ON MULTIPLE COMPLETENESS OF EIGEN AND ASSOCIATED VECTORS OF A CLASS OF OPERATOR PENCILS

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Abstract. In the paper we obtain conditions providing multiple completeness of eigen and associated elements of higher order operator pencils on a finite interval. These conditions are directly expressed by properties of coefficients of the operator pencil.

1. Introduction

On separable Hilbert space H we consider a polynomial operator pencil of n-th order

$$P(\lambda) = (-1)^k \lambda^n E + \lambda^{n-1} A_1 + ... + A_n + A^n, \qquad (1.1)$$

where $n = 2k$, $(k = 1, 2, ...)$, λ is a spectral parameter, E is a unit operator in H, the remaining coefficients of the operator pencil (1.1) satisfy the conditions:

1) A is a positive-definite operator with completely continuous inverse A^{-1};
2) The operators $B_j = A_j A^{-j}$ $(j = 1, ..., n)$ are bounded in H;
3) The operator $E + B_n$ is invertible in H.

Note that subject to conditions 1)-3), the operator pencil has a discrete spectrum with a unique limit point at infinity. Indeed,

$$P(\lambda) = (E + B_n) \left(\sum_{j=1}^{n-1} \lambda^{n-j} (E + B_n)^{-1} (A_j A^{-j}) \cdot A^{-n+j} \right.$$

$$\left. + (-1)^k \lambda^n (E + B_n)^{-1} A^{-n} + E \right) A^n = (E + B_n)(E + L(\lambda)) A^n,$$

where

$$L(\lambda) = (-1)^k \lambda^n (E + B_n)^{-1} A^{-n} + \sum_{j=1}^{n-1} \lambda^{n-j} (E + B_n)^{-1} B_j A^{-n+j}.$$

As $L(0) = 0$, the coefficients $(E + B_n)^{-1} A^{-n}$ and $(E + B_n)^{-1} B_n A^{-n+j}$ $(j = \overline{1, \ n-1})$ are completely continuous in H, then $E + L(\lambda)$ by the Keldysh lemma [4] has only a discrete spectrum with a unique limit point at infinity. Then the operator pencil $P(\lambda) = (E + B_n)^{-1}(E + L(\lambda)) A^n$ also possesses this property.

Obviously, the domain of definition of the operator A^γ ($\gamma \geq 0$) is Hilbert space H_γ with respect to the scalar product $(x, y)_\gamma = (A^\gamma x, A^\gamma y)$, $x, y \in H_\gamma$. For $\gamma = 0$ we assume $H_0 = H$.

Denote by $L_2(R_+ : H)$ Hilbert space of all vector-functions $f(t)$ determined almost everywhere in the interval $(0, 1)$ with values in H, for which

$$\|f\|_{L_2((0,1):H)} = \left(\int_0^1 \|f(t)\|^2 \, dt \right)^{1/2} < +\infty.$$

Further, following the monograph [5] we introduce Hilbert space

$$W_2^n((0,1) : H) = \left\{ u : \ u^{(n)} \in L_2((0,1) : H), \ \ A^n u \in L_2((0,1) : H) \right\}$$

with the norm

$$\|u\|_{W_2^n((0,1):H)} = \left(\|A^n u\|_{L_2((0,1):H)}^2 + \|u^n\|_{L_2((0,1):H)}^2 \right)^{1/2}.$$

Here and in the sequel, the derivatives are understood in the sense of theory of distributions in abstract Hilbert spaces [5].

Associate the pencil (1.1) with the boundary value problem

$$P(d/dt)u(t) = (-1)^k u^{(n)} + A^n u + \sum_{j=1}^{n-1} A_{n-j} u^{(j)} = 0, \ \ t \in (0, 1) \qquad (1.2)$$

$$u^{(2\nu)}(0) = \varphi_\nu, u^{(2\nu)}(1) = \psi_\nu, \nu = \overline{0, \ k-1}. \qquad (1.3)$$

Definition 1.1. If $u(t) \in W_2^n((0,1) : H)$ satisfies the equation (1.2) almost everywhere in $(0, 1)$, then $u(t)$ it said to be a regular solution of equation (1.2).

Definition 1.2. If for any collection of n vectors $\varphi_\nu \in H_{n-2\nu-1/2}$, $\psi_\nu \in H_{n-2\nu-1/2}$, ($\nu = \overline{0, \ k-1}$) there exists the regular solution $u(t)$ of equation (1.2) satisfying boundary conditions (1.3) in the sense of convergence

$$\lim_{t \to +0} \left\| u^{(2\nu)}(t) - \psi_\nu \right\|_{n-2\nu-1/2} = 0, \lim_{t \to 1-0} \left\| u^{(2\nu)}(t) - \psi_\nu \right\|_{n-2\nu-1/2} = 0, \nu = \overline{0, k-1}$$

and the estimation

$$\|u(t)\|_{W_2^n((0,1):H)} \leq const \sum_{\nu=0}^{k-1} \left(\|\varphi_\nu\|_{n-2\nu-1/2} + \|\psi_\nu\|_{n-2\nu-1/2} \right),$$

then problem (1.2), (1.3) is said to be regularly solvable.

Definition 1.3. If the equation $P(\lambda_i) x_{0,i,j} = 0$ has nonzero solution $x_{0,i,j}$, then λ_i is called a characteristic number of $P(\lambda)$, and $x_{0,i,j}$ an eigen-vector of the operator pencil $P(\lambda)$, corresponding to λ_i. If the vectors $x_{0,i,j}, \ x_{1,i,j}, \ ..., \ x_{h,i,j}$, $h = \overline{0, \ m_{i,j}}, j = \overline{1, \ q_i}$ satisfy the equations

$$\sum_{p=0}^{h} \frac{\partial^p P(\lambda)}{\partial \lambda^p} \bigg|_{\lambda=\lambda_i} x_{h-p,i,j} = 0, h = \overline{0, \ m_{i,j}}, j = \overline{1, \ q_i},$$

then $x_{0,i,j}, \ ..., \ x_{h,i,j}$ are said to be eigen and associated vectors of the pencil $P(\lambda)$, corresponding to the characteristic number λ_i.

The vector-functions

$$u_{h,i,j}(t) = e^{\lambda_i t} \left(\frac{t^h}{h!} x_{0,i,j} + \frac{t^{h-1}}{(h-1)!} x_{1,i,j} + \dots + x_{h,i,j} \right), h = \overline{0, \ m_{i,j}}, j = \overline{1, \ q_i}$$

belong to the space $W_2^n((0, \ 1) : H)$, satisfy equation (1.2) and are called elementary solutions of equation (1.2).

In the space $\overset{k-1}{\underset{\nu=0}{\oplus}} H_{n-2\nu-1/2} \times \overset{n-1}{\underset{\nu=0}{\oplus}} H_{n-2\nu-1/2}$ let us construct the system

$$\{\tilde{x}_{h,i,j}\}_{i=1, h=\overline{0, m_{i,j}}, j=\overline{1, \ q_m}}^{\infty} \equiv \left\{ u_{h,i,j}^{(2\nu)}(0), u_{h,i,j}^{(2\nu)}(1) \right\}_{i=1, h=\overline{0, m_{i,j}}, j=\overline{1, \ q_i}}^{\infty}.$$

Definition 1.4. If the system $\{\tilde{x}_{h,i,j}\}_{i=1, h=\overline{0, m_{i,j}}, j=\overline{1, \ q_i}}^{\infty}$ is complete in the space $\overset{k-1}{\underset{\nu=0}{\oplus}} H_{n-2\nu-1/2} \times \overset{n-1}{\underset{\nu=0}{\oplus}} \bar{H}_{n-2\nu-1/2}$, we say that the system of eigen and associated vectors $P(\lambda)$ is n-fold complete in the space of traces of regular solutions of equation (1.2).

In the present paper we find conditions on the coefficients of the operator pencil $P(\lambda)$, that provide n-fold completeness of the system of eigen and associated vectors of the pencil $P(\lambda)$ in the space of traces of solutions.

To this end we study regular solvability of problem (1.2), (1.3) and estimate the norms of the resolvent $P^{-1}(\lambda)$ on some rays.

Similar problems were studied for example in the papers [1]-[4], [6]-[8], [10]-[12].

2. On regular solvability of problem (1.2), (1.3).

At first we consider the problem

$$(-1)^k u^{(n)}(t) + A^n u(t) = 0, t \in (0, \ 1) \tag{2.1}$$

$$u^{(2\nu)}(0) = \varphi_\nu, u^{(2\nu)}(1) = \psi_\nu, \nu = \overline{0, \ k-1} \tag{2.2}$$

It holds

Theorem 2.1. *Problem (2.1), (2.2) is regularly solvable.*

Proof. Let $\omega_0, \omega_1, \dots, \omega_{n-1}$ be the solution of the equation $(-1)^n \omega^n + 1 = 0$, moreover $Re\omega_l < 0$ $(l = 0, \dots, k-1)$, $Re\omega_l > 0$ $(l = k, \dots, n-1)$. Then the general solution of equation (2.1) from the space $W_2^n((0, \ 1) : H)$ has the form

$$u_0(t) = \sum_{l=0}^{k-1} e^{\omega_l t A} c_l + \sum_{l=k}^{n-1} e^{\omega_l(t-1)A} c_l, \tag{2.3}$$

where the vectors $c_l \in H_{n-1/2}$. From condition (2.2) it follows that

$$\sum_{l=0}^{k-1} \omega_l^{2\nu} A^{2\nu} c_l + \sum_{l=k}^{n-1} (\omega_e)^{2\nu} A^{2\nu} e^{-\omega_l A} c_l = \varphi_\nu, \nu = \overline{0, \ k-1},$$

$$\sum_{l=0}^{k-1} \omega_l^{2\nu} A^{2\nu} l^{\omega_l A} c_l + \sum_{l=k}^{n-1} \omega_l^{2\nu} A^{2\nu} c_l = \psi_\nu, \nu = \overline{0, \ k-1},$$

or

$$\sum_{l=0}^{k-1} \omega_l^{2\nu} c_l + \sum_{l=k}^{n-1} \omega_l^{2\nu} A^{-\omega_l A} c_l = A^{-2\nu} \varphi_\nu, \nu = \overline{0,\ k-1},$$

$$\sum_{l=0}^{k-1} \omega_l^{2\nu} e^{\omega_l A} c_l + \sum_{l=k}^{n-1} \omega_l^{2\nu} c_l = A^{-2\nu} \psi_\nu, \nu = \overline{0,\ k-1}.$$

Thus, we get the equation $\Delta(A)\,\tilde{c} = \bar{\theta}$, $\tilde{\theta} = \left(\varphi_0, ..., A^{-2\nu}\varphi_{k-1},\ \psi_0, ..., A^{-2\nu}\psi_{n-1}...\right)$, $\tilde{c} = (c_0,\ c_1,\ ...,\ c_{n-1})$, where

$$\Delta(A) = \begin{pmatrix} E & & E & e^{-\omega_k A} & \cdots & e^{-\omega_{n-1} A} \\ \omega_0^2 E & \cdots & \omega_{k-1}^2 E & \omega_k^2 e^{-\omega_k A} & \cdots & \omega_{n-1}^2 e^{\omega_{n-1} A} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \omega_0^{2(k-1)} E & \omega_{k-1}^{2(k-1)} E & \omega_k^{2(k-1)} e^{-\omega_k A} & \omega_{n-1}^{2(k-1)} e^{-\omega_{n-1} A} \\ e^{\omega_0 A} & e^{\omega_{k-1} A} & E & E \\ \omega_0^2 e^{\omega_0 A} & \omega_{k-1}^2 e^{\omega_{k-1} A} & \omega_k^2 E & \omega_{n-1}^2 E \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \omega_0^{2(k-1)} e^{\omega_0 A} & \cdots & \omega_{k-1}^{2(k-1)} e^{\omega_{k-1} A} & \omega_k^{2(k-1)} E & \cdots & \omega_{n-1}^2 E \end{pmatrix} \quad (2.4)$$

Let $\sigma \geq \mu_0$, i.e. $\sigma \in [\mu_0, \infty)$, where μ_0 is the lower bound of spectrum A and consider the scalar matrix $\Delta(\sigma)$ in the domain $[\mu_0, \infty)$. Obviously, as $\sigma \to \infty$ $|\det \Delta(\sigma)| = \left|wronskian(\omega_0^2, ..., \omega_{k-1}^2)\right| \times \left|wronskian(\omega_k^2, ..., \omega_{n-1}^2)\right| \neq 0$. Therefore, there exists $R_0 > 0$ such that for $\sigma \in [R_0, \infty)$ $|\det \Delta(\sigma)| > c > 0$. Now show that for any $\sigma \in [\mu_0, R_0]$ $\det \Delta(\sigma) \neq 0$. If it is not so, then $\det \Delta(\sigma_0) = 0$ for some $\sigma_0 \in [\mu_0, R_0]$.

Then the equation $\Delta(\sigma_0)\bar{\xi} = 0$ has the nonzero solution $\bar{\xi} = (\xi_0, \xi_1, ..., \xi_{n-1})$. This means that the scalar function

$$\xi(t) = \sum_{l=0}^{k-1} e^{\omega_l t \sigma_0} \xi_l + \sum_{l=k}^{n-1} e^{\omega_l (t-1)\sigma_0} \xi_l$$

satisfies the equation

$$(-1)^k \xi^{(n)}(t) + \sigma_0^n \xi(t) = 0, t \in (0,\ 1) \quad (2.5)$$

and the boundary condition

$$\xi^{(2\nu)}(0) = 0, \xi^{(2\nu)}(1) = 0,\ \nu = \overline{0, k-1}. \quad (2.6)$$

Multiplying the equation (2.5) scalarly by the function $\xi(t)$ in $L_2(0,\ 1)$ and taking into account condition (2.6), after integrating by parts, we get

$$\left\|\xi^{(k)}\right\|_{L_2(0,\ 1)}^2 + \sigma_0^n \|\xi\|_{L_2(0,\ 1)}^2 = 0.$$

Hence we get $\xi(t) = 0$, i.e. $\bar{\xi} = (\xi_0, ..., \xi_{n-1}) = 0$ and this contradicts the condition $\bar{\xi} \neq 0$. As $|\det \Delta(\sigma)|$ is a continuous function on $[\mu_0,\ R_0]$, then $\inf_\sigma |\det \Delta(\sigma)| \geq d_0 > 0$. Then we get that for all $\sigma \in [\mu_0, \infty)$ the inequality $|\det \Delta(\sigma)| > d > 0$ is valid.

Using spectral expansion of A we get that $\Delta^{-1}(A)$ exists and takes the space $\oplus H_{n-1/2}$ to $\oplus H_{n-1/2}$ and $\left\|\Delta^{-1}(A)\right\| \leq const$. Here the n copies of direct sum of

spaces $H_{n-1/2}$ is taken. Then, obviously

$$\sum_{l=0}^{n-1} \|c_l\|_{n-1/2} \le const \, \|\Delta^{-1}(A)\| \, \left(\sum_{l=0}^{n-1} \|A^{-2\nu}\varphi_\nu\| + \sum_{l=0}^{n-1} \|A^{-2\nu}\psi_\nu\|_{n-1/2}^2 \right)$$

$$\le const \, \left(\sum_{l=0}^{n-1} \|\varphi_\nu\|_{n-2\nu-1/2} + \sum_{l=k}^{k+1} \|\psi_\nu\|_{n-2\nu-1/2} \right).$$

Hence we get

$$\|u_0(t)\|_{W_2^n((0,1):H)} \le const \, \left(\sum_{l=0}^{k} \|\varphi_\nu\|_{n-2\nu-1/2} + \sum_{l=0}^{n-1} \|\psi_\nu\|_{n-2\nu-1/2} \right). \qquad (2.7)$$

The theorem is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now solve the problem (1.2), (1.3).
Denote by

$$\overset{\circ}{W}_2^n = \left\{ u: \, u \in W_2^n((0,\,1)H), \, u^{(2\nu)}(0) = u^{(2\nu)}(1) = 0 \right\}$$

and consider the operators

$$P_0 u = P_0(d/dt)u = (-1)^{(k)} u^{(n)}(t) + A^n u(t),$$

$$P_1 u = P_1(d/dt)u = \sum_{j=0}^{n-1} A_{n-j} u^{(j)}(t), u \in \overset{\circ}{W}_2^n((0,\,1):H).$$

After substitution $u(t) = \omega(t) + u_0(t)$, where $\omega(t)$ is an unknown function from $\overset{\circ}{W}_2^n((0,\,1):H)$, and $u_0(t)$ is determined from (2.3) as the solution of problem (2.1), (2.2). From boundary condition (1.2), (1.3) we get

$$(P_0(d/dt) + P_1(d/dt))\,\omega(t) = g(t), t \in (0,\,1) \qquad\qquad (2.8)$$

$$\omega^{(2\nu)}(0) = 0, \omega^{(2\nu)}(1) = 0, \qquad\qquad\qquad\qquad (2.9)$$

where $\omega \in \overset{\circ}{W}_2^n((0,\,1):H)$, while $g(t) \in L_2((0,\,1):H)$.

Indeed, after substitution $\omega(t) = u(t) - u_0(t)$ we have:

$$(P_0(d/dt) + P_1(d/dt))\,\omega(t) = -P_1(d/dt)u_0(t),$$

$$\omega^{(2\nu)}(0) = 0, \omega^{(2\nu)}(1) = 0$$

moreover

$$\|g(t)\|_{L_2((0,1):H)} = \|P_1(d/dt)u_0(t)\|_{L_2((0,1):H)}$$

$$\le \sum_{j=0}^{n-1} \|B_j\| \cdot \left\| A^{n-j} u_0^{(j)}(t) \right\| \le const \, \|u_0(t)\|_{W_2^n((0,1):H)}$$

$$\le const \, \left(\sum_{\nu=0}^{k-1} \|\varphi_\nu\|_{n-2\nu-1/2} + \sum_{\nu=k}^{n-1} \|\psi_\nu\|_{n-2\nu-1/2} \right).$$

Here we used the theorem on intermediate derivatives [5] and inequality (2.7). Thus, we obtained problem (2.8), (2.9).

Now we will use the following result from the paper [9].

Theorem 2.2. [9] *Let conditions 1), 2) be fulfilled, and the inequality*

$$q = \sum_{j=0}^{n-1} d_{n,j} \|B_j\| < 1, \tag{2.10}$$

where

$$d_{n,j} = \begin{cases} \left(\frac{n-j}{n}\right)^{\frac{n-j}{n}} \left(\frac{j}{n}\right)^{j/n}, & j = 1, \dots n-1 \\ 1, & j = 0 \end{cases}, \tag{2.11}$$

hold. Then for any $g(t) \in L_2((0,1) : H)$ there exists $\omega(t) \in \overset{\circ}{W}_2^n((0,1) : H)$ that satisfies equation (2.8) almost everywhere and the estimation

$$\|\omega\|_{\dot{W}_2^n((0,1):H)} \leq const \|g(t)\|_{L_2((0,1):H)}.$$

Thus, we obtain

Theorem 2.3. *Let all conditions of theorem 2.2 be fulfilled. Then problem (1.2), (1.3) is regularly solvable.*

3. On completeness of eigen and associated vectors of the pencil $P(\lambda)$

At first we prove a theorem on estimation of the resolvent $P^{-1}(\lambda)$.

Theorem 3.1. *Let conditions 1), 2) be fulfilled, $\omega_0, \dots, \omega_{n-1}$ be the roots of the equation $(-1)^k \lambda^n + 1 = 0$ ($n = 2k$, $k = 1, 2, \dots$) and the number $0 < \alpha \leq \pi/n$. Then subject to the condition*

$$q = \sum_{j=0}^{n-1} d_{n,j} \|B_j\| < \sin\frac{n\alpha}{2}, \tag{3.1}$$

on the rays $\Gamma_{s,\alpha}^{\pm} = \{\lambda : \arg\lambda = \arg\omega_s \pm \alpha\}$, $s = \overline{0, \; n-1}$ there exists the resolvent $P^{-1}(\lambda)$, and on these rays it holds the estimation

$$\left\| A^\beta P^{-1}(\lambda) \right\| \leq const \left(|\lambda| + 1\right)^{-n+\beta}, 0 \leq \beta \leq n.$$

Here the numbers $d_{n,j}$ $(j = \overline{0, \; n-1})$ are determined from equality (2.11).

Proof. Obviously $\|B_n\| < 1$, i.e. $E + B_n$ has a bounded inverse in H. Then $P(\lambda)$ has a discrete spectrum.

At first we show that on the rays $\Gamma_{s,\alpha}^{\pm} = \{\lambda : \arg\lambda = \arg\omega_s \pm \alpha\}$ $s = \overline{0, \; n-1}$ the estimations

$$\left\| A^{n-j}\lambda^j \left((-1)^k \lambda^n E + A^n\right)^{-1} \right\| \leq d_{n,j} \left(\sin\frac{n\alpha}{2}\right)^{-1} \tag{3.2}$$

hold.

Indeed, for $\lambda \in \Gamma_{s,\alpha}^{\pm}$ we have:

$$\left\| A^{n-j}\lambda^j \left((-1)^k \lambda^n E + A^n\right)^{-1} \right\| = \left\| A^{n-j} r^j \left(-r^n e^{\pm in\alpha} E + A^n\right)^{-1} \right\|$$

$$= \sup_{\mu \in \sigma(A)} \left| \mu^{n-j} r^j (-r^n e^{\pm in\alpha} + \mu^n)^{-1} \right| = \sup_{\mu \in \sigma(A)} \left| \mu^{n-j} r^j (r^{2n} + \mu^{2n} - 2r^n \mu^n \cos n\alpha)^{-1/2} \right|.$$

After substitution $\tau = r/\mu$ and using the Cauchy inequality, we get that for $\lambda \in \Gamma_\alpha^\pm$ the following inequalities

$$\left\| A^{n-j} \lambda^j \left((-1)^k \lambda^n E + A^n \right)^{-1} \right\| \leq \sup_{\tau > 0} \left| \tau^j (\tau^{2n} + 1 - 2\tau^n \cos n\alpha)^{-1/2} \right|$$

$$= \sup_{\tau > 0} \left| \tau^j ((\tau^n + 1)^2 - 2\tau^n (1 + \cos n\alpha))^{-1/2} \right|$$

$$= \sup_{\tau > 0} \left| \tau^j (\tau^n + 1)^{-1} \left(1 - \frac{4\tau^n}{(\tau^n + 1)^2} \cos^2 \frac{n\alpha}{2} \right)^{-1/2} \right|$$

$$\leq \sup_{\tau > 0} \left| \tau^j (\tau^n + 1)^{-1} \right| \cdot \left(1 - \cos^2 \frac{n\alpha}{2} \right)^{-1/2}$$

$$= \left(\sin \frac{n\alpha}{2} \right)^{-1} \sup_{\tau > 0} \left| \tau^j (\tau^n + 1)^{-1} \right| = d_{n,j} \left(\sin \frac{n\alpha}{2} \right)^{-1}$$

hold.

Then from the equality

$$P(\lambda) = P_0(\lambda) + P_1(\lambda) = (E + P_1(\lambda) P_0^{-1}(\lambda)) P_0(\lambda) \tag{3.3}$$

we get that on the rays $\Gamma_{s,\alpha}^\pm$ there hold the estimations

$$\left\| P_1(\lambda) P_0^{-1}(\lambda) \right\| = \left\| \sum_{j=0}^{n-1} \lambda^j A_{n-j} ((-1)^k \lambda^n E + A^n)^{-1} \right\|$$

$$= \sum_{j=0}^{n-1} \| B_{n-j} \| \, \left\| A^{n-j} \lambda^j ((-1)^k \lambda^n E + A^n)^{-1} \right\| \leq \sum_{j=0}^{n-1} d_{n,j} \left(\sin \frac{n\alpha}{2} \right)^{-1} \| B_{n-j} \|$$

$$= \sum_{j=0}^{n-1} d_{n,j} \left(\sin \frac{n\alpha}{2} \right)^{-1} \| B_j \| < 1.$$

Then from equality (3.3) it follows that

$$\left\| A^\beta P^{-1}(\lambda) \right\| \leq \left\| A^\beta P_0^{-1}(\lambda) \right\| \, \left\| (E + P_1(\lambda) P_0^{-1}(\lambda)) \right\| \leq const \, (|\lambda|^n + 1)^{-n+\beta} .$$

Now using the method of the papers [10], [11] we prove a theorem on n-fold completeness of eigen and associated vectors in the space of traces. $\qquad \square$

Theorem 3.2. *Let conditions 1), 2), $A^1 \in \sigma_\rho$ ($0 < \rho < \infty$) be fulfilled and the inequality*

$$q(\rho) = \sum_{j=0}^{n-1} d_{n,j} \| B_j \| < \begin{cases} 1, & \text{for} \quad 0 < \rho \leq k \\ \sin \frac{\pi n}{4\rho}, & \text{for} \quad k \leq \rho < \infty \end{cases} \tag{3.4}$$

hold. Then the system of eigen and associated vectors of the pencil $P(\lambda)$ is n-fold complete in the space of traces.

Proof. As $A^{-1} \in \sigma_\rho$ $(0 < \rho < \infty)$, the operators $B_j = A_j A^{-j}$ $(j = 1, n)$ are completely continuous in H, then from the Keldysh theorem [4],[9] it follows that $A^n P^{-1}(\lambda)$ is represented in the form of ratio of two entire functions of order ρ and of minimal type of order ρ. Assume that the system of eigen and associated vectors of the operator-pencil $P(\lambda)$ is not n-fold complete in the space of traces. Then from the results of the paper of M.V. Keldysh [1], M.G. Gasymov and S.S. Mirzoyev [6] it follows that there exist the vectors $\chi_\nu \in H_{n-2\nu-1/2}$ $(\nu = \overline{0, \ n-1})$ such that $\sum_{\nu=0}^{n-1} \|\chi_\nu\|_{n-2\nu-1/2} \neq 0$, and the operator-function

$$g(\lambda) = \sum_{\nu=0}^{k-1} \lambda^{2\nu} \left(A^{n-2\nu-1/2} P_\nu^{-1}(\bar{\lambda}) \right)^* A^{n-\nu-1/2} \chi_\nu$$

$$+ \sum_{\nu=k}^{n-1} \lambda^{2\nu} e^\lambda \left(A^{n-2\nu-1/2} P^{-1}(\bar{\lambda}) \right)^* A^{n-2\nu-1/2} \chi_\nu$$

is an entire function. From the theorem [2] on estimation of the resolvent for $0 < \rho \leq k$ it follows that on the rays $\Gamma^{\pm}_{s, \pi/2k} = \left\{ \lambda : \arg \lambda = \arg \omega_k + \frac{\pi}{2k} \right\}$ the estimations

$$\|g(\lambda)\| \leq const \cdot \lambda^{-1/2} (1 + e^{Re\lambda}) \tag{3.5}$$

hold.

Then taking into account that the angle between these rays equals $\frac{\pi}{n}$, then for $\frac{\pi}{n} \leq \frac{\pi}{2p}$, i.e. when $0 < \rho \leq \frac{n}{2} = k$ applying the Fragmen-Lindeloff theorem we get that estimation (3.5) holds on the whole of complex plane. If $k \leq \rho < \infty$, then again using theorem 5 on estimation of the resolvent, and the Fragmen-Lindeloff theorem, we get that estimation (3.5) holds for all λ from the complex plane. Thus, in both cases, inequality (3.5) holds for all λ. Further it is obvious that for $\lambda = i\xi$, $\xi \in R$, $\lim_{\xi \to \infty} \|g(i\xi)\| = 0$.

As the boundary value problem (1.2), (1.3) is regularly solvable, we denote its solution by $u(t)$ and assume

$$\hat{u}(\lambda) = \int_0^1 u(t) e^{-\lambda t} dt.$$

Obviously, $\hat{u}(\lambda)$ is an entire function. Further repeating all the reasonings from the paper [11, pp. 95-99] we get $\chi_\nu = 0$ $(\nu = \overline{0, \ n-1})$. And this contradicts the condition $\sum_{\nu=0}^{n-1} |\chi_\nu| \neq 0$. The theorem is proved. $\square$

References

[1] M.G. Gasymov, S.S. Mirzoyev, On solvability of boundary value problems for second order operator-differential equation of elliptic type, *Diff. uravn.*, **28** (1992), no. 4, 651-661.

[2] M.G. Gasymov, On multiple completeness of a part of eigen and associated vectors of polynomial operator pencils. *Izv. AS Arm. SSR, ser. Mathematics* **6** (1971), no. 2-3, 131-147.

[3] M.G. Gasymov, To theory of polynomial operator pencils. *DAN SSSR*, **149** (1971), no. 44, 747-750.

[4] M.V. Keldysh, On completeness of eigen functions of some classes of not self-adjoint operators, *UMN*, **27** (1971), no. 1, 15-47.

[5] J.L. Lions, E. Magenes, *Inhomogeneous boundary value problems and their applications.* M.Mir, 1971.

[6] S.S. Mirzoyev, M.D. Karaaslan, R.Z. Gumbataliyev, To theory of second order operator-differential equations, *Dokl. RAN*, **453** (2013), 610-612.

[7] S.S. Mirzoyev, M.S. Salimov, On completeness of elementary solutions of a class of second order operator-differential equations, *Sibirskiy mat. zh.*, **51** (2010), no. 4, 815-828.

[8] S.S. Mirzoyev, M.S. Salimov, On solvability of a boundary value problem with operator coefficient in the boundary condition, *Matem. zametki*, **91** (2012), no 6, 861-869.

[9] S.S. Mirzoyev, G.I. Zamanov, On solvability of a boundary value problem for higher order operator-differential equations, *Vestnik Bakinskogo Universiteta*, (2015), no. 4, 13-17.

[10] G.V. Radzievskiy, Problem on completeness of root vectors in spectral theory of operator-functions, *UMN*, **37** (1982), issue. 2(221), 81-145.

[11] G.V. Radzievskiy, On completeness of derivative chains responding to boundary problems on finite interval, *Ukr. Mat. Zhurn.*, **34** (1979), no 4, 407-416.

[12] A.I. Vbrozub, On energetic completeness of the system of elementary solutions of differential equation in Hilbert space, *Funct. Analysis i ego priloz.*, **9** (1975), no 1, 52-53.

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