4.** *Subject to conditions 1), 2), for small values of $\varepsilon$, the following estimation is valid*

$$N_2\,(\varepsilon) < n_{(2)\,i} + \frac{1}{\pi}\sum_{j=1}^{l_\varepsilon}\int_{0_1}^{\tilde{\varphi}_j(\varepsilon)}\sqrt{\frac{\alpha_j\,(x)-\varepsilon}{p\,(x)}}\,dx + l_3\cdot\delta^{-1}\cdot\psi_1\,(\varepsilon)\,.$$

*Proof.* By R. Courant's variational principle and theorem 3.3, we have

$$N_2\,(\varepsilon) \le \sum_{i=1}^{M} n_{(2)\,i} + \sum_{i=1}^{M}\sum_{j}\left[\frac{1}{\pi}\int_{x_{i-2}}^{x_{i-1}}\sqrt{\frac{\alpha_j\,(x)-\varepsilon}{p\,(x)}}\,dx+1\right]$$
$$= \sum_{\substack{j \\ \alpha_j\,(x_i)\,>\,\varepsilon}}\ \sum_{\substack{i\ge 2 \\ \alpha_j(x_{i-1})>\varepsilon}}\left[\frac{1}{\pi}\int_{x_{i-2}}^{x_{i-1}}\sqrt{\frac{\alpha_j\,(x)-\varepsilon}{p\,(x)}}\,dx+1\right]$$
$$= \sum_{\substack{j \\ \alpha_j\,(x_i)\,>\,\varepsilon}}\left[\frac{1}{\pi}\int_{0}^{x_1}\sqrt{\frac{\alpha_j\,(x)-\varepsilon}{p\,(x)}}\,dx + \frac{1}{\pi}\int_{x_1}^{x_2}\sqrt{\frac{\alpha_j\,(x)-\varepsilon}{p\,(x)}}\,dx + ...+\right.$$

$$+\frac{1}{\pi}\int_{x_{i_0-1}}^{x_{i_0}}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\,dx+i_0\Bigg].\tag{3.34}$$

Here $i_0$ is a natural number for which the conditions $\alpha_j(x_{i_0})>\varepsilon$ and $\alpha_j(x_{i_0+1})\le\varepsilon$ are fulfilled. From (3.12) it follows $x_{i_0}\le\tilde{\psi}_j(\varepsilon)$. On the other hand, if we take into account $i_0=\frac{x_{i_0}}{\delta}$, then from (3.34) we get

$$N_2(\varepsilon)<n_{2(1)}+\sum_{\substack{j\\ \alpha_j(x_i)>\varepsilon}}\left[\frac{1}{\pi}\int_0^{\tilde{\psi}_j(\varepsilon)}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}+\frac{\psi_j(\varepsilon)}{\delta}\right]$$

$$\le n_{(3)\,1}+\sum_{j=1}^{l_\varepsilon}\left[\frac{1}{\pi}\int_0^{\tilde{\psi}_j(\varepsilon)}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}+\frac{\psi_j(\varepsilon)}{\delta}\right]$$

$$<n_{(2)\,1}+\frac{1}{\pi}\sum_{j=1}^{l_\varepsilon}\int_0^{\tilde{\psi}_j(\varepsilon)}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\,dx+l_3\cdot\frac{\psi_1(\varepsilon)}{\delta}.\tag{3.35}$$

The theorem is proved. □

**Theorem 3.5.** *If conditions 1), 2) are fulfilled, then for small values of $\varepsilon$, it holds the estimation*

$$N(\varepsilon)<\frac{1}{\pi}\sum_{j=1}^{l_\varepsilon}\int_0^{\tilde{\psi}_j(\varepsilon)}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\,dx+const\cdot l_3\int_0^{\delta}\sqrt{\alpha_1(x)}\,dx+const\cdot l_3\psi_1^k(\varepsilon).$$

*Proof.* In inequality (3.35) $n_{(2)\,1}$ denotes the number of negative eigen values of less than the number $\varepsilon$ of the operator $L_{(2)\,1}$ generated by expression (1.1) and boundary conditions $y'(0)=y'(\delta)=0$ in space $L_2(H;\,[0,\delta])$. Partition $[0,\delta]$ into the intervals of identical length

$$\delta=\frac{\delta_0}{\left[\delta_0\psi_1^{2k-1}(\varepsilon)\right]+1}\quad(\delta_0=\delta).$$

In the same way as it was done in the proof of theorem 3.3, we can get the estimation

$$n_{(2)\,i}<n_{(2)\,1}^{(1)}+\frac{1}{\pi}\sum_{\tilde{\psi}_j(\varepsilon)<\delta_0}\int_0^{\tilde{\psi}_j(\varepsilon)}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\,dx$$

$$+\frac{1}{\pi}\sum_{\psi_j(\varepsilon)\ge\delta_0}\int_0^{\delta_0}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\,dx+l_3\cdot\delta_1^{-1}\cdot\delta_0.\tag{3.36}$$

Here $n_{(2)\,i}^{(1)}$ denotes the number of negative values of the operator $L_{(2)\,i}^{(1)}$ generated by expression (1.1) and the boundary conditions $y'(0)=y'(\delta)=0$ in space $L_2(H:[0,\delta])$.

From (3.36) we get

$$n_{(2)\,i}<n_{(2)\,1}^{(1)}+const\cdot l_3\int_0^{\delta}\sqrt{\alpha_1(x)}\,dx+l_3\cdot\delta^{-1}\cdot\delta_0.\tag{3.37}$$

Denote

$$\delta_m = \frac{\delta_{m-1}}{\left[\delta_{m-1} \cdot \psi_1^{(m+1)k-1}(\varepsilon)\right] + 1}, \ (m = 1, 2, ...). \tag{3.38}$$

If we apply inequality (3.37) to the operator $L_{(2)\,1}^{(m)}$ generated by differential expression (1.1) and boundary conditions $y'(0) = y'(\delta_m) = 0$ in space $L_2(H; [0, \delta_m])$ and denote by $n_{(2)\,1}^{(m)}$ the number of eigen values less than the given number of $\varepsilon$, then we get:

$$n_{(2)\,1}^{(m)} \leq n_{(2)\,1}^{(m+1)} + const \cdot l_3 \int_0^{\delta_m} \sqrt{\alpha_1(x)}\, dx + l_3 \cdot \delta_{m+1}^{-1} \delta_m \quad (m = 1, 2, ...). \tag{3.39}$$

If we take into account $\delta_1 = \psi_1(\varepsilon)$ from equality (3.2) we get:

$$\frac{\delta_{m-1}}{\delta_m} = \left[\delta_{m-1}\psi_1^{(m+1)k-1}\right] + 1 \leq \delta_{m-1}\psi_1^{(m+1)} + 1$$

$$= \frac{\delta_{m-2}}{\left[\delta_{m-2} \cdot \psi_1^{m_{k-1}}(\varepsilon)\right] + 1} \cdot \psi_1^{(m+1)\,k-1} + 1$$

$$< \frac{\delta_{m-2}\psi_1^{(m+1)\,k-1}(\varepsilon)}{\delta_{m-2}\psi_1^{(m+1)\,k-1}} + 1 = \psi_1^{(k)}(\varepsilon) + 1, \qquad m = 1, 2, .... \tag{3.40}$$

Hence for all $\varepsilon$ satisfying the condition $\psi_1^{(k)}(\varepsilon) > 2$ we get:

$$\frac{\delta_{m-1}}{\delta} < 2\psi_1^k(\varepsilon). \tag{3.41}$$

From (3.39) and (3.41) we get

$$n_{(2)\,1}^{(m)} \leq n_{(2)\,1}^{(m+1)} + const \cdot l_3 \int_0^{\delta_m} \sqrt{\alpha_k(x)}\, dx + 2l_3 \cdot \psi_1^k(\varepsilon). \tag{3.42}$$

Write inequality (3.29) for $i = 1$. Substituting $\delta_m$ instead of $\delta_{m+1}$ and taking into account inequality (3.32), we get:

$$n_{(2)\,1}^{(m+1)} < \sum_{\alpha_j(0) > \varepsilon} \left[\frac{\delta_{m+1}}{\pi} \sqrt{\frac{\alpha_j(\varepsilon) - \varepsilon}{p(0)}} + 1\right]$$

$$< \sum_{\alpha_j(0) \geq \varepsilon} \left[\frac{\delta_{m+1}}{\pi} \sqrt{\frac{\alpha_j(0)}{p(0)}} + 1\right] < const \cdot l_3(\delta_{m+1}). \tag{3.43}$$

For $m_0 \in \Lambda$ that satisfy the condition $m_0 \geq \frac{1}{k}$ from (3.38) we get:

$$\delta_{m_0+1} \leq 1. \tag{3.44}$$

From (3.43) and (3.44) we find:

$$n_{(2)\,1}^{(m_0+1)} < const \cdot l_\varepsilon. \tag{3.45}$$

From (3.35), (3.37), (3.39), (3.42) and (3.45) we have:

$$N_2(\varepsilon) \leq \frac{1}{\pi} \sum_{j=1}^{l_3} \int_0^{\tilde{\psi}_j(\varepsilon)} \sqrt{\frac{\alpha_j(x) - \varepsilon}{p(x)}}\, dx + l_3 \cdot \delta^{-1}\psi_1(\varepsilon)$$

$$+const \cdot l_3 \sum_{m=0}^{m_0} \int_0^{\delta_m} \sqrt{\alpha_1 (x)}\, dx + const \cdot l_3 \cdot \psi_1^k (\varepsilon) . \tag{3.46}$$

Using lemma 3.2, from relations (3.4) and (3.46) we finally get

$$N(\varepsilon) \leq \frac{1}{\pi} \sum_{j=1}^{l_3} \int_0^{\tilde{\psi}_j(\varepsilon)} \sqrt{\frac{\alpha_j (x) - \varepsilon}{p(x)}}\, dx$$

$$+const \cdot l_3 \int_0^{\delta_m} \sqrt{\alpha_1 (x)}\, dx + const \cdot l_3 \cdot \psi_1^k (\varepsilon) .$$

The theorem is proved. □

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## References

[1] E. Abdukadirov, Asymptotic distribution of eigen values of Sturm-Liouville operator problem, *Dokl. AN Uzb. SSR.*, (1970), no. 12, 3-5.

[2] A.A. Adygezalov, On asymptotics of negative part of the spectrum of Sturm-Liouville operator problem, *Izv. AN Azerb. SSR. ser. phys. techn. i math. sci.* (1980), no. 6, 8-12.

[3] H.I. Aslanov, Asymptotics of the number of eigen values of ordinary differential equations with operator coefficients on a semi-axis, *Dokl. AN Az. SSR*, **32** (1976), no. 3, 3-7.

[4] M. Bayramoglu, Asymptotics of the number of eigen values of ordinary ifferential operator with operator coefficients, *In Funk. anal. i ee prim. Baku*, "Elm", (1971), 144-166.

[5] A.M. Bayramov, Asymptotic behavior of the number of negative eigen values of one Sturm-Liouville operator, *Dep. in Az. NIINTI*, (1966), no. 2404 -Az. 20 p.

[6] M.Sh. Birman, M.Z. Solomiak, Asymptotics of spectrum of differential equations, *Mathematical analysis.* "Itogi nauki i tekhniki", M. (1977), 5-58.

[7] R. Courant, D. Hilbert, *Mathematical physics methods*, vol. 1, 1951, 476 pp.

[8] M.G. Gasymov, V.V. Zhikov, V.M. Levitan, Discreteness and finiteness conditions of the negative spectrum of Shordinger's operator equation, *Matem-zametki.* **2** (1967), no. 5, 531-538.

[9] I.M. Glazman, *Direct methods of qualitative spectral analysis of singular differential operators,* Physmath. 1963, 339 pp.

[10] V.I. Gorbachuk , M.L. Gorbachuk, On some classes of problems for sturm-Liouville equation with operator potential, *UMZh.* **24** (1972), no. 3, 291-305.

[11] V.I. Gorbachuk, M.L. Gorbachuk, Some problems of spectral theory of elliptic type differential equations in the space of vector-functions on a finite interval, *UMZh.* **28** (1976), no. 1, 12-26.

[12] N. Ts. Hochberg, M.G. Krein, *Introduction to theory of not self-adjoint operators*, Moscow. Nauka, 1965, 448 pp.

[13] A.N. Kochubey, On negative part of spectrum of abstract differential operators. *Methods of functional analysis in mathematical physics problems,* Ins. Mathem. of AS Ukr. SSR, (1975), 88-101.

[14] A.G. Kostyuchenko, B.M. Levitan, On asymptotic behavior of eigen values of Sturm-Liouville's operator problem, *Funk. Analiz i ego pril.* **1** (1967), no. 1, 86-96.
[15] A.G. Kostyuchenko, N.S. Sargsyan, *Distribution of eigen values,* M. Nauke, 1979.
[16] G.I. Rosenbloom, Asymptotics of negative discrete spectrum of Shorodinger's operator, Matem. Zametki, **21** (1977), no. 3, 399-407.
[17] N.S. Rosenfeld, The eigen values of a class of singular differential operators, *Communs. Pure and Appl. Math.*, **13** (1960), no. 3, 35-405.
[18] B.Ya. Skachek, Asymptotics of the negative part of the spectrum of one-dimensional differential operators, *Approximative methods for solving differential equations*, Kiev 1963, 96-109.
[19] B.Ya. Skachek, On asymptotics of the negative part of the spectrum of many-dimensional differential operators, *Dokl. AN. Ukr. SSR*, (1964), I, 14-17.
[20] V.I. Smirnov, *Course of higher mathematics*, M. Fizmatgiz, vol. 5, 1959, 655 pp.
[21] D.R. Yafaev, On negative spectrum of Scrodinger's operator equation, *Matem. zametki*, **7** (1970), no. 6, 753-763.
[22] J.A. Zeynalov, *On negative part of the spectrum of sturm-Liouville operator with singularity*, Dep. in VINITI, no. 6 /59-82 Dep.

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