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PROBLEMS OF g-LIFTS

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Abstract. The main purpose of this paper is to transfer complete lifts from the tangent bundle to the cotangent bundle by using a musical isomorphism between these bundles. The g-lifts of tensor fields are described in this study newly. We study some problems of g-lifts that are constituted by transferring the tensor fields via musical isomorphism. The results obtained give new remarks about g-lifts on the cotangent bundle.

1. Introduction

Let M be a Riemannian manifold of dimension n with a pseudo-Riemannian metric g. We denote by $TM = \bigcup_{x\in M_n} T_xM$ the tangent bundle over M with local coordinates (x^i, y^i) where $y_x = y^i\frac{\partial}{\partial x^i} \in T_xM$,$\forall x \in M$. We denote by $T^*M = \bigcup_{x\in M_n} T^*_xM$ the cotangent bundle over M with local coordinates (x^i, p_i) where $p_x = p_i dx^i \in T^*_xM$, $\forall x \in M$. Throughout this paper we assume the manifolds, tensor fields and connections to be differentiable of class C^∞. We always use the ranges of the index i being $\{1, ..., n\}$ and the index $\bar{i}$ being $\{n + 1, ..., 2n\}$.

Any pseudo-Riemannian metric g is that it supplies a musical isomorphisms $g^\flat : TM \to T^*M$ from the tangent to cotangent bundles and $g^\sharp : T^*M \to TM$ from the cotangent to tangent bundles. Some properties of geometric structures on cotangent bundle with respect to the musical isomorphism are studied in [2], [1].

The musical isomorphisms $g^\flat$ and $g^\sharp$ are expressed by

$$g^\flat : x^I = (x^i, x^{\bar{i}}) = (x^i, y^i) \to \tilde{x}^K = (x^k, \tilde{x}^{\bar{k}}) = (\delta^k_i x^i, p_k = g_{ki}y^i)$$

and

$$g^\sharp : \tilde{x}^K = (x^k, \tilde{x}^{\bar{k}}) = (x^k, p_k) \to x^I = (x^i, x^{\bar{i}}) = (\delta^i_k x^k, y^i = g^{ik}p_k)$$

with respect to the local coordinates, respectively. The Jacobian matrices of $g^\flat$ and $g^\sharp$ are given by

$$(g^\flat_*) = \left(\frac{\partial \tilde{x}^K}{\partial x^I}\right) = \begin{pmatrix} \delta^k_i & 0 \\ y^s\partial_i g_{ks} & g_{ki} \end{pmatrix} \tag{1.1}$$

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and

$$(g_*^\sharp) = \left(\frac{\partial x^I}{\partial \tilde{x}^K}\right) = \begin{pmatrix} \delta_k^i & 0 \\ p_s \partial_k g^{is} & g^{ik} \end{pmatrix}, \tag{1.2}$$

respectively, where δ is the Kronecker delta.

We denote by $\Im_q^p(M)$ the set of all differentiable tensor fields of type (p,q) on M. Let ${}^C X_T \in \Im_0^1(TM)$, ${}^C\varphi_T \in \Im_1^1(TM)$, ${}^C S_T \in \Im_2^1(TM)$ and ${}^C g_T \in \Im_2^0(TM)$ be complete lifts of tensor fields $X \in \Im_0^1(M)$, $\varphi \in \Im_1^1(M)$,$S \in \Im_2^1(M)$ and $g \in \Im_2^0(M)$ to the tangent bundle TM, respectively.

When we transferred the complete lifts of the tensor fields from the tangent bundles to the cotangent bundles we obtained the g-lift of tensor fields to cotangent bundles. The aim of this paper is to study the g-lift problems of cotangent bundles of Riemannian manifolds. The results are significant for a better understanding of g-lifts on cotangent bundles.

2. Problems of g-lifts of vector fields

Let $X = X^a \partial_a$ be the local expression in $U \subset M$ of a vector field $X \in \Im_0^1(M)$. Then the complete lift ${}^C X_T$ of X to the tangent bundle TM is given by [5, p.15]

$${}^C X_T = X^a \partial_a + y^s \partial_s X^a \partial_{\overline{a}} \tag{2.1}$$

with respect to the natural frame $\{\partial_a, \partial_{\overline{a}}\}$.

Using (1.1) and (2.1), we have

$$g_*^\flat\,{}^C X_T = \begin{pmatrix} \delta_a^k & 0 \\ y^s \frac{\partial g_{ks}}{\partial x^a} & g_{ka} \end{pmatrix} \begin{pmatrix} X^a \\ y^s \partial_s X^a \end{pmatrix} = \begin{pmatrix} X^k \\ y^s \left(L_X g\right)_{sk} - p_a\left(\partial_k X^a\right) \end{pmatrix}, \tag{2.2}$$

where $p_i = g_{is} y^s$ and L_X is the Lie derivation of g with respect to the vector field X:

$$\left(L_X g\right)_{sk} = X^a \partial_a g_{sk} + \left(\partial_s X^a\right) g_{ak} + \left(\partial_k X^a\right) g_{sa}.$$

The vector field in the form (2.2) is called a $g-lift$ of X to cotangent bundle T^*M.

It is well known that the complete lift ${}^C X_{T^*}$ of X to the cotangent bundle T^*M is given by [5, p.236]

$${}^C X_{T^*} = X^a \partial_a - p_s \partial_a X^s \partial_{\overline{a}}\,.$$

From (2.2) we find $g-lift$ $g_*^\flat\,{}^C X_T$ of X:

$$g_*^\flat\,{}^C X_T = {}^C X_{T^*} + \gamma(L_X g)\,,$$

where the vertical vector field $\gamma(L_X g)$ in T^*M is defined by

$$\gamma(L_X g) = \begin{pmatrix} 0 \\ y^s\left(L_X g\right)_{sa} \end{pmatrix}.$$

In a manifold (M,g), a vector field X is called a Killing vector field if $L_X g = 0$. Thus we have

Theorem 2.1. *Let (M,g) be a pseudo-Riemannian manifold. Let ${}^C X_T$ and ${}^C X_{T^*}$ be complete lifts of a vector field X to the tangent and cotangent bundles, respectively. Then the differential (pushforward) of ${}^C X_T$ by $g^\flat$, i.e. a $g-lift$ $g_*^\flat\,{}^C X_T$ in the cotangent bundle T^*M is a complete lift ${}^C X_{T^*}$ to the cotangent bundle if X is a Killing vector field.*

Using (1.1) and (2.1) we have

$$\left(g_*^\flat\ \left[{}^C X_T, {}^C Y_T\right]\right)^i =$$
$$={}^C X^a \partial_a {}^C Y^i - {}^C Y^a \partial_a {}^C X^i + {}^C X^{\overline{a}} \partial_{\overline{a}} {}^C Y^i - {}^C Y^{\overline{a}} \partial_{\overline{a}} {}^C X^i$$
$$X^a \partial_a Y^i - Y^a \partial_a X^i = [X,Y]^i ,$$
$$\left(g_*^\flat\ \left[{}^C X_T, {}^C Y_T\right]\right)^{\overline{i}}$$
$$={}^C X^a \partial_a {}^C Y^{\overline{i}} -{}^C Y^a \partial_a {}^C X^{\overline{i}} +{}^C X^{\overline{a}} \partial_{\overline{a}} {}^C Y^{\overline{i}} -{}^C Y^{\overline{a}} \partial_{\overline{a}} {}^C X^{\overline{i}}$$
$$= X^a \partial_a \left(y^s (L_Y g)_{si} - p_k \left(\partial_i Y^k\right)\right)$$
$$+ \left(y^s \left(L_X g\right)_{sa} - p_k \left(\partial_a X^k\right)\right) \partial_{\overline{a}} \left(y^s (L_Y g)_{si} - p_k \left(\partial_i Y^k\right)\right)$$
$$-Y^a \partial_a \left(y^s \left(L_X g\right)_{si} - p_k \left(\partial_i X^k\right)\right)$$
$$- \left(y^s (L_Y g)_{sa} - p_k \left(\partial_a Y^k\right)\right) \partial_{\overline{a}} \left(y^s \left(L_X g\right)_{si} - p_k \left(\partial_i X^k\right)\right)$$
$$= y^s X^a \partial_a (L_Y g)_{si} - p_k X^a \partial_a \left(\partial_i Y^k\right)$$
$$+ \left(y^s \left(L_X g\right)_{sa} - p_k \left(\partial_a X^k\right)\right) \partial_{\overline{a}} \left(g^{ts} p_t (L_Y g)_{sa} - p_k \left(\partial_i Y^k\right)\right)$$
$$-y^s Y^a \partial_a \ \left(L_X g\right)_{si} + p_k Y^a \partial_a \left(\partial_i X^k\right)$$
$$- \left(y^s (L_Y g)_{sa} - p_k \left(\partial_a Y^k\right)\right) \partial_{\overline{a}} \left(g^{st} p_t \left(L_X g\right)_{si} - p_k \left(\partial_i X^k\right)\right)$$
$$= y^m X^a \partial_a (L_Y g)_{mi} - X^a \partial_a p_k \left(\partial_i Y^k\right)$$
$$+ \left(y^m \left(L_X g\right)_{ma} - p_k \left(\partial_a X^k\right)\right) \left(g^{as} (L_Y g)_{si} - (\partial_i Y^a)\right)$$
$$-y^m Y^a \partial_a \left(L_X g\right)_{mi} + Y^a \partial_a p_k \left(\partial_i X^k\right)$$
$$- \left(y^m (L_Y g)_{ma} - p_k \left(\partial_a Y^k\right)\right) \left(g^{sa} \left(L_X g\right)_{si} - \left(\partial_i X^i\right)\right)$$
$$= y^m X^a \partial_a (L_Y g)_{mi} - X^a \partial_a p_k \left(\partial_i Y^k\right)$$
$$+y^m g^{as} \left(L_X g\right)_{ma} \left(L_Y g\right)_{si} - y^m \left(L_X g\right)_{ma} \left(\partial_i Y^a\right)$$
$$-p_k g^{as} \left(\partial_a X^k\right) \left(L_Y g\right)_{si} + p_k \left(\partial_a X^k\right) \left(\partial_i Y^a\right)$$
$$-y^m Y^a \partial_a \left(L_X g\right)_{mi} + Y^a \partial_a p_k \left(\partial_i X^k\right)$$
$$-y^m g^{as} (L_Y g)_{ma} \left(L_X g\right)_{si} + y^m (L_Y g)_{ma} \left(\partial_i X^a\right)$$
$$+p_k g^{sa} \left(\partial_a Y^k\right) \left(L_X g\right)_{si} - p_k \left(\partial_a Y^k\right) \left(\partial_i X^a\right)$$
$$= -p_k \left(\partial_i \left[X,Y\right]^k\right) + y^m X^a \partial_a (L_Y g)_{mi}$$
$$+y^m g^{as} \left(L_X g\right)_{ma} \left(L_Y g\right)_{si} - y^m \left(L_X g\right)_{ma} \left(\partial_i Y^a\right)$$
$$-p_k g^{as} \left(\partial_a X^k\right) \left(L_Y g\right)_{si} - y^m Y^a \partial_a \left(L_X g\right)_{mi}$$
$$-y^m g^{as} (L_Y g)_{ma} \left(L_X g\right)_{si} + y^m (L_Y g)_{ma} \left(\partial_i X^a\right)$$
$$+p_k g^{sa} \left(\partial_a Y^k\right) \left(L_X g\right)_{si}$$

$$= p_k \left(\partial_i [X,Y]^k\right) - y^m g^{sa}(L_Y g)_{ma} (L_X g)_{si}$$
$$+ y^m \left(X^a \partial_a (L_Y g)_{mi} + (\partial_i X^a)(L_Y g)_{ma}\right)$$
$$+ (\partial_m X^a)(L_Y g)_{ai} - y^m (\partial_m X^a)(L_Y g)_{ai}$$
$$+ y^m (\partial_m Y^a)(L_X g)_{ai} + y^m g^{as}(L_X g)_{ma}(L_Y g)_{si}$$
$$- p_k g^{as}\left(\partial_a X^k\right)(L_Y g)_{si} + p_k g^{sa}\left(\partial_a Y^k\right)(L_X g)_{si}$$
$$- y^m \left(Y^a \partial_a (L_X g)_{mi} + (\partial_i Y^a)(L_X g)_{ma} + (\partial_m Y^a)(L_X g)_{ai}\right)$$
$$= -p_k \left(\partial_i [X,Y]^k\right) + y^m L_X (L_Y g)_{mi} - y^m L_Y (L_X g)_{mi}$$
$$+ y^m g^{as}(L_X g)_{ma}(L_Y g)_{si} - y^m g^{sa}(L_Y g)_{ma}(L_X g)_{si} - y^m (\partial_m X^a)(L_Y g)_{ai}$$
$$+ y^m (\partial_m Y^a)(L_X g)_{ai} - p_k g^{as}\left(\partial_a X^k\right)(L_Y g)_{si} + p_k g^{sa}\left(\partial_a Y^k\right)(L_X g)_{si}$$
$$= -p_k \left(\partial_i [X,Y]^k\right) + y^m [L_X, L_Y] g_{mi} + y^m g^{as}(L_X g)_{ma}(L_Y g)_{si}$$
$$- y^m g^{sa}(L_Y g)_{ma}(L_X g)_{si} - y^m (\partial_m X^a)(L_Y g)_{ai} + y^m (\partial_m Y^a)(L_X g)_{ai}$$
$$- p_k g^{as}\left(\partial_a X^k\right)(L_Y g)_{si} + p_k g^{sa}\left(\partial_a Y^k\right)(L_X g)_{si}.$$

Thus we have

Theorem 2.2. *Let (M, g) be a pseudo-Riemannian manifold, and let ${}^C X_T$, ${}^C Y_T$ and ${}^C X_{T^*}$, ${}^C Y_{T^*}$ be complete lifts of the vector fields X and Y to the tangent and cotangent bundles, respectively. Then the differential (pushforward) of $\left[{}^C X_T, {}^C Y_T\right]$ by $g^\flat$, i.e. a $g - lift$ $g^\flat_* \left[{}^C X_T, {}^C Y_T\right]$ in the cotangent bundle T^*M is a complete lift ${}^C [X, Y]_{T^*}$ to the cotangent bundle if X and Y are Killing vector fields.*

Since ${}^C [X,Y]_{T^*} = \left[{}^C X_{T^*}, {}^C Y_{T^*}\right]$ (see [5, p.238]), from Theorem 2.2 we have

Theorem 2.3. *If X, Y are Killing vector fields, then $\left[{}^C X_T, {}^C Y_T\right]$ is $g^\flat - related$ to $\left[{}^C X_{T^*}, {}^C Y_{T^*}\right]$, where ${}^C X_T, {}^C Y_T$ and ${}^C X_{T^*}, {}^C Y_{T^*}$ are complete lifts of X and Y to tangent and cotangent bundles, respectively.*

3. Problems of g-lifts of affinor fields

Let M_{2n} be a pseudo-Riemannian manifold with a neutral metric, i.e. with a pseudo-Riemannian metric g of signature (n, n). We say (M_{2n}, φ) is an almost complex manifold if M_{2n} can be endowed with an affinor field $\varphi \in \Im^1_1(M_{2n})$ such that $\varphi^2 = -I$,where I is a field of identity endomorphisms. If the Nijenhuis tensor field $N_\varphi \in \Im^1_2(M_{2n})$ vanishes, then φ is a complex structure and moreover M_{2n} is a C-holomorphic manifold X_n (C) whose transition functions are C-holomorphic mappings. A metric g is a Norden metric if

$$g(\varphi X, Y) = g(X, \varphi Y)$$

for any $X, Y \in \Im^1_0(M_{2n})$, i.e. g is pure with respect to φ [3].

Let $\varphi = \varphi^i_j \partial_i \otimes dx^j$ be the local expression in $U \subset M$ of an affinor field φ. Then the complete lift ${}^C\varphi_T$ of φ to the tangent bundle TM is given by [5, p.21]

$$ {}^C\varphi_T = ({}^C\varphi^I_J) = \begin{pmatrix} \varphi^i_j & 0 \\ y^s \partial_s \varphi^i_j & \varphi^i_j \end{pmatrix} \tag{3.1}$$

with respect to the induced coordinates $(x^i, x^{\bar{i}}) = (x^i, y^i)$ in TM. It is well known that ${}^C\varphi_T$ defines an almost complex structure on TM, if and only if so does φ on M.

Using (1.1), (1.2) and (3.1), we have

$$g_*^\flat\, {}^C\varphi_T = (\widetilde{\varphi}\,_L^J) = (\frac{\partial x^J}{\partial \tilde{x}^I}\frac{\partial \tilde{x}^K}{\partial x^L}\, {}^C\varphi_K^I)$$

$$= \begin{pmatrix} \varphi_l^j & 0 \\ y^s \phi_l g_{sj} + p_s(\partial_l \varphi_j^s - \partial_j \varphi_l^s) & g_{ji} g^{kl} \varphi_k^i \end{pmatrix}, \qquad (3.2)$$

where

$$\phi_k g_{ij} = \varphi_k^m \partial_m g_{ij} - \partial_k\left(g \circ \varphi\right)_{ij} + g_{mj}\partial_i \varphi_k^m + g_{im}\partial_j \varphi_k^m \,.$$

The expression (3.2) is called a $g-lift$ $g_*^\flat\, {}^C\varphi_T$ of φ to cotangent bundle T^*M. It is well known that the complete lift ${}^C\varphi_{T^*}$ of $\varphi \in \Im_1^1(M)$ to the cotangent bundle T^*M is given by [5, p.242]

$${}^C\varphi_{T^*} = \begin{pmatrix} \varphi_l^j & 0 \\ p_s(\partial_l \varphi_j^s - \partial_j \varphi_l^s) & \varphi_j^l \end{pmatrix}$$

with respect to the induced coordinates in T^*M.

From (3.2) equation we find $g-lift$ $g_*^\flat\, {}^C\varphi_T$ of φ

$$g_*^\flat\, {}^C\varphi_T = {}^C\varphi_{T^*} + \gamma(\phi_\varphi g)\,,$$

where $\gamma(\phi_\varphi g)$ is defined by

$$\gamma(\phi_\varphi g) = \begin{pmatrix} 0 & 0 \\ y^s \phi_l g_{sj} & 0 \end{pmatrix}.$$

From here, we have

Theorem 3.1. *Let (M, g) be a pseudo-Riemannian manifold, and let ${}^C\varphi_T$ and ${}^C\varphi_{T^*}$ be complete lifts of an affinor field φ to the tangent and cotangent bundles, respectively. Then the differential (pushforward) of ${}^C\varphi_T$ by $g^\flat$,i.e. a $g-lift$ $g_*^\flat\, {}^C\varphi_T$ in the cotangent bundle T^*M is a complete lift ${}^C\varphi_{T^*}$ if (M, g, φ) is a Kahler-Norden manifold.*

Let ${}^CX_T$ be a complete lift of a vector field X and ${}^C\varphi_T$ be a complete lift of an affinor field $\varphi$to the tangent bundle. We have

$$g_*^{\flat C}\varphi_T{}^C X_T = \widetilde{\varphi}_L^J \widetilde{X}^L$$

$$= \begin{pmatrix} \varphi_l^j & 0 \\ y^s\Phi_l g_{sj} + p_s\left(\partial_l\varphi_j^s - \partial_j\varphi_l^s\right) & \varphi_j^l \end{pmatrix}\begin{pmatrix} X^l \\ y^k\left(L_X g\right)_{kl} - p_i\left(\partial_l X^i\right) \end{pmatrix}$$

$$= \begin{pmatrix} \varphi_l^j X^l \\ X^l y^s\Phi_l g_{sj} + X^l p_s\left(\partial_l\varphi_j^s - \partial_j\varphi_l^s\right) + \varphi_j^l y^k\left(L_X g\right)_{kl} - \varphi_j^l p_i\left(\partial_l X^i\right) \end{pmatrix}$$

$$= \begin{pmatrix} (\varphi X)^j \\ X^l y^s\Phi_l g_{sj} + X^l p_s\partial_l\varphi_j^s - X^l p_s\partial_j\varphi_l^s + y^k\varphi_j^l\left(L_X g\right)_{kl} - p_i\varphi_j^l\left(\partial_l X^i\right) \end{pmatrix}$$

$$= \begin{pmatrix} (\varphi X)^j \\ p_s\left(X^l\partial_l\varphi_j^s + \partial_j X^l\varphi_l^s - \partial_l X^s\varphi_j^l\right) + y^k\varphi_j^l\left(L_X g\right)_{kl} - p_s\partial_j\left(\varphi_l^s X^l\right) \end{pmatrix}$$

$$+\begin{pmatrix} 0 \\ y^s X^l\Phi_l g_{sj} \end{pmatrix}$$

$$= \begin{pmatrix} (\varphi X)^j \\ y^s X^l\Phi_l g_{sj} + p_s\left(L_X\varphi\right)_j^s + y^k\varphi_j^l\left(L_X g\right)_{kl} - p_s\partial_j\left(\varphi X\right)^s \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ y^s\left(X^l\Phi_l g_{sj} + \varphi_j^l\left(L_X g\right)_{sl}\right) \end{pmatrix} + \begin{pmatrix} 0 \\ p_s\left(L_X\varphi\right)_j^s \end{pmatrix} + \begin{pmatrix} \varphi_l^j X^l \\ -p_s\partial_j\left(\varphi_l^s X^l\right) \end{pmatrix}$$

Theorem 3.2. *Let (M,g) be a pseudo-Riemannian manifold, let ${}^C\varphi_T$ and ${}^C\varphi_{T^*}$ be complete lifts of an affinor field φ to the tangent and cotangent bundles, let ${}^C X_T$ and ${}^C X_{T^*}$ be complete lifts of a vector field X to the tangent and cotangent bundles, respectively. Then the differential (pushforward) of ${}^C\varphi_T{}^C X_T$ by $g^\flat$,i.e. a $g-lift$ $g_*^{\flat C}\varphi_T{}^C X_T$ in the cotangent bundle T^*M is a vector field ${}^C\varphi_{T^*}{}^C X_{T^*}$ if (M,g,φ) is a Kahler-Norden manifold and $X$is a Killing vector field.*

Using (1.1), (1.2) and (3.1), we have

$$g_*^\flat\left({}^C\varphi\right)_T^2 = \widetilde{\varphi}_M^J\widetilde{\varphi}_K^M$$

$$= \begin{pmatrix} \varphi_m^j & 0 \\ y^s\phi_m g_{sj} + p_s\left(\partial_m\varphi_j^s - \partial_j\varphi_m^s\right) & \varphi_j^m \end{pmatrix}\begin{pmatrix} \varphi_k^m & 0 \\ y^t\phi_k g_{tm} + p_t\left(\partial_k\varphi_m^t - \partial_m\varphi_k^t\right) & \varphi_m^k \end{pmatrix}$$

$$\widetilde{\varphi}_M^j\widetilde{\varphi}_k^M = \varphi_m^j\varphi_k^m$$

$$\widetilde{\varphi}_M^j\widetilde{\varphi}_{\overline{k}}^M = 0$$

$$\widetilde{\varphi}_M^{\overline{j}}\widetilde{\varphi}_{\overline{k}}^M = \varphi_j^m\varphi_m^k$$

$$
\begin{aligned}
\widetilde{\varphi}_M^{\bar{j}}\widetilde{\varphi}_k^M &= y^s\varphi_k^m\phi_m g_{sj} + p_s\varphi_k^m\partial_m\varphi_j^s - p_s\varphi_k^m\partial_j\varphi_m^s + y^t\varphi_j^m\phi_k g_{tm} \\
&\quad + p_t\varphi_j^m\partial_k\varphi_m^t - p_t\varphi_j^m\partial_m\varphi_k^t + p_t\varphi_m^t\partial_k\varphi_j^m - p_t\varphi_m^t\partial_k\varphi_j^m \\
&\quad + p_s\varphi_m^s\partial_j\varphi_k^m - p_s\varphi_m^s\partial_j\varphi_k^m \\
&= \left(p_t\varphi_j^m\partial_k\varphi_m^t + p_t\varphi_m^t\partial_k\varphi_j^m - p_s\varphi_k^m\partial_j\varphi_m^s - p_s\varphi_m^s\partial_j\varphi_k^m\right) \\
&\quad + \left(p_s\varphi_k^m\partial_m\varphi_j^s - p_t\varphi_j^m\partial_m\varphi_k^t - p_t\varphi_m^t\partial_k\varphi_j^m + p_s\varphi_m^s\partial_j\varphi_k^m\right) \\
&\quad + y^s\varphi_k^m\phi_m g_{sj} + y^t\varphi_j^m\phi_k g_{tm} \\
&= {}^C\left(\varphi^2\right)_{T^*} + \left(\gamma N_\varphi\right)_{T^*} + y^s\varphi_k^m\phi_m g_{sj} + y^t\varphi_j^m\phi_k g_{tm}\,.
\end{aligned}
$$

Since in Kahler-Norden manifold $N_\varphi = 0$, we have

Theorem 3.3. *Let (M,g) be a pseudo-Riemannian manifold, let ${}^C\varphi_T$ and ${}^C\varphi_{T^*}$ be complete lifts of an affinor field φ to the tangent and cotangent bundles, respectively. The differential (pushforward) of $\left({}^C\varphi\right)_T^2$ by $g^\flat$,i.e. a $g-lift$ $g_*^\flat\left({}^C\varphi\right)_T^2$ in the cotangent bundle T^*M is an affinor field $\left({}^C\varphi\right)_{T^*}^2$ if (M,g,φ) is a Kahler-Norden manifold.*

4. Problems of g-lifts of vector-valued 2-forms

Let S be a vector-valued 2-form on M. A semi-Riemannian metric g is called pure with respect to S if

$$g(S_Y X_1, X_2) = g(X_1, S_Y X_2)$$

for any $X_1, X_2, Y \in \Im_0^1(M)$, where S_Y denotes a tensor field of type $(1,1)$ such that $S_Y(Z)= S(Y,Z) = -S(Z,Y) = -S_Z(Y)$ for any $Y,Z \in \Im_0^1(M)$. The condition of purity of g may be expressed in terms of the local components as follows:

$$g_{mi_2}S_{i_1 l}^m = g_{i_1 m}S_{i_2 l}^m\,.$$

We now define the Yano-Ako operator $\Phi_S : \Im_2^0(M) \to \Im_4^0(M)$ associated with S and applied to a pure tensor field g by (see [1], [4])

$$
\begin{aligned}
\left(\Phi_S g\right)(X_1,X_2,Y_1,Y_2) &= \left(L_{S(X_1,X_2)}g\right)(Y_1,Y_2) - \left(L_{X_1}\left(g\circ S\right)\right)(Y_1,X_2,Y_2) \\
&\quad - \left(L_{X_2}\left(g\circ S\right)\right)(X_1,Y_1,Y_2) + \left(g\circ S\right)\left(\left[X_1,X_2\right],Y_1,Y_2\right),
\end{aligned}
$$

where $\left(g\circ S\right)(X,Y_1,Y_2) = g(S(X,Y_1),Y_2)$. The Yano-Ako operator has following components with respect to the natural coordinate system:

$$
\begin{aligned}
\left(\Phi_S g\right)_{jihs} &= S_{ji}^m\partial_m g_{hs} - \left(\partial_j S_{hi}^m\right)g_{ms} - \left(\partial_j g_{ms}\right)S_{hi}^m - \left(\partial_i S_{jh}^m\right)g_{ms} \\
&\quad - \left(\partial_i g_{ms}\right)S_{jh}^m + \left(\partial_h S_{ji}^m\right)g_{ms} + \left(\partial_s S_{ji}^m\right)g_{hm}.
\end{aligned}
\tag{4.1}
$$

The non-zero components of the complete lift ${}^C S_T$ of S to the tangent bundle TM are given by [5, p.22]

$${}^C S_{ji}^h = {}^C S_{\bar{j}\,i}^{\bar{h}} = {}^C S_{j\,\bar{\imath}}^{\bar{h}} = S_{ji}^h\,,\quad {}^C S_{ji}^{\bar{h}} = x^{\bar{m}}\partial_m S_{ji}^h\,.$$

Using (1.1) and (1.2), we can easily verify that $g-lift\ g^{\flat}_{*}\,{}^{C}S_{T}=(\widetilde{S}^{H}_{JI})=(\frac{\partial x^{H}}{\partial\tilde{x}^{M}}\frac{\partial\tilde{x}^{K}}{\partial x^{J}}\frac{\partial\tilde{x}^{P}}{\partial x^{I}}{}^{C}S^{M}_{KP}),\ I,J,...=1,...,2n$ has non-zero components of the form

$$\begin{aligned}
&\widetilde{S}\,{}^{h}_{ji}=\delta^{h}_{m}\delta^{k}_{j}\delta^{t}_{i}{}^{C}S^{m}_{kt}=S^{h}_{ji}\,,\\
&\widetilde{S}\,{}^{\bar h}_{\bar j i}=g_{hm}g^{kj}\delta^{t}_{i}{}^{C}S^{\bar m}_{\bar k t}=g_{mk}g^{kj}S^{m}_{hi}=\delta^{j}_{m}S^{m}_{hi}=S^{j}_{hi}\,,\\
&\widetilde{S}\,{}^{\bar h}_{j\bar i}=g_{hm}\delta^{k}_{j}g^{ti}{}^{C}S^{\bar m}_{k\bar t}=g_{mt}g^{ti}S^{m}_{jh}=\delta^{i}_{m}S^{m}_{jh}=S^{i}_{jh}\,,\\
&\widetilde{S}^{\bar h}_{ji}=y^{s}(\partial_{m}g_{hs})\delta^{k}_{j}\delta^{t}_{i}{}^{C}S^{m}_{kt}+g_{hm}\delta^{k}_{j}\delta^{t}_{i}{}^{C}S^{\overline{m}}_{kt}+g_{hm}p_{s}(\partial_{j}g^{ks}){}^{C}S^{\overline{m}}_{\overline{k}t}\\
&+g_{hm}\delta^{k}_{j}p_{s}(\partial_{i}g^{ts}){}^{C}S^{\overline{m}}_{k\overline{t}}\\
&=y^{s}(\partial_{m}g_{hs})\delta^{k}_{j}\delta^{t}_{i}S^{m}_{kt}+g_{hm}\delta^{k}_{j}\delta^{t}_{i}y^{s}\partial_{s}S^{m}_{kt}+g_{hm}p_{s}(\partial_{j}g^{ks})S^{m}_{kt}\\
&+g_{hm}\delta^{k}_{j}p_{s}(\partial_{i}g^{ts})S^{m}_{kt}\\
&=y^{s}(\partial_{m}g_{hs})S^{m}_{ji}+g_{hm}y^{s}\partial_{s}S^{m}_{ji}+g_{hm}p_{s}(\partial_{j}g^{ks})S^{m}_{ki}+g_{hm}p_{s}(\partial_{i}g^{ts})S^{m}_{jt}\\
&=y^{s}\left(\Phi_{S}g\right)_{jihs}+y^{s}\left(\partial_{j}S^{m}_{hi}\right)g_{ms}+y^{s}\left(\partial_{j}g_{ms}\right)S^{m}_{hi}+y^{s}\left(\partial_{i}S^{m}_{jh}\right)g_{ms}\\
&+y^{s}\left(\partial_{i}g_{ms}\right)S^{m}_{jh}-y^{s}\left(\partial_{h}S^{m}_{ji}\right)g_{ms}\\
&-y^{k}\left(\partial_{j}g_{km}\right)S^{m}_{hi}-y^{t}\left(\partial_{i}g_{tm}\right)S^{m}_{jh}=y^{s}\left(\Phi_{S}g\right)_{jihs}+y^{s}\left(\partial_{j}S^{m}_{hi}\right)g_{ms}\\
&+y^{s}\left(\partial_{i}S^{m}_{jh}\right)g_{ms}-y^{s}\left(\partial_{h}S^{m}_{ji}\right)g_{ms}\\
&=y^{s}\left(\Phi_{S}g\right)_{jihs}-p_{m}(\partial_{j}S^{m}_{ih}+\partial_{i}S^{m}_{hj}+\partial_{h}S^{m}_{ji})\,.
\end{aligned}\tag{4.2}$$

From equation (4.2) we find $g-lift\ g^{\flat}_{*}\,{}^{C}S_{T}$ of S:

$$g^{\flat}_{*}\,{}^{C}S_{T}={}^{C}S_{T^{*}}+\gamma(\Phi_{S}g)\,.$$

From here, we have

Theorem 4.1. *Let (M,g) be a pseudo-Riemannian manifold, and let ${}^{C}S_{T}$ and ${}^{C}S_{T^{*}}$ be complete lifts of a vector-valued 2-form S to the tangent and cotangent bundles, respectively. Then the differential (pushforward) of ${}^{C}S_{T}$ by $g^{\flat}$,i.e. a $g-lift\ g^{\flat}_{*}\,{}^{C}S_{T}$ in the cotangent bundle $T^{*}M$is a complete lift ${}^{C}S_{T^{*}}$ if $\Phi_{S}g=0$.*

5. Problems of g-lifts of metrics

Let ${}^{C}g$ be a complete lift of a pseudo-Riemannian metric g to TM:

$${}^{C}g=({}^{C}g_{IJ})=\begin{pmatrix} y^{s}\partial_{s}g_{ij} & g_{ij}\\ g_{ij} & 0\end{pmatrix}.\tag{5.1}$$

Let ∇ be a torsion-free connection on M_{n} . A new pseudo-Riemannian metric ${}^{\nabla}g\in\Im^{0}_{2}(T^{*}M)$ on $T^{*}M$ is defined by the equation [5, p.268]

$${}^{\nabla}g\left({}^{C}X,{}^{C}Y\right)=-\gamma(\nabla_{X}Y+\nabla_{Y}X)$$

for any $X,Y\in\Im^{1}_{0}(M)$, where $\gamma(\nabla_{X}Y+\nabla_{Y}X)$ is a function in $\pi^{-1}(U)\subset T^{*}M$ with local expression $\gamma\left(\nabla_{X}Y+\nabla_{Y}X\right)=p_{h}\left(X^{i}\nabla_{i}Y^{h}+Y^{i}\nabla_{i}X^{h}\right)$, and is called a Riemannian extension of ∇ to $T^{*}M$. The Riemannian extension ${}^{\nabla}g$ has components of the form

$${}^{\nabla}g=({}^{\nabla}g_{IJ})=\begin{pmatrix} -2p_{m}\Gamma^{m}_{ij} & \delta^{j}_{i}\\ \delta^{i}_{j} & 0\end{pmatrix}\tag{5.2}$$

with respect to the to the natural frame $\{\partial_{i},\partial_{\bar{i}}\}$, Γ^{m}_{ij} where are components of ∇.

Using (1.2) and (5.1) we see that the pullback of ${}^C g$ by $g^\sharp$ is the $(0,2)$-tensor field $g-$ lift $(g^\sharp)^*\, {}^C g$ on T^*M and has components

$$
\begin{aligned}
(((g^\sharp)^*\, {}^C g)_{KL}) &= \left(\frac{\partial x^I}{\partial \tilde{x}^K}\frac{\partial x^J}{\partial \tilde{x}^L}\, {}^C g_{IJ}\right) \\
&= \begin{pmatrix} \frac{\partial x^i}{\partial \tilde{x}^k}\frac{\partial x^j}{\partial \tilde{x}^l}\, {}^C g_{i\,j} + \frac{\partial x^{\bar{i}}}{\partial \tilde{x}^k}\frac{\partial x^j}{\partial \tilde{x}^l}\, {}^C g_{\bar{i}\,j} + \frac{\partial x^i}{\partial \tilde{x}^k}\frac{\partial x^{\bar{j}}}{\partial \tilde{x}^l}\, {}^C g_{i\,\bar{j}} & \frac{\partial x^i}{\partial \tilde{x}^k}\frac{\partial x^{\bar{j}}}{\partial \tilde{x}^{\bar{l}}}\, {}^C g_{i\,\bar{j}} \\ \frac{\partial x^{\bar{i}}}{\partial \tilde{x}^{\bar{k}}}\frac{\partial x^j}{\partial \tilde{x}^l}\, {}^C g_{\bar{i}\,j} & 0 \end{pmatrix} \\
&= \begin{pmatrix} \delta_k^i \delta_l^j y^s \partial_s g_{ij} + p_s(\partial_k g^{is})\delta_l^j g_{i\,j} + \delta_k^i p_s(\partial_l g^{js}) g_{i\,j} & \delta_k^i g^{jl} g_{i\,j} \\ g^{ik}\delta_l^j g_{i\,j} & 0 \end{pmatrix} \\
&= \begin{pmatrix} y^s \partial_s g_{kl} + p_s((\partial_k g^{is}) g_{i\,l} + (\partial_l g^{js}) g_{k\,j}) & \delta_k^l \\ \delta_l^k & 0 \end{pmatrix} \\
&= \begin{pmatrix} p_t g^{st}\partial_s g_{kl} - p_s(g^{is}\partial_k g_{i\,l} + g^{js}\partial_l g_{k\,j}) & \delta_k^l \\ \delta_l^k & 0 \end{pmatrix} \\
&= \begin{pmatrix} -p_s g^{ts}(\partial_l g_{tk} + \partial_k g_{lt} - \partial_t g_{kl}) & \delta_k^l \\ \delta_l^k & 0 \end{pmatrix} \\
&= \begin{pmatrix} -2p_s \left\{ \begin{matrix} s \\ kl \end{matrix} \right\} & \delta_k^l \\ \delta_l^k & 0 \end{pmatrix},
\end{aligned} \tag{5.3}
$$

where $\left\{ \begin{matrix} s \\ kl \end{matrix} \right\} = \frac{1}{2} g^{ts}(\partial_l g_{tk} + \partial_k g_{lt} - \partial_t g_{kl})$. Thus, from (5.2) and (5.3) we obtain $(g^\sharp)^*\, {}^C g = {}^\nabla g$, if $\left\{ \begin{matrix} s \\ kl \end{matrix} \right\} = \Gamma_{kl}^s$, i.e. we have

Theorem 5.1. *The Riemannian extension* ${}^\nabla g \in \Im_2^0(T^*M)$ *coincides with* $g-$ *lift* $(g^\sharp)^*\, {}^C g$ *on cotangent bundle* T^*M *if the torsion-free connection* ∇ *is a Riemann connection.*

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