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ON THE WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR A CLASS OF DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DELAY AND THE CONTINUOUS INITIAL CONDITION

PH. DVALISHVILI, A. NACHAOUI, M. NACHAOUI, AND T. TADUMADZE

Abstract. Theorems on the continuous dependence of solutions on perturbations of the initial data and the right-hand side of equations are proved. Under initial data we imply the collection of an initial moment, delay and initial functions. Perturbations of the initial data are small in a standard norm and perturbations of the right-hand side of equation are small in the integral sense.

1. Introduction and Formulation of the Main Results

In this work, for the differential equation (1.2) with the initial condition (1.3) we prove the theorems on the continuous dependence of solutions when perturbations of the initial moment t_0, the delay function $\tau(t)$ and the initial function $\varphi(t)$ are small in the Euclidean topology, and a perturbation of the right-hand side of equation is small in the integral topology (see (1.1). Theorems 1.1-1.3 given here and their like theorems play an important role, in general, in studying optimal control problems [6–8, 11, 13, 16–19, 21], in proving variation formulas of solution [7, 8, 11–13, 21, 22] and in the sensitivity analysis of differential equations and optimal control problems [4, 5, 20]. We note that in [2] is proposed a mathematical model of the immune response like to Marchuk's model [4] that is a particular case of the differential equation (1.2). Theorems on the continuous dependence of solutions when a perturbation of the right-hand side is small in the integral sense were proved for various classes of ordinary differential equations in many works, including in [7, 8, 15], and for differential equations with concentrated delay in [11–13, 20, 21]. The problem on the continuous dependence of solutions for differential equations with deviated argument and for functional differential equations in the case where a perturbation of the right-hand side is small in the Euclidean topology was considered in [1, 3, 9, 14, 19]. Theorem 1.1 proved in this work is an analog of the theorems presented in [7, 8]. Finally, we note that the continuity of the initial condition means that the values of the initial function and trajectory always coincide at the initial moment of time.

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Let $I=[a,b]$ be a finite interval and $\mathbb{R}^n$ be the n -dimensional vector space of points $x=(x^1,...,x^n)^T$, with norm $|x|^2=\sum_{i=1}^n(x^i)^2$, where T denotes transposition. Suppose that $O\subset\mathbb{R}^n$ is an open set, and E_f is the space of functions $f:I\times O\times\mathbb{R}^n\to\mathbb{R}^n$ satisfying the following conditions: for each fixed $(x_1,x_2)\in O\times\mathbb{R}^n$ the function $f(\cdot,x_1,x_2):I\to\mathbb{R}^n$ is measurable; for each $f\in E_f$ and compact set $K\subset O$ there exist functions $m_{f,K}(t),L_{f,K}(t)\in L_1(I,\mathbb{R}_+),\mathbb{R}_+=[0,\infty)$, such that for almost all $t\in I$,

$$\mid f(t,x_1,x_2)\mid\leq m_{f,K}(t)\ \forall(x_1,x_2)\in K\times\mathbb{R}^n, \mid f(t,x_1,x_2)-f(t,y_1,y_2)\mid$$

$$\leq L_{f,K}(t)\sum_{i=1}^{2}\mid x_i-y_i\mid\forall(x_1,y_1)\in K^2,\ \forall(x_2,y_2)\in\mathbb{R}^n\times\mathbb{R}^n.$$

Two functions $f_1,f_2\in E_f$ are said to be equivalent if for every fixed $(x_1,x_2)\in O\times\mathbb{R}^n$ and for almost all $t\in I, f_1(t,x_1,x_2)-f_2(t,x_1,x_2)=0$. The equivalence classes of functions of the space E_f compose a vector space , which is also denoted by E_f; these classes are also called functions and denoted by f again. We introduce a topology in E_f using the following base of neighborhoods of the origin

$$\Big\{V_{K,\delta}:K\subset O\ \text{ is a compact set and }\ \delta>0\ \text{ is an arbitrary number}\Big\},$$

where

$$V_{K,\delta}=\Big\{\delta f\in E_f:H(\delta f;K)\leq\delta\Big\}\text{ and }H(\delta f;K)=\sup\Big\{\Big|\int_{t'}^{t''}\delta f(t,x_1,$$

$$x_2)dt\Big|:t',t''\in I,x_1\in K,x_2\in\mathbb{R}^n\Big\}. \tag{1.1}$$

Let D be the set of continuous differentiable scalar functions (delay functions) $\tau(t),t\in[a,\infty)$, satisfying the conditions: $\tau(t)<t,\dot{\tau}(t)>0$ and $\inf\{\tau(a):\tau\in D\}:=\hat{\tau}>-\infty$.

Let $C(I_1)$ be the space of continuous functions $\varphi(t)\in\mathbb{R}^n,t\in I_1=[\hat{\tau},b]$ equipped with the norm $\|\varphi\|_{I_1}=\sup\{|\varphi(t)|:t\in I_1\}$. By $\Phi=\{\varphi\in C(I_1):\varphi(t)\in O,t\in I_1\}$ we denote the set of initial functions.

To each element $\mu=(t_0,\tau,\varphi,f)\in\Lambda=[a,b)\times D\times\Phi\times E_f$ we assign the differential equation with distributed delay on the interval $[\tau(t),t]$

$$\dot{x}(t)=f\Big(t,x(t),\int_{\tau(t)}^{t}\sigma(s,x(s))ds\Big) \tag{1.2}$$

with the continuous initial condition

$$x(t)=\varphi(t),t\in[\hat{\tau},t_0], \tag{1.3}$$

where $\sigma(s,x_1),(s,x_1)\in I_1\times O$ is a given function satisfying the following conditions: for each fixed $x_1\in O$ the function $\sigma(\cdot,x_1):I_1\to\mathbb{R}^n$ is measurable; for each compact set $K\subset O$ there exist functions $m_K(s),L_K(s)\in L_1(I,\mathbb{R}_+)$, such that for almost all $s\in I_1$

$$\mid\sigma(s,x_1)\mid\leq m_K(s),\mid\sigma(s,x_1)-\sigma(s,y_1)\mid\leq L_K(s)\mid x_1-y_1\mid\forall(x_1,y_1)\in K^2.$$

Definition 1.1. *Let $\mu=(t_0,\tau,\varphi,f)\in\Lambda$. A function $x(t)=x(t;\mu)\in O,t\in[\hat{\tau},t_1],t_1\in(t_0,b]$, is called a solution of the equation (1.2) with the initial condition (1.3) or a solution corresponding to the element μ and defined on the interval*

$[\hat{\tau}, t_1]$, *if it satisfies the condition (1.3), is absolutely continuous on the interval* $[t_0, t_1]$ *and satisfies the equation (1.2) almost everywhere (a.e.) on* $[t_0, t_1]$.

We introduce the following sets:

$$W(K;\alpha) = \Big\{\delta f \in E_f : \exists\, m_{\delta f,K}(t), L_{\delta f,K}(t) \in L_1(I, R_+),$$

$$\int_I [m_{\delta f,K}(t) + L_{\delta f,K}(t)]dt \le \alpha\Big\},$$

where $K \subset O$ is a compact set and $\alpha > 0$ is a fixed number not dependent on δf; the set $W(K;\alpha)$ is called the class of perturbations of the right-hand side of the equation (1.2);

$$B(t_{00};\delta) = \{t_0 \in I : |\, t_0 - t_{00} \,| < \delta\},\ V(\tau_0;\delta) = \{\tau \in D : \|\, \tau - \tau_0 \,\|_I < \delta\},$$

$$V_1(\varphi_0;\delta) = \{\varphi \in \Phi : \|\, \varphi - \varphi_0 \,\|_{I_1} < \delta\},$$

where $t_{00} \in [a, b)$ is a fixed point, $\tau_0 \in D$ and $\varphi_0 \in \Phi$ are fixed functions, $\delta > 0$ is a fixed number.

Theorem 1.1. L*et* $x_0(t)$ *be the solution corresponding to* $\mu_0 = (t_{00}, \tau_0, \varphi_0, f_0) \in \Lambda$ *and defined on* $[\hat{\tau}, t_{10}], t_{10} < b$. *Let* $K_1 \subset O$ *be a compact set containing a certain neighborhood of the set* $K_0 = \varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$. *Then the following conditions hold:*

(1) *there exist numbers* $\delta_i > 0, i = 0, 1$ *such that to each element*

$$\mu = (t_0, \tau, \varphi, f_0 + \delta f) \in V(\mu_0; K_1, \delta_0, \alpha) = B(t_{00};\delta_0) \times V(\tau_0;\delta_0)$$

$$\times V_1(\varphi_0;\delta_0) \times \Big[f_0 + \Big(W(K_1;\alpha) \cap V_{K_1,\delta_0}\Big)\Big]$$

corresponds solution $x(t;\mu)$ *defined on the interval* $[\hat{\tau}, t_{10}+\delta_1] \subset I_1$ *and satisfying the condition* $x(t;\mu) \in K_1$;

(2) *for an arbitrary* $\varepsilon > 0$ *there exists a number* $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ *such that the following inequality holds for any* $\mu \in V(\mu_0; K_1, \delta_2, \alpha)$:

$$|\, x(t;\mu) - x(t;\mu_0) \,| \le \varepsilon\ \forall t \in [\theta, t_{10} + \delta_1], \theta = \max\{t_0, t_{00}\};$$

(3) *for an arbitrary* $\varepsilon > 0$ *there exists a number* $\delta_3 = \delta_3(\varepsilon) \in (0, \delta_0)$ *such that the following inequality holds for any* $\mu \in V(\mu_0; K_1, \delta_3, \alpha)$:

$$\int_{\hat{\tau}}^{t_{10}+\delta_1} |\, x(t;\mu) - x(t;\mu_0) \,|\, dt \le \varepsilon.$$

Obviously, the solution $x(t;\mu_0)$ is the continuation of the solution $x_0(t)$ and to the element $\mu = (t_0, \tau, \varphi, f_0 + \delta f) \in V(\mu_0; K_1, \delta_0, \alpha)$ corresponds the perturbed differential equation

$$\dot{x}(t) = f_0\Big(t, x(t), \int_{\tau(t)}^t \sigma(s, x(s))ds\Big) + \delta f\Big(t, x(t), \int_{\tau(t)}^t \sigma(s, x(s))ds\Big)$$

with the perturbed initial condition $x(t) = \varphi(t), t \in [\hat{\tau}, t_0]$.

In the space $E_\mu = \mathbb{R} \times D \times C(I_1) \times E_f$ we introduce the set of variations

$$\Im = \Big\{\delta\mu = (\delta t_0, \delta\tau, \delta\varphi, \delta f) \in E_\mu - \mu_0 : |\, \delta t_0 \,| \le \beta, \|\, \delta\tau \,\|_I \le \beta,$$

$$\| \delta\varphi \|_{I_1} \leq \beta, \delta f = \sum_{i=1}^{m} \lambda_i \delta f_i, \mid \lambda_i \mid \leq \beta, i = \overline{1,m} \Big\},$$

where $\beta > 0$ is a fixed number and $\delta f_i \in E_f - f_0, i = \overline{1,m}$ are fixed functions.

Theorem 1.2. *Let $x_0(t)$ be the solution corresponding to $\mu_0 \in \Lambda$ and defined on $[\hat{\tau}, t_{10}], t_{i0} \in (a,b), i = 0,1$. Let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set K_0. Then the following conditions hold:*

(4) *there exist numbers $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that for an arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times \Im$ the element $\mu_0 + \varepsilon\delta\mu \in \Lambda$, and there corresponds the solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\hat{\tau}, t_{10} + \delta_1] \subset I_1$. Moreover, $x(t; \mu_0 + \varepsilon\delta\mu) \in K_1$;*

(5) *the following relations fulfilled:*

$$\lim_{\varepsilon \to 0} \Big[\sup \Big\{ \mid x(t; \mu_0 + \varepsilon\delta\mu) - x(t; \mu_0) \mid : t \in [\theta, t_{10} + \delta_1] \Big\} \Big] = 0$$

and

$$\lim_{\varepsilon \to 0} \int_{\hat{\tau}}^{t_{10}+\delta_1} \mid x(t; \mu_0 + \varepsilon\delta\mu) - x(t; \mu_0) \mid dt = 0$$

uniformly in $\delta\mu \in \Im$, where $\theta = \max\{t_{00}, t_{00} + \varepsilon\delta t_0\}$.

Theorem 1.2 is a simple corollary of Theorem 1.1.

Let $U_0 \subset \mathbb{R}^r$ be an open set and Ω be the set of measurable functions $u(t) \in U_0, t \in I$, satisfying the conditions: $clu(I)$ is a compact set in $\mathbb{R}^r$ and $clu(I) \subset U_0$.

To each element $\varrho = (t_0, \tau, \varphi, u) \in \Lambda_1 = [a,b) \times D \times \Phi \times \Omega$ we assign the controlled differential equation with distributed delay

$$\dot{x}(t) = p\Big(t, x(t), \int_{\tau(t)}^{t} \sigma(s, x(s))ds, u(t)\Big) \tag{1.4}$$

with the initial condition (1.3). Here the function $p(t, x_1, x_2, u)$ is defined on $I \times O \times \mathbb{R}^n \times U_0$ and satisfies the following conditions: for each fixed $(x_1, x_2, u) \in O \times \mathbb{R}^n \times U_0$ the function $p(\cdot, x_1, x_2, u) : I \to \mathbb{R}^n$ is measurable; for each compact sets $K \subset O$ and $U \subset U_0$ there exist a functions $m_{K,U}(t), L_{K,U}(t) \in L_1(I, R_+)$ such that for almost all $t \in I$,

$$\mid p(t, x_1, x_2, u) \mid \leq m_{K,U}(t) \; \forall (x_1, x_2, u) \in K \times \mathbb{R}^n \times U,$$

$$\mid p(t, x_1, x_2, u_1) - p(t, y_1, y_2, u_2) \mid \leq L_{K,U}(t) \Big[\sum_{i=1}^{2} \mid x_i - y_i \mid + \mid u_1 - u_2 \mid \Big]$$

$$\forall (x_1, y_1) \in K^2, \forall (x_2, y_2) \in \mathbb{R}^n \times \mathbb{R}^n \text{ and } (u_1, u_2) \in U^2.$$

Definition 1.2. Let $\varrho = (t_0, \tau, \varphi, u) \in \Lambda_1$. A function $x(t) = x(t; \varrho) \in O, t \in [\hat{\tau}, t_1], t_1 \in (t_0, b]$, is called a solution of the equation (1.4) with the initial condition (1.3) or a solution corresponding to the element ϱ and defined on the interval $[\hat{\tau}, t_1]$, if it satisfies condition (1.3), is absolutely continuous on the interval $[t_0, t_1]$ and satisfies the equation (1.4) a.e. on $[t_0, t_1]$.

Theorem 1.3. *Let $x_0(t)$ be the solution corresponding to $\varrho_0 = (t_{00}, \tau_0, \varphi_0$*

$, u_0) \in \Lambda_1$ and defined on $[\hat{\tau}, t_{10}], t_{10} < b$. Let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $\varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$. Then the following conditions hold:

(6) *there exist numbers $\delta_i > 0, i = 0, 1$ such that to each element*

$$\varrho \in \hat{V}(\varrho_0; \delta_0) = B(t_{00}; \delta_0) \times V(\tau_0; \delta_0) \times V_1(\varphi_0; \delta_0) \times V_2(u_0; \delta_0)$$

corresponds solution $x(t; \varrho)$ defined on the interval $[\hat{\tau}, t_{10} + \delta_1] \subset I_1$ and satisfying the condition $x(t; \varrho) \in K_1$, here $V_2(u_0; \delta_0) = \{u \in \Omega :$ $\| u - u_0 \|_I < \delta_0\}$;

(7) *for an arbitrary $\varepsilon > 0$ there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ such that the following inequality holds for any $\varrho \in \hat{V}(\varrho_0; \delta_2)$:*

$$| x(t; \varrho) - x(t; \varrho_0) | \leq \varepsilon \ \forall t \in [\theta, t_{10} + \delta_1], \theta = \max\{t_0, t_{00}\};$$

(8) *for an arbitrary $\varepsilon > 0$ there exists a number $\delta_3 = \delta_3(\varepsilon) \in (0, \delta_0)$ such that the following inequality fulfilled for any $\varrho \in \hat{V}(\varrho_0; \delta_3)$:*

$$\int_{\hat{\tau}}^{t_{10}+\delta_1} | x(t; \varrho) - x(t; \varrho_0) | \, dt \leq \varepsilon.$$

2. Proof of theorem 1.1

On the continuous dependence of solution for a class of functional differential equations. To each element $\mu \in \Lambda$ we assign the functional differential equation

$$\dot{y}(t) = f\Big(t, y(t), \int_{\tau(t)}^{t} \sigma(s, h(t_0, \varphi, y)(s))ds\Big) \tag{2.1}$$

with the initial condition

$$y(t_0) = \varphi(t_0), \tag{2.2}$$

where $h : I \times \Phi \times C(I) \to C(I_1)$ is the operator given by the formula

$$h(t_0, \varphi, y)(t) = \begin{cases} \varphi(t) \text{ for } t \in [\hat{\tau}, t_0), \\ y(t) \text{ for } t \in [t_0, b]. \end{cases} \tag{2.3}$$

Definition 2.1. A*n absolutely continuous function $y(t) = y(t; \mu) \in O, t \in [r_1, r_2] \subset I$, is called a solution of the equation (2.1) with the initial condition (2.2) or the solution corresponding to the element $\mu \in \Lambda$ and defined on $[r_1, r_2]$, if $t_0 \in [r_1, r_2]$, $y(t_0) = \varphi(t_0)$ and satisfies the equation (2.1) a.e. on the interval $[r_1, r_2]$.*

Remark 2.1. *Let $y(t; \mu), t \in [r_1, r_2], \mu \in \Lambda$ be the solution of the equation (2.1) with the initial condition (2.2). Then, as is easily seen, the function*

$$x(t; \mu) = h(t_0, \varphi, y(\cdot; \mu))(t), t \in [\hat{\tau}, r_2]$$

is the solution of the equation (1.2) with the initial condition (1.3).

Theorem 2.1. *Let $y_0(t)$ be the solution corresponding to $\mu_0 \in \Lambda$ defined on $[r_1, r_2] \subset (a, b)$ and let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $K_0 = \varphi_0(I_1) \cup y_0([r_1, r_2])$. Then the following conditions hold:*

(9) *there exist numbers* $\delta_i > 0, i = 0, 1$ *such that a solution* $y(t;\mu)$ *defined on* $[r_1 - \delta_1, r_2 + \delta_1] \subset I$ *corresponds to each element*

$$\mu = (t_0, \tau, \varphi, f_0 + \delta f) \in V(\mu_0; K_1, \delta_0, \alpha).$$

Moreover, $\varphi(t) \in K_1, t \in I_1; y(t;\mu) \in K_1, t \in [r_1 - \delta_1, r_2 + \delta_1]$, *for arbitrary* $\mu \in V(\mu_0; K_1, \delta_0, \alpha)$;

(10) *for an arbitrary* $\varepsilon > 0$ *there exists a number* $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ *such that the following inequality holds for any* $\mu \in V(\mu_0; K_1, \delta_2, \alpha)$

$$| y(t;\mu) - y(t;\mu_0) | \leq \varepsilon, \forall t \in [r_1 - \delta_1, r_2 + \delta_1]. \tag{2.4}$$

Proof. Let $\varepsilon_0 > 0$ be so small that a closed ε_0-neighborhood of the set K_0 : $K_0(\varepsilon_0) = \{x \in \mathbb{R}^n : \exists \hat{x} \in K_0 \quad | x - \hat{x} | \leq \varepsilon_0\}$ lies in $intK_1$. There exist a compact set $Q : K_0(\varepsilon_0) \subset Q \subset K_1$ and a continuously differentiable function $\chi : \mathbb{R}^n \to [0, 1]$ such that,

$$\chi(x_1) = \begin{cases} 1 \ for \ x_1 \in Q, \\ 0 \ for \ x_1 \notin K_1 \end{cases} \tag{2.5}$$

(see [7, p. 41]).

To each element $\mu \in \Lambda$, we assign the functional differential equation

$$\dot{z}(t) = g\Big(t, z(t), \int_{\tau(t)}^{t} w(s, h(t_0, \varphi, z)(s))ds\Big) \tag{2.6}$$

with the initial condition

$$z(t_0) = \varphi(t_0), \tag{2.7}$$

where $g = \chi f, w = \chi\sigma$. The functions $g(t, x_1, x_2)$ and $w(s, x_1)$ satisfy the conditions :

$$| g(t, x_1, x_2) | \leq m_{f,K_1}(t), \forall x_i \in \mathbb{R}^n, i = 1, 2, \forall t \in I, \tag{2.8}$$

for $\forall x_i', x_i'' \in \mathbb{R}^n, i = 1, 2$ and for the almost all $t \in I$

$$| g(t, x_1', x_2') - g(t, x_1'', x_2'') | \leq L_f(t) \sum_{i=1}^{2} | x_i' - x_i'' |, \tag{2.9}$$

where

$$L_f(t) = L_{f,K_1}(t) + \alpha_1 m_{f,K_1}(t), \alpha_1 = \sup\Big\{ | \chi_{x_1}(x_1) | : x_1 \in \mathbb{R}^n\Big\}; \tag{2.10}$$

next

$$| w(s, x_1) | \leq m_{K_1}(s), \forall x_1 \in \mathbb{R}^n, \forall s \in I_1, \tag{2.11}$$

for $\forall x_1', x_2'' \in \mathbb{R}^n$ and for the almost all $s \in I_1$

$$| w(s, x_1') - w(s, x_1'') | \leq L(s) | x_1' - x_1'' |, \tag{2.12}$$

where $L(s) = L_{K_1}(s) + \alpha_1 m_{K_1}(s)$ (see [13, p. 5]). It is clear that if $f = f_0 + \delta f$ then

$$\begin{cases} L_{f,K_1}(t) = L_{f_0,K_1}(t) + L_{\delta f,K_1}(t), \\ m_{f,K_1}(t) = m_{f_0,K_1}(t) + m_{\delta f,K_1}(t). \end{cases} \tag{2.13}$$

The solution of the equation (2.6) with the initial condition (2.7) depends on the parameter $\mu = (t_0, \tau, \varphi, f_0 + \delta f) \in \Lambda_0 = [a, b) \times D \times \Phi \times \Big(f_0 + W(K_1; \alpha)\Big) \subset E_\mu$.

The topology in Λ_0 is induced from the vector space E_μ. On the complete metric space $C(I)$ with the distance $d(y_1, y_2) = \|y_1 - y_2\|_I$ we introduce a family

$$F(\cdot;\mu) : C(I) \to C(I) \tag{2.14}$$

of mapping depending on the parameter $\mu \in A_0$ by the formula

$$\zeta(t) = \zeta(t; z, \mu) = \varphi(t_0) + \int_{t_0}^{t} g\Big(\xi, z(\xi), \int_{\tau(\xi)}^{\xi} w(s, h(t_0, \varphi, z)(s))ds\Big)d\xi,$$

$t \in I$,where $g = \chi(f_0 + \delta f)$. Clearly, every fixed point $z(t;\mu), t \in I$, of mapping (2.14) is a solution of the equation (2.6) with the initial condition (2.7). Define the kth iteration $F^k(z;\mu)$ by

$$\zeta_k(t) = \zeta_k(t; z, \mu) = \varphi(t_0) + \int_{t_0}^{t} g\Big(\xi, \zeta_{k-1}(\xi), \int_{\tau(\xi)}^{\xi} w(s, h(t_0, \varphi,$$

$$\zeta_{k-1})(s))ds\Big)d\xi, k = 1, 2, ..., \zeta_0(t) = z(t).$$

Now let us prove that for a sufficiently large k, the family of mappings $F^k(z;\mu)$ is uniformly contractive. For this purpose, we estimate the difference

$$| \zeta_k^{'}(t) - \zeta_k^{''}(t) |=| \zeta_k(t; z^{'}, \mu) - \zeta_k(t; z^{''}, \mu) |\leq \int_a^t \Big|g\Big(\xi, \zeta_{k-1}^{'}(\xi), \int_{\tau(\xi)}^{\xi} w(s,$$

$$h(t_0, \varphi, \zeta_{k-1}^{'})(s))ds\Big) - g\Big(\xi, \zeta_{k-1}^{''}(\xi), \int_{\tau(\xi)}^{\xi} w(s, h(t_0, \varphi, \zeta_{k-1}^{''})(s))ds\Big)\Big|d\xi$$

$$\leq \int_a^t L_f(\xi)\Big\{ | \zeta_{k-1}^{'}(\xi) - \zeta_{k-1}^{''}(\xi) | + \int_{\tau(\xi)}^{\xi} L(s)|h(t_0, \varphi, \zeta_{k-1}^{'})(s)$$

$$-h(t_0, \varphi, \zeta_{k-1}^{''})(s)|ds\Big\}d\xi, k = 1, 2, ... \tag{2.15}$$

(see (2.9) and (2.12)), where a function $L_f(\xi)$ has the form (2.10),i.e.,

$$L_f(\xi) = L_{f_0+\delta f,K_1}(\xi) + \alpha_1 m_{f_0+\delta f,K_1}(\xi) = L_{f_0,K_1}(\xi) + L_{\delta f,K_1}(\xi)$$

$$+\alpha_1[m_{f_0,K_1}(\xi) + m_{\delta f,K_1}(\xi)] \tag{2.16}$$

(see (2.13)). Here , it is assumed that $\zeta_0^{'} = z^{'}(t)$ and $\zeta_0^{''} = z^{''}(t)$. It follows from the definition of the operator $h(\cdot)$ that $h(t_0, \varphi, \zeta_{k-1}^{'})(s) - h(t_0, \varphi, \zeta_{k-1}^{''})(s) = h(t_0, 0, \zeta_{k-1}^{'} - \zeta_{k-1}^{''})(s)$. Using the last equality from relation (2.15) it follows

$$| \zeta_k^{'}(t) - \zeta_k^{''}(t) |\leq \int_a^t L_f(\xi)\Big\{ \max_{\theta\in[a,\xi]} | \zeta_{k-1}^{'}(\theta) - \zeta_{k-1}^{''}(\theta) |$$

$$+ \int_{\tau(\xi)}^{\xi} L(s) \max_{\theta\in[a,\xi]} | \zeta_{k-1}^{'}(\theta) - \zeta_{k-1}^{''}(\theta) | ds\Big\}d\xi$$

$$= \int_a^t \hat{L}_f(\xi) \max_{\theta\in[a,\xi]} | \zeta_{k-1}^{'}(\theta) - \zeta_{k-1}^{''}(\theta) | d\xi,$$

where

$$\hat{L}_f(\xi) = L_f(\xi)\Big(1 + \int_{\tau(\xi)}^{\xi} L(s)ds\Big).$$

Furthermore,

$$\max_{\theta\in[a,\xi]} \mid \zeta'_{k-1}(\theta)-\zeta''_{k-1}(\theta) \mid \leq \int_a^{\xi} \hat{L}_f(\xi_1) \max_{\theta\in[a,\xi_1]} \mid \zeta'_{k-2}(\theta)-\zeta''_{k-2}(\theta) \mid d\xi_1.$$

Therefore

$$\mid \zeta'_k(t)-\zeta''_k(t) \mid \leq \int_a^t \hat{L}_f(\xi_1)d\xi_1 \int_a^{\xi_1} \hat{L}_f(\xi_2) \max_{\theta\in[a,\xi_2]} \mid \zeta'_{k-2}(\theta)-\zeta''_{k-2}(\theta) \mid d\xi_2$$

By continuing this procedure, we obtain $\mid \zeta'_k(t)-\zeta''_k(t) \mid \leq \alpha_k(t)\|z'-z''\|_I$, where

$$\alpha_k(t)=\int_a^t \hat{L}_f(\xi_1)d\xi_1 \int_a^{\xi_1} \hat{L}_f(\xi_2)d\xi_2 ... \int_a^{\xi_{k-1}} \hat{L}_f(\xi_k)d\xi_k.$$

By induction, one can readily show that

$$\alpha_k(t)=\frac{1}{k!}\Big(\int_a^t \hat{L}_f(\xi)d\xi\Big)^k.$$

Thus,

$$d(F^k(z';\mu),F^k(z'';\mu))=\|\zeta'_k-\zeta''_k\|_I \leq \frac{1}{k!}\Big(\int_I \hat{L}_f(\xi)d\xi\Big)^k\|z'-z''\|_I$$

$$=\hat{\alpha}_k\|z'-z''\|_I.$$

Let us prove the existence of a number $\alpha_2>0$ such that

$$\int_I \hat{L}_f(\xi)d\xi \leq \alpha_2, \forall f\in f_0+W(K_1;\alpha).$$

Indeed, by (2.16) we have

$$\int_I \hat{L}_f(\xi)d\xi \leq \Big(1+\int_{I_1} L(s)ds\Big)\int_I L_f(\xi)d\xi$$

$$\leq \Big(1+\int_{I_1} L(s)ds\Big)\Big(\alpha(\alpha_1+1)+\int_I [L_{f_0,K_1}(\xi)+\alpha_1 m_{f_0,K_1}(\xi)]d\xi\Big)=\alpha_2.$$

Taking into account this estimate, we obtain $\hat{\alpha}_k \leq \alpha_2^k/k!$. Consequently, there exists a positive integer k_1 such that $\hat{\alpha}_{k_1}<1$. Therefore, the k_1st iteration of the family (2.14) is contracting. By the fixed point theorem for contraction mappings (see [7, p. 61], [10, p. 608]), the mapping (2.14) has a unique fixed point for each μ. Hence it follows that the equation (2.6) with the initial condition (2.7) has a unique solution $z(t;\mu), t\in I$.

Let us prove that the mapping $F^k(z(\cdot;\mu_0);\cdot):\Lambda_0\to C(I)$ is continuous at the point $\mu=\mu_0$ for an arbitrary $k=1,2,....$ For this purpose, it suffices to show that if a sequence $\mu_i=(t_{0i},\tau_i,\varphi_i,f_i)\in\Lambda_0, i=1,2,...,$ where $f_i=f_0+\delta f_i$, converges to $\mu_0=(t_{00},\tau_0,\varphi_0,f_0)$, i.e., if

$$\lim_{i\to\infty}\Big(\mid t_{0i}-t_{00} \mid +\|\tau_i-\tau_0\|_I+\|\varphi_i-\varphi_0\|_{I_1}+H(\delta f_i;K_1)\Big)=0$$

then

$$\lim_{i\to\infty} F^k(z(\cdot;\mu_0);\mu_i)=F^k(z(\cdot;\mu_0);\mu_0)=z(\cdot;\mu_0). \tag{2.17}$$

We prove relation (2.17) by induction. Let $k=1$, then we have

$$
\begin{aligned}
\mid \zeta_1^i(t)-z_0(t)\mid \leq \mid \varphi_i(t_{0i})-\varphi_0(t_{00})\mid + \Big| \int_{t_{0i}}^{t} g_i\Big(\xi, z_0(\xi), \int_{\tau_i(\xi)}^{\xi} w(s, h(t_{0i},\varphi_i,\\
z_0)(s))ds\Big)d\xi - \int_{t_{00}}^{t} g_0\Big(\xi, z_0(\xi), \int_{\tau_0(\xi)}^{\xi} w(s,h(t_{00},\varphi_0,z_0)(s))ds\Big)d\xi\Big|\\
\leq \alpha_1^i+\alpha_2^i(t), \qquad (2.18)
\end{aligned}
$$

where $\zeta_1^i(t)=\zeta_1(t;z_0,\mu_i), z_0(t)=z(t;\mu_0), g_i=\chi f_i=g_0+\delta g_i, g_0=\chi f_0, \delta g_i=\chi\delta f_i$;

$$
\begin{aligned}
\alpha_1^i = |\varphi_i(t_{0i})-\varphi_0(t_{00})| + \Big|\int_{t_{00}}^{t_{0i}} \Big|g_0\Big(\xi, z_0(\xi), \int_{\tau_0(\xi)}^{\xi} w(s,h(t_{00},\varphi_0,\\
z_0)(s))ds\Big)\Big|d\xi\Big|, \alpha_2^i(t) = \Big|\int_{t_{0i}}^{t}\Big\{\Big|g_i\Big(\xi,z_0(\xi),\int_{\tau_i(\xi)}^{\xi} w(s,h(t_{0i},\varphi_i,z_0)(s))ds\Big)\\
-g_0\Big(\xi,z_0(\xi),\int_{\tau_0(\xi)}^{\xi} w(s,h(t_{00},\varphi_0,z_0)(s))ds\Big)\Big|\Big\}d\xi\Big|.
\end{aligned}
$$

According to (2.8) we have

$$
\alpha_1^i \leq \mid \varphi_i(t_{0i})-\varphi_0(t_{00})\mid + \Big|\int_{t_{0i}}^{t_{00}} m_{f_0,K_1}(\xi)d\xi\Big|,
$$

therefore,

$$
\lim_{i\to\infty}\alpha_1^i = 0 \qquad (2.19)
$$

After elementary transformation we obtain

$$
\begin{aligned}
\alpha_2^i(t) \leq \Big|\int_{t_{0i}}^{t}\Big\{\Big|g_0\Big(\xi,z_0(\xi),\int_{\tau_i(\xi)}^{\xi} w(s,h(t_{0i},\varphi_i,z_0)(s))ds\Big)\\
-g_0\Big(\xi,z_0(\xi),\int_{\tau_0(\xi)}^{\xi} w(s,h(t_{00},\varphi_0,z_0)(s))ds\Big)\Big|\Big\}d\xi\Big|\\
+\Big|\int_{t_{0i}}^{t}\Big\{\Big|\delta g_i\Big(\xi,z_0(\xi),\int_{\tau_i(\xi)}^{\xi} w(s,h(t_{0i},\varphi_i,z_0)(s))ds\Big)\\
-\delta g_i\Big(\xi,z_0(\xi),\int_{\tau_0(\xi)}^{\xi} w(s,h(t_{00},\varphi_0,z_0)(s))ds\Big)\Big|\Big\}d\xi\Big|\\
+\Big|\int_{t_{0i}}^{t}\delta g_i\Big(\xi,z_0(\xi),\int_{\tau_0(\xi)}^{\xi} w(s,h(t_{00},\varphi_0,z_0)(s))ds\Big)d\xi\Big|\\
\leq \alpha_3^i+\alpha_4^i+\alpha_5^i, \qquad (2.20)
\end{aligned}
$$

where

$$
\alpha_3^i = \int_I L_{f_0}(\xi)\rho^i(\xi)d\xi,\ \alpha_4^i = \int_I L_{\delta f_i}(\xi)\rho^i(\xi)d\xi
$$

$$
\begin{aligned}
\rho^i(\xi) = \Big|\int_{\tau_i(\xi)}^{\xi} w(s,h(s,h(t_{0i},\varphi_i,z_0)(s))ds - \int_{\tau_0(\xi)}^{\xi} w(s,h(s,h(t_{00},\varphi_0,\\
z_0)(s))ds\Big|, \alpha_5^i = \max_{t',t''\in I}\Big|\int_{t'}^{t''}\delta g_i\Big(\xi,z_0(\xi),\int_{\tau_0(\xi)}^{\xi} w(s,h(t_{00},\varphi_0,z_0)(s))ds\Big)d\xi\Big|.
\end{aligned}
$$

Introduce notation $s_{1i} = \min(t_{0i}, t_{00}), s_{2i} = \max(t_{0i}, t_{00})$. It easy to see that $\lim_{i\to\infty}(s_{2i} - s_{1i}) = 0$. Now we estimate $\rho^i(\xi)$. We have

$$\rho^i(\xi) \le \Big|\int_{\tau_i(\xi)}^{\tau_0(\xi)} m_{K_1}(s)ds\Big| + \int_{\tau_0(\xi)}^{\xi} L(s)|h(t_{0i}, \varphi_i, z_0)(s) - h(t_{00}, \varphi_0, z_0)(s)|ds$$
$$\le \rho_1^i + \int_{\hat\tau}^{s_{1i}} L(s)|\varphi_i(s) - \varphi_0(s)|ds + \int_{s_{1i}}^{s_{2i}} L(s)\{|z_0(s) - \varphi_i(s)|$$
$$+|z_0(s) - \varphi_0(s)|\}ds \le \rho_1^i + \rho_2^i,$$

where

$$\rho_1^i = \max_{\xi\in I}\Big|\int_{\tau_i(\xi)}^{\tau_0(\xi)} m_{K_1}(s)ds\Big|, \rho_2^i = \|\varphi_i - \varphi_0\|_{I_1}\int_{I_1} L(s)ds$$
$$+2\|z_0 - \varphi_0\|_I \int_{s_{1i}}^{s_{2i}} L(s)ds$$

(see (2.11)). Thus ,

$$\alpha_3^i \le (\rho_1^i + \rho_2^i)\int_I L_{f_0}(\xi)d\xi$$

Analogously we prove that

$$\alpha_4^i \le (\rho_1^i + \rho_2^i)\int_I L_{\delta f_i}(\xi)d\xi \le (\rho_1^i + \rho_2^i)\alpha(1 + \alpha_1)$$

(see (2.10)). Consequently,

$$\lim_{i\to\infty} \alpha_3^i = \lim_{i\to\infty} \alpha_4^i = 0. \tag{2.21}$$

Obviously, $H(\delta g_i; K_1) = H(\chi\delta f_i; K_1) \le H(\delta f_i; K_1)$ (see (1.1),(2.5)). Since $H(\delta f_i; K_1) \to 0$ as $i \to \infty$, we have $\lim_{i\to\infty} H(\delta g_i; K_1) = 0$. This allows us to use Lemma 1.1.5 given in [13, p. 7], which in turn, implies

$$\lim_{i\to\infty} \alpha_5^i = 0. \tag{2.22}$$

Conditions (2.21) and (2.22) yield

$$\lim_{i\to\infty} \alpha_2^i(t) = 0. \tag{2.23}$$

uniformly in $t \in I$ (see (2.20)). Taking into account (2.19) and (2.23) we get $\|\zeta_1^i - z_0\|_I = 0$ (see (2.18)). Relation (2.17) is proved for $k = 1$. Let (2.17) holds for a certain $k > 1$; we will prove it for $k + 1$. Elementary transformations yield:

$$|\zeta_{k+1}^i(t) - z_0(t)| \le |\varphi_i(t_{0i}) - \varphi_0(t_{00})| + \Big|\int_{t_{0i}}^{t} g_i\Big(\xi, \zeta_k^i(\xi), \int_{\tau_i(\xi)}^{\xi} w(s, h(t_{0i}, \varphi_i,$$
$$\zeta_k^i)(s))ds\Big)d\xi - \int_{t_{00}}^{t} g_0\Big(\xi, z_0(\xi), \int_{\tau_0(\xi)}^{\xi} w(s, h(t_{00}, \varphi_0, z_0)(s))ds\Big)d\xi\Big|$$
$$\le \alpha_1^i + \alpha_2^i(t) + \alpha_{3k}^i(t),$$

where

$$\alpha_{3k}^i(t) = \Big|\int_{t_{0i}}^{t}\Big\{\Big|g_i\Big(\xi, \zeta_k^i(\xi), \int_{\tau_i(\xi)}^{\xi} w(s, h(t_{0i}, \varphi_i, \zeta_k^i)(s))ds\Big)$$
$$-g_i\Big(\xi, z_0(\xi), \int_{\tau_i(\xi)}^{\xi} w(s, h(t_{0i}, \varphi_i, z_0)(s))ds\Big)\Big|\Big\}d\xi\Big|.$$

The quantities α_1^i and $\alpha_2^i(t)$ have been estimated in the preceding, and it remains to estimate $\alpha_{3k}^i(t)$. We have

$$\alpha_{3k}^i \leq \Big|\int_{t_{0i}}^t L_{f_i}(\xi)\Big\{|\zeta_k^i(\xi)-z_0(\xi)|+\int_{\tau_i(\xi)}^{\xi} L(s)|h(t_{0i},0,\zeta_k^i-z_0)(s)|ds\Big\}d\xi\Big|$$

$$\leq \|\zeta_k^i-z_0\|_I\int_I L_{f_i}(\xi)\Big(1+\int_{I_1}L(s)ds\Big)d\xi \leq \|\zeta_k^i-z_0\|_I\Big(1+\int_{I_1}L(s)ds\Big)\alpha_2.$$

Since $\lim_{i\to\infty}\|\zeta_k^i-z_0\|_I=0$ it follows that

$$\lim_{i\to\infty}\alpha_{4k}^i=0 \tag{2.24}$$

According to (2.19),(2.23) and (2.24), we have $\lim_{i\to\infty}\|\zeta_{k+1}^i-z_0\|_I=0$. Relation (2.17) is proved for every $k=1,2,....$

Let a number $\delta_1>0$ be so small that $[r_1-\delta_1,r_2+\delta_1]\subset I$ and $|z(t;\mu_0)-z(r_1;\mu_0)|\leq\varepsilon_0/2$ for $t\in[r_1-\delta_1,r_1]$ and $|z(t;\mu_0)-z(r_2;\mu_0)|\leq\varepsilon_0/2$ for $t\in[r_2,r_2+\delta_1]$. We can conclude from the uniqueness of the solution $z(t;\mu_0)$ that $z(t;\mu_0)=y_0(t)$ for $t\in[r_1,r_2]$. Taking into account the above inequalities, we have $z(t;\mu_0)\in K_0(\varepsilon_0/2), t\in[r_1-\delta_1,r_2+\delta_1], h(t_{00},\varphi_0,z(\cdot;\mu_0))(s))\in K_0(\varepsilon_0/2), s\in[\tau_0(t),t]$. Hence $\chi(z(t;\mu_0))=1, t\in[r_1-\delta_1,r_2+\delta_1], \chi(h(t_{00},\varphi_0,z(\cdot;\mu_0))(s))=1, s\in[\tau_0(t),t]$ and the function $z(t;\mu_0)$ satisfies the equation

$$\dot y(t)=f_0\Big(t,y(t),\int_{\tau_0(t)}^t\sigma(s,h(t_{00},\varphi_0,y)(s))ds\Big), t\in[r_1-\delta_1,r_2+\delta_1]$$

and the initial condition $y(t_{00})=\varphi_0(t_{00})$. Therefore, $y(t;\mu_0)=z(t;\mu_0)$, $t\in[r_1-\delta_1,r_2+\delta_1]$. According to the fixed point theorem, for $\varepsilon_0/2$ there exists a number $\delta_0\in(0,\varepsilon_0)$ such that a solution $z(t;\mu)$ satisfying the condition $|z(t;\mu)-z(t;\mu_0)|\leq\varepsilon_0/2, t\in I$, corresponds to each element $\mu\in V(\mu_0;K_1,\delta_0,\alpha)$. Therefore, for $t\in[r_1-\delta_1,r_2+\delta_1], z(t;\mu)\in K_0(\varepsilon_0)\forall\mu\in V(\mu_0;K_1,\delta_0,\alpha)$. Taking into account that $\varphi(t)\in K(\varepsilon_0)$, we see that for $t\in[r_1-\delta_1,r_2+\delta_1]$ and $s\in[\tau(t),t]$ this implies $\chi(z(t;\mu))=1, \chi(h(t_0,\varphi,z(\cdot;\mu))(s))=1\forall\mu\in V(\mu_0;K_1,\delta_0,\alpha)$. Hence the function $z(t;\mu)$ satisfies the equation (2.1) and condition (2.2), i.e.,

$$y(t;\mu)=z(t;\mu)\in K_1, t\in[r_1-\delta_1,r_2+\delta_1], \mu\in V(\mu_0;K_1,\delta_0,\alpha). \tag{2.25}$$

The first part of Theorem 2.1 is proved. By the fixed point theorem , for an arbitrary $\varepsilon>0$, there exists a number $\delta_2=\delta_2(\varepsilon)\in(0,\delta_0)$ such that for each $\mu\in V(\mu_0;K_1,\delta_0,\alpha)$ we have $|z(t;\mu)-z(t;\mu_0)|\leq\varepsilon, t\in I$. Whence using (2.25), we obtain (2.4). □

Proof of Theorem 1.1. In Theorem 2.1 , let $r_1=t_{00}$ and $r_2=t_{10}$. Obviously, the solution $x_0(t)$ satisfies the following equation on the interval $[t_{00},t_{10}]$:

$$\dot y(t)=f_0\Big(t,y(t),\int_{\tau_0(t)}^t\sigma(s,h(t_{00},\varphi_0,y)(s))ds\Big).$$

Therefore, in Theorem 2.1, as the solution $y_0(t)$ we can take the function $x_0(t), t\in[t_{00},t_{10}]$. By Theorem 2.1, there exist numbers $\delta_i>0, i=0,1$, and for an arbitrary $\varepsilon>0$, there exists a number $\delta_2=\delta_2(\varepsilon)\in(0,\delta_0]$ such that the solution $y(t;\mu), t\in$

$[t_{00}-\delta_1, t_{10}+\delta_1]$, corresponds to each $\mu \in V(\mu_0; K_1, \delta_0, \alpha)$. Moreover, the following conditions hold:

$$\begin{cases} \varphi(t) \in K_1, t \in I_1; y(t;\mu) \in K_1, \\ |y(t:\mu) - y(t;\mu_0)| \le \varepsilon, t \in [t_{00} - \delta_1, t_{10} + \delta_1], \\ \mu \in V(\mu_0; K_1, \delta_0, \alpha). \end{cases} \tag{2.26}$$

For an arbitrary $\mu \in V(\mu_0; K_1, \delta_0, \alpha)$, the function

$$x(t;\mu) = \begin{cases} \varphi(t), t \in [\hat{\tau}, t_0), \\ y(t;\mu), t \in [t_0, t_{10} + \delta_1]. \end{cases}$$

is the solution corresponding to μ. Moreover, if $t \in [\theta, t_{10} + \delta_1]$, where $\theta = \max\{t_0, t_{00}\}$ then $x(t;\mu_0) = y(t;\mu_0)$ and $x(t;\mu) = y(t;\mu)$. Taking into account (2.26), we see that this implies (1) and (2). It is easy to note that for an arbitrary $\mu \in V(\mu_0; K_1, \delta_0, \alpha)$, we have

$$\int_{\hat{\tau}}^{t_{10}+\delta_1} |x(t;\mu) - x(t;\mu_0)|dt = \int_{\hat{\tau}}^{\theta_0} |\varphi(t) - \varphi_0(t)|dt + \int_{\theta_0}^{\theta} |x(t;\mu)$$

$$-x(t;\mu_0)|dt + \int_{\theta}^{t_{10}+\delta_1} |x(t;\mu) - x(t;\mu_0)|dt \le \| \varphi - \varphi_0 \|_{I_1} (b - \hat{\tau})$$

$$+M|t_0 - t_{00}| + \max_{t\in[\theta, t_{10}+\delta_1]} |x(t;\mu) - x(t;\mu_0|(b - \hat{\tau}),$$

where $\theta_0 = \min\{t_0, t_{00}\}, M = \sup\{|x^{'} - x^{''}| : x^{'}, x^{''} \in K_1\}$. By (1) and (2) this inequality implies (3). □

3. Proof of theorem 1.3

To each element $\varrho \in \Lambda_1$ we will set in correspondence the functional differential equation

$$\dot{y}(t) = p\Big(t, y(t), \int_{\tau(t)}^{t} \sigma(s, h(t_0, \varphi, y))(s))ds, u(t)\Big), \tag{3.1}$$

with the initial condition (2.2).

Theorem 3.1. *Let $y_0(t)$ be a solution corresponding to $\varrho_0 = (t_{00}, \tau_0, \varphi_0, u_0) \in \Lambda_1$ defined on $[r_1, r_2] \subset (a, b)$ and let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $K_0 = \varphi_0(I_1) \cup y_0([r_1, r_2])$. Then the following conditions hold:*

(11) *there exist numbers $\delta_i > 0, i = 0, 1$ such that to each element $\varrho = (t_0, \tau, \varphi, u) \in \hat{V}(\varrho_0; \delta_0) = B(t_{00}; \delta_0) \times V(\tau_0; \delta_0) \times V_1(\varphi_0; \delta_0) \times V_2(u_0; \delta_0)$ corresponds solution $y(t;\varrho)$ defined on the interval $[r_1 - \delta_1, r_2 + \delta_1] \subset I$ and satisfying the condition $y(t;\varrho) \in K_1$;*

(12) *for an arbitrary $\varepsilon > 0$, there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ such that the following inequality holds for any $\varrho \in \hat{V}(\varrho_0; \delta_2)$:*

$$| y(t;\varrho) - y(t;\varrho_0) | \le \varepsilon \ \forall t \in [r_1 - \delta_1, r_2 + \delta_1].$$

Proof. Rewrite the equation (3.1) in the form

$$\dot{y}(t) = p_0\Big(t, y(t), \int_{\tau(t)}^{t} w(s, h(t_0, \varphi, y)(s))ds\Big)$$

$$+\delta p_u\Big(t, y(t), \int_{\tau(t)}^{t} w(s, h(t_0, \varphi, y)(s)ds\Big),$$

where

$$p_0(t, x_1, x_2) = p(t, x_1, x_2, u_0(t)) \in E_f,$$

$$\delta p_u(t, x_1, x_2) = p(t, x_1, x_2, u(t)) - p_0(t, x_1, x_2) \in E_f.$$

Let $\hat{\delta}_0 > 0$ be a number so small that $V_2(u_0; \hat{\delta}_0) \subset \Omega$. There exists a compact set $U \subset U_0$ such that any function from the neighborhood $V_2(u_0; \hat{\delta}_0)$ assumes its values in U.

Let $K \subset O$ be a compact set. There exists a function $L_{K,U}(t) \in L_1(I, \mathbb{R}_+)$ such that for almost all $t \in I$ and $\forall x_1^{'}, x_1^{''} \in K, \forall x_2^{'}, x_2^{''} \in \mathbb{R}^n, \forall u^{'}, u^{''} \in U$ the following inequality holds:

$$|p(t, x_1^{'}, x_2^{'}, u^{'}) - p(t, x_1^{''}, x_2^{''}, u^{''})| \le L_{K,U}(t)\Big[\sum_{i=1}^{2} |x_i^{'} - x_i^{''}| + |u^{'} - u^{''}|\Big].$$

Hence

$$|\delta p_u(t, x_1, x_2)| \le L_{K,U}(t)|u(t) - u_0(t)| \le \hat{\delta}_0 L_{K,U}(t)$$

$$\forall x_i \in K, i = 1, 2, \forall u \in V_2(u_0; \hat{\delta}_0)$$

and

$$|\delta p_u(t, x_1^{'}, x_2^{'}) - \delta p_u(t, x_1^{''}, x_2^{''})| \le 2L_{K,U}(t) \sum_{i=1}^{2} |x_i^{'} - x_i^{''}|,$$

$$\forall x_i^{'}, x_i^{''} \in K, i = 1, 2.$$

It is easy to see that for $\delta \in (0, \hat{\delta}_0]$ the following inclusions hold $\{\delta p_u(t, x_1, x_2) : u \in V_2(u_0; \delta)\} \subset W(K; \alpha), \{\delta p_u(t, x_1, x_2) : u \in V_2(u_0; \delta)\}$
$\subset V_{K,\hat{\delta}_1}$, where

$$\alpha = (2 + \hat{\delta}_0) \int_I L_{K,U}(t)dt, \hat{\delta}_1 = \delta \int_I L_{K,U}(t)dt.$$

Now we can use Theorem 2.1, which, is turn, proves Theorem 3.1. □

Proof of Theorem 1.3. In Theorem 3.1, let $r_1 = t_{00}$ and $r_2 = t_{10}$. Obviously, the solution $x_0(t)$ satisfies the following equation on the interval $[t_{00}, t_{10}]$:

$$\dot{y}(t) = p(t, y(t), \int_{\tau(t)}^{t} \sigma(s, h(t_0, \varphi_0, y)(s))ds, u_0(t)\Big).$$

Therefore, in Theorem 3.1, as the solution $y_0(t)$, we can take the function $x_0(t), t \in [t_{00}, t_{10}]$. After that, the proof of the theorem completely coincides with that Theorem 1.1; for this purpose, it suffices to replace the element μ by the element ϱ and the set $V(\mu_0; K_1, \delta_0, \alpha)$ by the set $\hat{V}(\varrho_0; \delta_0)$ everywhere.

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Ph. Dvalishvili
Department of Computer Sciences, Tbilisi State University, Tbilisi, Georgia.
E-mail address: pridon.dvalishvili@tsu.ge

A. Nachaoui
Jean Leray Laboratory of Mathematics , University of Nantes, Nantes, France.
E-mail address: nachaoui@math.cnrs.fr

M. Nachaoui
Faculty of Sciences and Techniques, Sultan Moulay Slimane University, Beni-Mellal, Morocco.
E-mail address: nachaoui@gmail.com

T. Tadumadze
Department of Mathematics & *I. Vekua Institute of Applied Mathematics, Tbilisi State University, Tbilisi, Georgia.*
E-mail address: tamaz.tadumadze@tsu.ge

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