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ON APPROXIMATIVE PROPERTIES OF LOCALLY CHEBYSHEV SETS

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Abstract. A locally Chebyshev set is a set whose intersection with some closed neighbourhood with center at any point of this set is a Chebyshev set. The paper is concerned with local and global approximative properties of sets. A number of new properties of locally Chebyshev sets and local strict suns are put forward. We give an elementary proof of the recent Flerov's result to the effect that in a two-dimensional normed linear space a connected locally Chebyshev set is a Chebyshev set.

1. Introduction

Below X is a real normed linear space. We shall follow the definitions from the survey [3]. The main definitions will be given below.

A set M is a *Chebyshev set* if it is a set of existence and uniqueness (see [8], [3]); that is, for any $x \in X$ the set $P_M x$ of nearest points from M to x is a singleton.

The best approximation, that is, the distance of a given element x in a normed linear space X from a given non-empty set $M \subset X$ is, by definition $\rho(x, M) := \inf_{y\in M} \|x - y\|$. The set of all *nearest points* (elements of best approximation) in M for a given $x \in X$ is denoted by $P_M x$. So,

$$P_M x := \bigl\{y \in M \mid \rho(x, M) = \|x - y\|\bigr\}.$$

In what follows, X is a normed linear space, $B(x, r)$ is the closed ball with center x and radius r, $\mathring{B}(x, r)$ is the open ball; $S(x, r)$ is the sphere with center x and radius r. For brevity, we put $S := S(0, 1)$.

For a set $\varnothing \neq M \subset X$, a point $x \in X \setminus M$ is called a *solar point* if there exists a point $y \in P_M x \neq \varnothing$ (a *luminosity point*) such that

$$y \in P_M\bigl((1-\lambda)y + \lambda x\bigr) \quad \text{for all} \quad \lambda \geq 0 \tag{1.1}$$

(geometrically, this means that there is a 'solar' ray emanating from y and passing through x such that y is a point of best approximation in M for any point from this ray).

A point $x \in X \setminus M$ is called a *strict solar point* if $P_M x \neq \varnothing$ and condition (1.1) holds for any point $y \in P_M x$. A set M is an *existence set* (or proximinal) if $P_M x \neq \varnothing$ for any $x \in X$.

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A set $M \subset X$ is a *sun* (respectively, a *strict sun*) if any point $x \in X \setminus M$ is a solar point (respectively, strict solar point) for M.

In a finite-dimensional space any Chebyshev set is a sun (that is, a Chebyshev sun). This is no longer true in the infinite-dimensional setting [3].

The concept of a *locally Chebyshev set* was proposed by M. V. Balashov in analogy with that of a locally convex set: a set $M \subset X$ is called *locally convex* [12, § 1.4] if, for any point $y \in \overline{M}$, there exists a number $r = r(y) > 0$ such that the set $M \cap B(y, r)$ is convex (see, for example, [12, § 1.10]). A connected locally convex set is well known to be convex. Perusing this analogy further, a set $M \subset X$ will be called a *locally Chebyshev set* if, for any $y \in M$, there exists a closed solid neighbourhood $\mathcal{O}(y)$ of the point y such that the set $M \cap \mathcal{O}(y)$ is a Chebyshev set.

A locally Chebyshev set is necessarily closed and possesses a Chebyshev layer (of possibly nonuniform size). Recall, that a set M in a normed space X has Chebyshev layer of size $R > 0$ if, for any point $u \in \mathring{\mathcal{U}}(R, M) := \{x \in X \mid 0 < \rho(x, M) < R\}$ the set $P_M u$ consists of one point [11]. Clearly, in the finite-dimensional (or reflexive setting), in the definition of a locally Chebyshev set it suffices to consider only convex neighbourhoods.

Remark 1.1. Our definition of the local Chebyshev property is more general than that by Balashov and Flerov [9], in which the local Chebyshev property of a set is defined in terms of intersections with balls; however, this cannot be considered fairly natural, because this definition drastically narrows the class of spaces in which locally Chebyshev sets may exist: if a point lies inside a set, then Balashov–Flerov's definition implicitly requires that the ball of the space be strictly convex [9, § 4]. Flerov (using the definition of the local Chebyshev property in terms of intersection with balls) proved that in a two-dimensional space any connected locally Chebyshev set is a Chebyshev set. The requirement that a set be connected cannot be dropped [9]: a two-point set is a locally Chebyshev set which is not a Chebyshev set.

In analogy with a locally Chebyshev set a *local sun* (*local strict sun*, respectively) is defined as a set M such that, for any $y \in M$, there exists a closed neighbourhood $\mathcal{O}(y)$ such that the intersection $M \cap \mathcal{O}(y)$ is a sun (strict sun). (A local (strict) sun is necessarily closed.)

Remark 1.2. A connected local sun which is not a sun can be easily constructed even in the two-dimensional setting: the unit sphere in ℓ^∞_2.

In the present paper, we shall establish some new properties of locally Chebyshev sets in normed linear spaces (Theorems 3.2 and 3.1) and involve the machinery of geometric approximation theory to give an elementary proof of Flerov's result [9] to the effect that in a two-dimensional normed linear space a connected locally Chebyshev set is a Chebyshev set (Theorem 3.2).

It is also worth mentioning that local approximative properties of sets proved to be useful in the study of their global approximative and geometric properties and of stability of best and near-best approximation operators [7].

2. Definitions and notation

Following L. P. Vlasov, if Q is some property (for example, 'connectedness'), then we say that a closed set M has the property

P-Q if, for all $x \in X$, the set $P_M x$ is nonempty and has the property Q.

For example, a set M is P-connected if $P_M x$ is nonempty and connected for any $x \notin M$.

Let $k(\tau)$, $0 \leqslant \tau \leqslant 1$, be a continuous curve in a normed linear space X. A curve $k(\,\cdot\,)$ is called *monotone* if $f(k(\tau))$ is a monotone function in τ for any $f \in \operatorname{ext} S^*$, where $\operatorname{ext} S^*$ is the set of all extreme points of the unit sphere S^* of the dual space (see [3]). A set $M \subset X$ is *monotone path-connected* [3] if any two points from M can be joined by a continuous monotone curve (arc) $k(\,\cdot\,)$ from M. A monotone path-connected set is always extreme monotone path-connected (that is, its intersection with any intersection of hyper-layers generated by extreme functionals of the dual unit sphere is monotone path-connected [3]).

The concept of monotone path-connectedness was found to be important in many problems of the approximation theory. For example, with the help of monotone path-connectedness the solarity of Chebyshev sets was first proved under connection-type constraints (see [3, § 9.2]).

We recall that a set M is *locally compact* if every point in M has a neighbourhood in M which is a compact set. A set is *boundedly compact* if its intersection with any closed ball is compact. A set M will be called *locally monotone path-connected* if any point $x \in M$ has a neighbourhood whose intersection with M is monotone path-connected.

3. Main results and proofs

Theorem 3.1. *In a normed linear space a locally Chebyshev set M is a Chebyshev set if M is P-connected, locally compact and locally monotone path-connected.*

Note that in Theorem 3.1 M is an existence set.

Remark 3.1. The well-known Dunham's example of a disconnected Chebyshev set in $C[0,1]$ was modified by Flerov to construct a connected locally Chebyshev set M in $C[0,1]$ which is not a Chebyshev set. Recall Dunham's construction. In $C[0,1]$ consider the set

$$M = \{f_0\} \cup \{f_a \mid a > 0\},$$

$$\text{where} \quad f_a(t) = (1+a)e^{-t/a}, \quad a > 0, \quad f_0(t) \equiv 0$$

(as $f(\,\cdot\,)$ one may also take $f(t) = (1+t)^{-1}$). Such a set M has an isolated point $\{f_0\}$ and hence is not a sun. The set M is locally compact (but not boundedly compact). Flerov [10] proved that $M_1 := M \setminus \{f_0\}$ is a locally Chebyshev set (and hence a locally Chebyshev sun, since a boundedly compact Chebyshev set is a sun). The author proved in [2] that a boundedly compact strict sun in the space $C(Q)$ is monotone path-connected. Hence, the set M_1 (and, of course, M) is locally monotone path-connected. Besides, a direct verification shows that M_1 itself is monotone path-connected (but not M) and hence M_1 is a strict protosun. However, M_1 is not an existence set, since $P_{M_1}(f_0) = \emptyset$. It is also worth noting that M_1 is P-connected on $C[0,1] \setminus \{f_0\}$, because M is a Chebyshev set and

$P_{M_1}(f_0) = \emptyset$. This example of the set M_1 shows that the proximinality condition in Theorem 3.1 cannot be discarded.

Remark 3.2. *In a normed linear space a locally Chebyshev set is a Chebyshev sun if it is boundedly compact and monotone path-connected.* Consider such a set M. To prove the claim, we first note that a monotone path-connected set is P-monotone path-connected, and hence, P-connected (see [3, § 9.1]). Hence, the set M satisfies all the hypotheses of Theorem 3.1, and therefore, is a Chebyshev set. The solarity of the set M in Theorem 3.1 follows from the fact that a boundedly compact monotone path-connected set is a sun [3, § 9.1]. In the above fact we can replace the monotone path-connectedness condition by the Menger-connectedness condition; see [3, § 9.1].

Remark 3.3. Theorem 3.1 turns out to be futile in the finite-dimensional setting, since in this case a locally Chebyshev set M is necessarily P-finite (that is, $P_M x$ consists of finitely many points for any point $x \in X$). This fact, which was noticed by A. Flerov, easily follows from compactness arguments.

The next theorem was proved by A. Flerov's under a different more restrictive definition of a locally Chebyshev set (see Remark 1.1). We give an elementary geometric proof of this result. Theorem 3.3 extends Theorem 3.2 to the more general setting of local strict suns.

Theorem 3.2. *In a two-dimensional Banach space a connected locally Chebyshev set is a Chebyshev set* (*a Chebyshev sun*).

Theorem 3.3. *In a two-dimensional Banach space a connected local strict sun is a strict sun.*

Remark 3.4. As was already mentioned above, a connected local sun need to ne a sun. It would be interesting to find conditions to guarantee that a connected local sun in a plane is a sun (cf. Remark 1.2).

Proof of Theorem 3.1. Assume on the contrary that for some point x there are at least two nearest points in M, let y be one such a point. Note that $P_M x \neq \emptyset$, since M is an existence set. We assume without loss of generality that $x = 0$, $\rho(0, M) = 1$.

Let $\mathcal{O}_1(y)$ be a neighbourhood of y whose intersection with M is monotone path-connected, let $B(y, r_1)$ be a ball lying in $\mathcal{O}_1(y)$, $r_1 > 0$. Next, let $\mathcal{O}_2(y)$ be a neighbourhood of y whose intersection with M is compact and let $B(y, r_2)$ be a ball lying in the neighbourhood $\mathcal{O}_2(y)$ and in the ball $B(y, r_1)$. Further, let $\mathcal{O}(y)$ be a (closed) neighbourhood of y whose intersection with M is a Chebyshev set and let $B(y, r)$ be a ball lying in this neighbourhood, $0 < r < r_2$.

It is well known that the intersection of a monotone path-connected set with a closed ball is monotone path-connected [3, § 9.1]. Hence the set

$$M_1 := \big(M \cap B(y, r)\big) \cap B(0, 1)$$

is monotone path-connected. Here $M \cap B(y, r)$ is monotone path-connected *qua* the intersection of the monotone path-connected set $\mathcal{O}_1(y) \cap M$ with the ball $B(y, r)$, and M_1 is monotone path-connected *qua* the intersection of the monotone path-connected set $M \cap B(y, r)$ with the ball $B(0, 1)$.

Since M_1 lies in the sphere $S(0,1)$, since by the hypotheses the set $P_M 0$ is connected and consists of at least two points, and since $y \in P_M 0$, it follows that M_1 is a nondegenerate continuum (by definition, a continuum is a nonsingleton connected compact set). Hence, for sufficiently small $0 < \varepsilon < r$, the intersection $B(y,\varepsilon) \cap M_1$ consists of at least two points $y, y' \in M_1$. We have $y, y' \in P_M 0$, $y, y' \in M_1$, $(\mathcal{O}(y) \cap M) \cap \mathring{B}(0,1) = \emptyset$, and hence

$$y, y' \in P_{M_1} 0, \quad \text{and hence} \quad y, y' \in P_{\mathcal{O}(y) \cap M}.$$

The last inclusion contradicts the fact that $\mathcal{O}(y) \cap M)$ is a Chebyshev set. So, our assumption was false and M is a Chebyshev set. This proves Theorem 3.1. $\square$

For a proof of Theorem 3.2 we require one result, which was obtained jointly by the author and E. V. Shchepin [6]. Here we need one more definition.

Given $y \in S$, we let Λ_y denote the set of limit points of the expression

$$(y-z)/\|y-z\| \quad \text{as} \quad z \to y, \qquad z \in S$$

(so, Λ_y is the *set of semi-tangent directions* to the sphere S at a point y). A direction a is called (*globally*) *tangent direction* to the sphere S if, for any point $y \in S$, the condition that a is a tangent direction at the point y implies that $a \in \Lambda_y$; that is, a is a tangent direction at y. For example, in the space ℓ^∞_n, $n \geq 2$, only the directions parallel to edges of the unit ball (cube) are the tangent directions to the sphere. In ℓ^1_n, $n \geq 3$, there are no tangent directions to the sphere. A set M is *convex in a direction* a if the condition $x, y \in M$, $(y-x) \parallel a$, implies that $[x,y] \subset M$.

Theorem A (see [6]). *Let X be a two-dimensional Banach space, $\emptyset \neq M \subset X$. Then M is a sun if and only if M is closed, connected and convex with respect to any tangent direction of the unit sphere S.*

Proof of Theorem 3.2. On a plane any sun (and hence, a Chebyshev set) is monotone path-connected [5]. By the hypotheses, M is a locally Chebyshev set, and hence is path-connected, since it is well-known that a connected and locally path-connected set is path-connected.

To prove the local monotone path-connectedness of M, we consider a sufficiently large ball $B(x,r)$ such that $M \cap B(x,r)$ is connected (this is possible since by the above M is path-connected). The intersection $M \cap B(x,r)$ is compact, since M is closed. It now suffices to cover the set $M \cap B(x,r)$ by open neighbourhoods $\mathcal{O}(y)$, $y \in M \cap B(x,r)$, such that $M \cap \overline{\mathcal{O}}(y)$ is a Chebyshev set, and extract a finite subcover.

Assume on the contrary that M is not a Chebyshev set. Then M is not a sun. By Theorem A there is a tangent direction a such that M is not convex in the direction a. So, we can take

$$x, y \subset M, \qquad (x-y) \parallel a, \qquad (x,y) \cap M = \emptyset.$$

Without loss of generality we may assume that $(x+y)/2 = 0$, $\|x\| = 1 = \|y\|$.

The open ball $\mathring{B}(0,1))$ is contained in one connected component Ω of the complement $X \setminus M$. Since M is connected, the domain Ω is simply connected and the closed interval $[x,y]$ divides Ω into two nonintersecting domains Ω_1 and Ω_2 of which at least once is bounded. Assume that Ω_1 is bounded.

Let $\omega_1 := \partial\Omega_1 \setminus (x, y)$, where $\partial\Omega_1$ is the boundary of Ω_1. The above argument shows that ω_1 is locally monotone path-connected. Hence, ω_1 can be looked upon as a curve $k(t)$, $0 \le t \le 1$, which joins x and y.

Clearly, any semi-tangent direction to S is generated by an extreme functional $f \in \operatorname{ext} S^*$ (and vice versa). Let $f \in \operatorname{ext} S^*$ be an extreme functional (one of the two) corresponding to the tangent direction a. We assume without loss of generality that $f(k(\,\cdot\,))$ is nonnegative. Let T be the set of points $t \in [0, 1]$ on which $f(k(t))$ assumes its maximum value, $T \ne \emptyset$. Let $t_0 \in T$. Let us examine the behaviour of $k(t)$ near t_0.

It is easily checked that t_0 cannot be a point of strict maximum. Indeed, if this were so, then we would consider a closed neighbourhood $\mathscr{O}_0 := \mathscr{O}(k(t_0))$ such that $\mathscr{O}_0 \cap M$ is a Chebyshev set. By Theorem A any Chebyshev set on a plane is convex with respect to any tangent direction of the sphere. But since the maximum is strict, the Chebyshev $\mathscr{O}_0 \cap M$ cannot be convex with respect to the direction a.

Assume now that t_0 is a point of nonstrict maximum: $f(k(t_0) = f(k(t))$ for some $t \in [t_0, t_1]$, where $t_1 := t_0 + \varepsilon_0$ and $\varepsilon_0 > 0$ is such that

$$f(k(\tau)) < f(k(t_0 + \varepsilon_0)) = f(k(t_0)) \quad \text{if} \quad \tau > t_0 + \varepsilon_0 =: t_1 \quad \text{or} \quad \tau < t_0. \tag{3.1}$$

We shall identify the curve $k(\,\cdot\,)$ and its trace.

Let $\varkappa$ be the portion of $k(\,\cdot\,)$ for $t \in [t_0, t_1]$. Since $f(k(t))$ is constant on $[t_0, t_1]$, $\varkappa$ is is the closed interval between $k(t_0)$ and $k(t_1)$, which parallel to the direction a, and hence to the interval $[x, y]$.

Lemma 2.2 of [1] asserts that, for a sun N in a finite-dimensional space Y, $N \ne Y$, any point y from the boundary of N is a point of luminosity; that is, there is a ray ℓ emanating from y and such that $y \in P_N x$ for all points $x \in \ell$. Hence, for any $v \in [k(t_0), k(t_1)] = \varkappa$, there is an $w \in \Omega$ (sufficiently close to v) such that $v \in P_M w$. Given $L \subset X$, $L \ne \emptyset$. Consider the mapping

$$\tau_v(w) = \tau(w) := \frac{v - w}{\rho(w, P_L w)} \subset S,$$

where $P_L w =: \{v\}$.

Since M is a locally Chebyshev set, for any $v \in \big(k(t_0), k(t_1)\big)$ the point $\tau_v(w)$ is a smooth and exposed point of the unit sphere S.

Consider the Chebyshev set $M_1 := M \cap \mathscr{O}(k(t_0))$ (where $\mathscr{O}(k(t_0))$ is chosen from the definition of a locally Chebyshev set). If $\varepsilon > 0$ is sufficiently small, then $k(t_0 + \varepsilon)$ is the nearest point in M_1 of for the corresponding point $w_\varepsilon \in \Omega$. Since $\tau(w_\varepsilon)$ is a smooth point of the unit sphere,

$$M_1 \cap \big\{u \mid f(u) < t(k(t_0))\big\} = \emptyset \tag{3.2}$$

by the well-known Kolmogorov criterion of best approximation (in the form for suns) [3, § 3,2]. Hence, (3.2) is possible only if there is $\varepsilon_1 > 0$ such that $f(k(\tau)) = f(k(t_0))$ for all $\tau \in (t_0 - \varepsilon_1, t_0)$. But this contradicts (3.1). $\square$

The proof of Theorem 3.3 repeats that of Theorem 3.2 and hence omitted.

Remark 3.5. The machinery of monotone path-connectedness is potent to prove that in the space ℓ^∞_n any connected locally Chebyshev *curve* is a Chebyshev set.

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