Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 44, Number 1, 2018, Pages 43–50

THE INVERSE PROBLEM OF FINDING THE INITIAL FUNCTION FOR THE STRING VIBRATION EQUATION

GUNAY ISMAYILOVA

Abstract. In this paper, we propose an approach to the solution of the inverse problem for finding the initial function for the string vibration equation. The search for unknown initial function is reduced to the minimization problem of the functional, constructed with the help of the additional information. A formula is obtained for the gradient of the functional and a necessary and sufficient condition for optimality is derived.

1. Introduction

Inverse problems for differential equations with partial derivatives are actual problems of modern mathematics because of their importance for applications. Such problems arise in the various fields of mathematics, shape optimization, geophysics, seismology, astronomy, ecology, etc. [1-3, 7].

In this paper, we propose an approach to the solution of the inverse problem for the string vibration equation. The search for the unknown initial function is reduced to the problem of minimization of the functional, built with the help of the additional information. A formula is obtained for the gradient of the functional and a necessary and sufficient condition for optimality is derived using the gradient of the functional.

2. Statement of the problem

We consider in the domain $Q = \{ (x,t) : 0 < x < l,\ 0 < t < T\}$ the following boundary value problem

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f\left(x,t\right), \quad (x,t) \in Q, \tag{2.1}$$

$$u(x,0) = u_0(x), \quad \frac{\partial u(x,0)}{\partial t} = v\left(x\right), \quad 0 \le x \le l, \tag{2.2}$$

$$u(0,t) = 0,\ u(l,t) = 0, \qquad 0 \le t \le T. \tag{2.3}$$

2010 *Mathematics Subject Classification.* 34L55.
Key words and phrases. inverse problem, equations of string vibration, optimality condition.

Here $l, T > 0$ are given numbers, $f \in L_2(Q)$, $u_0 \in \overset{0}{W}{}_2^1(0,l)$ are given functions, $v \in L_2(0,l)$ is an unknown function. In order to determine $v(x)$ we use the additional information

$$u(x,T) = \chi(x), \quad 0 \le x \le l, \tag{2.4}$$

where $\chi \in \left\{v(x) \in W_2^1(0,l) : v(0) = v(l) = 0\right\}$ is a given function.

We reduce this problem to the following optimal control problem: to minimize the functional

$$J_0(v) = \frac{1}{2}\int_0^l [u(x,T;v) - \chi(x)]^2\, dx \tag{2.5}$$

subject to (2.1)-(2.3). Here $u = u(x,t) = u(x,t;v)$ is a solution of problem (2.1)-(2.3), corresponding to the function $v = v(x)$. This problem we call problem (2.1)-(2.3), (2.5). The function $v(x)$ is called a control. If we find a control $v(x)$ that gives zero value to the functional (2.5) then the additional condition (2.4) is satisfied.

Note that for each fixed control $v = v(x) \in L_2(0,l)$, boundary problem (2.1)-(2.3) has a unique generalized solution $u = u(x,t;v)$ from $W_{2,0}^1(Q)$ [5].

3. On solvability of problem (2.1)-(2.3), (2.5).

Consider the problem: under which conditions

$$\inf_{v \in L_2(0,l)} J_0(v) = 0? \tag{3.1}$$

This question is equivalent to the problem of density in $L_2(0,l)$ of the image of $L_2(0,l)$ under the mapping

$$v \to u(x,T;v). \tag{3.2}$$

To solve this problem we use Hahn-Banach theorem [4].

Let $\varphi(x)$ be a given function from $L_2(0,l)$, that is orthogonal to the image of $L_2(0,l)$ under the mapping (3.2), i.e.

$$\int_0^l u(x,T;v)\varphi(x)dx = 0, \forall v \in L_2(0,l). \tag{3.3}$$

We want to find out whether it will follow from this that $\varphi(x) = 0$?

Let us introduce the function $W(x,t)$ as a solution of the problem

$$\frac{\partial^2 W}{\partial t^2} - \frac{\partial^2 W}{\partial x^2} = 0, \quad (x,t) \in Q, \tag{3.4}$$

$$W(x,T) = 0, \quad \frac{\partial W(x,T)}{\partial t} = \varphi(x), \quad 0 \le x \le l, \tag{3.5}$$

$$W(0,t) = 0, W(l,t) = 0, \qquad 0 \le t \le T. \tag{3.6}$$

Problem (3.4)-(3.6) has a unique generalized solution from $W_{2,0}^1(Q)$ [2].

Due to definition of the generalized solution of the problem (2.1)-(2.3) we have: at $t = 0$ the condition $u(x,0;v) = u_0(x)$ is satisfied and the integral identity

$$\iint_Q \left(-\frac{\partial u}{\partial t}\frac{\partial \eta}{\partial t}+\frac{\partial u}{\partial x}\frac{\partial \eta}{\partial x}-f\eta\right)dxdt+\int_0^l \frac{\partial u\left(x,T;v\right)}{\partial t}\eta\left(x,T\right)dx- \\ -\int_0^l v\left(x\right)\eta\left(x,0\right)dx=0 \tag{3.7}$$

is fulfilled for arbitrary function $\eta=\eta\left(x,t\right)\in W_{2,0}^1\left(Q\right)$.

By the definition of the generalized solution of problem (3.4)-(3.6) we have: at $t=T$ the condition $W(x,T)=0$ is satisfied and the integral identity

$$\iint_Q \left(-\frac{\partial W}{\partial t}\cdot\frac{\partial g}{\partial t}+\frac{\partial W}{\partial x}\frac{\partial g}{\partial x}\right)dxdt+\int_0^l \frac{\partial W(x,T)}{\partial t}g(x,T)dx- \\ -\int_0^l \frac{\partial W(x,0)}{\partial t}g(x,0)dx=0 \tag{3.8}$$

is fulfilled for arbitrary function $g\in W_{2,0}^1\left(Q\right)$.

Taking W as a function η in the equality (3.7), and u as a function g in (3.8), then subtracting (3.8) from (3.7) we get

$$-\iint_Q fWdxdt-\int_0^l v\left(x\right)W(x,0)dx-\int_0^l \frac{\partial W(x,T)}{\partial t}u\left(x,T;v\right)dx+ \\ +\int_0^l \frac{\partial W(x,0)}{\partial t}u_0\left(x\right)dx=0,\forall v\in L_2\left(0,l\right).$$

Considering here the second condition of (3.5) we obtain

$$\iint_Q fWdxdt+\int_0^l v\left(x\right)W(x,0)dx+\int_0^l u\left(x,T;v\right)\varphi\left(x\right)dx- \\ -\int_0^l \frac{\partial W(x,0)}{\partial t}u_0\left(x\right)dx=0,\forall v\in L_2\left(0,l\right).$$

Due to (3.3) from the last is follows that

$$\iint_Q fWdxdt+\int_0^l W(x,0)v\left(x\right)dx-\int_0^l \frac{\partial W(x,0)}{\partial t}u_0\left(x\right)dx=0, \qquad \forall v\in L_2\left(0,l\right).$$

If we write this relation for two arbitrary controls $v_1\left(x\right)\in L_2\left(0,l\right)$ and $v_2\left(x\right)\in L_2\left(0,l\right)$ and subtract the obtained equalities, we get

$$\int_0^l W\left(x,0\right)\left(v_1\left(x\right)-v_2\left(x\right)\right)dx=0, \quad \forall v_1,v_2\in L_2\left(0,l\right).$$

Hence by Lagrange lemma it follows that

$$W(x,0)=0, \quad 0\le x\le l.$$

Now let's consider the boundary problem

$$\frac{\partial^2 W}{\partial t^2} - \frac{\partial^2 W}{\partial x^2} = 0, \quad (x,t) \in Q, \tag{3.9}$$

$$W(0,t) = 0, \ \ W(l,t) = 0, \quad 0 \le t \le T, \tag{3.10}$$

$$W(x,0) = 0, \ \ W(x,T) = \ 0, \qquad 0 \le x \le l. \tag{3.11}$$

By the Fourier method the solution of the equation (3.9) with conditions (3.10) is obtained in the form

$$W(x,t) = \sum_{k=1}^{\infty} \left(a_k \cos \frac{\pi k}{l} t + b_k \sin \frac{\pi k}{l} t \right) \sin \frac{\pi k}{l} x.$$

Since the system of the functions $\left\{ \sin \frac{\pi k}{l} x \right\}_{k=1}^{\infty}$ forms a complete system in $L_2(0,l)$, from the first condition of (3.11) it follows that $a_k = 0, \ k = 1, 2, \dots$. Similarly from the second condition of (3.11) it follows that $b_k \sin \frac{\pi k}{l} T = 0, \ k = 1, 2, \dots$.
If $\sin \frac{\pi k}{l} T \ne 0$, i.e. $\frac{T}{l} \ne \frac{n}{k}, \ n \in Z, \ k \in N$, then $b_k = 0, \ k = 1, 2, \dots$.
From this we obtain that if the numbers T and l are incommensurable, then $b_k = 0, \ k = 1, 2, \dots$.
Therefore

$$W(x,t) \equiv 0, \quad (x,t) \in Q.$$

Then as follows from the condition (3.5), $\varphi(x) = 0$. Thus we prove the following theorem.

Theorem 3.1. *Let $f \in L_2(Q)$, $u_0 \in W_2^1(0,l), \chi \in L_2(0,l)$ and the numbers T and l are incommensurable. Then*

$$\inf_{v \in L_2(0,l)} J_0(v) = 0.$$

If the image of $L_2(0,l)$ by the mapping (3.2) is closed in $L_2(0,l)$, then there exists the element $v^*(x) \in L_2(0,l)$ such that

$$\min_{v \in L_2(0,l)} J_0(v) = J_0(v^*) = 0.$$

Now instead of the problem (2.1)-(2.3), (2.5) consider the problem: minimize the functional

$$J_\alpha(v) = J_0(v) + \frac{\alpha}{2} \|v\|^2_{L_2(0,l)} \tag{3.12}$$

on the convex closed set $V_d \subset L_2(0,l)$ subject to (2.1)-(2.3), where $\alpha > 0$ is a given number. This problem we call problem (2.1)-(2.3), (3.12), and V_d - a class of admissible controls. Due to the known theorem from [6, p.13] for the new problem (2.1)-(2.3), (3.12) there exists the only element from V_d that minimizes the functional (3.12).

4. Calculation of the differential of the functional (3.12) and optimality condition

Let's show that the functional (3.12) is differentiable in $L_2(0,l)$.
For this purpose we take two admissible controls v, $v+\delta v$. Corresponding solutions of the problem (2.1)-(2.3) denote by $u(x,t;v)$ and $u(x,t;v+\delta v)$.

Let $\delta u(x,t)=u(x,t;v+\delta v)-u(x,t;v)$. It is clear that $\delta u(x,t)$ is a generalized solution from $W_{2,0}^1(Q)$ of the following boundary problem

$$\frac{\partial^2\delta u}{\partial t^2}-\frac{\partial^2\delta u}{\partial x^2}=0,\qquad (x,t)\in Q, \tag{4.1}$$

$$\delta u(x,0)=0\ ,\quad \frac{\partial\delta u(x,0)}{\partial t}=\delta v(x),\qquad 0\le x\le l, \tag{4.2}$$

$$\delta u(0,t)=\delta u(l,t)=0,\qquad\qquad 0\le t\le T, \tag{4.3}$$

i.e. at $t=0$ it fulfills the condition $\delta u(x,0)=0$ and the integral identity

$$\begin{aligned}&\iint_Q\left(-\frac{\partial\delta u}{\partial t}\frac{\partial\eta}{\partial t}+\frac{\partial\delta u}{\partial x}\frac{\partial\eta}{\partial x}\right)dxdt+\int_0^l\frac{\partial\delta u\left(x,T\right)}{\partial t}\eta\left(x,T\right)dx-\\&-\int_0^l\delta v\left(x\right)\eta\left(x,0\right)dx=0.\end{aligned} \tag{4.4}$$

fulfils for any function $\eta\in W_{2,0}^1(Q)$.

Let $\psi=\psi\left(x,t\right)=\psi\left(x,t;v\right)$ be a generalized solution from $W_{2,0}^1(Q)$ of the adjoint problem

$$\frac{\partial^2\psi}{\partial t^2}-\frac{\partial^2\psi}{\partial x^2}=0,\qquad (x,t)\in Q, \tag{4.5}$$

$$\psi(x,T;v)=0,\quad \frac{\partial\psi(x,T;v)}{\partial t}=-\left[u(x,T;v)-\chi\left(x\right)\right],\qquad 0\le x\le l, \tag{4.6}$$

$$\psi(0,t;v)=0,\qquad \psi(l,t;v)=0,\qquad\qquad 0\le t\le T. \tag{4.7}$$

It is clear that boundary problem (4.5)-(4.7) for each $v\in V_d$ has a unique generalized solution from $W_{2,0}^1(Q)$ [5]. Then at $t=T$ the first condition of (4.6) and the integral identity

$$\begin{aligned}&\iint_Q\left(-\frac{\partial\psi}{\partial t}\frac{\partial g}{\partial t}+\frac{\partial\psi}{\partial x}\frac{\partial g}{\partial x}\right)dxdt-\int_0^l\left[u(x,T;v)-\chi\left(x\right)\right]g\left(x,T\right)dx-\\&-\int_0^l\frac{\partial\psi\left(x,0\right)}{\partial t}g\left(x,0\right)dx=0\end{aligned} \tag{4.8}$$

is satisfied for any function $g\in W_{2,0}^1(Q)$.

If we put $\eta=\psi\left(x,t;v\right)$, in (4.4) and $g=\delta u\left(x,t\right)$ in (4.8) and subtract the obtained relations we get

$$\int_0^l [u(x,T;v)-\chi(x)]\,\delta u(x,T)\,dx - \int_0^l \delta v(x)\,\psi(x,0;v)\,dx = 0\,. \tag{4.9}$$

Now we calculate the increment of the functional (3.12):

$$\begin{aligned}\Delta J_\alpha(v) = J_\alpha(v+\delta v) - J_\alpha(v) = \int_0^l [u(x,T;v)-\chi(x)]\,\delta u(x,T)\,dx + \\ +\alpha\int_0^l v(x)\,\delta v(x)\,dx + R,\end{aligned} \tag{4.10}$$

where R is a remainder term and has a form

$$R = \frac{1}{2}\int_0^l |\delta u(x,T)|^2\,dx + \frac{\alpha}{2}\int_0^l |\delta v(x)|^2\,dx. \tag{4.11}$$

Considering (4.9) in (4.10) one can get

$$\Delta J_\alpha(v) = \int_0^l [\psi(x,0;v)+\alpha v(x)]\,\delta v(x)\,dx + R\,. \tag{4.12}$$

From the boundary problem (4.1)-(4.3) as in $[5, pp.213-215]$ it is not difficult to obtain the estimate

$$\int_0^l \left[|\delta u(x,t)|^2 + \left|\frac{\partial \delta u(x,t)}{\partial x}\right|^2 + \left|\frac{\partial \delta u(x,t)}{\partial t}\right|^2 \right] dx \le c\,\|\delta v\|^2_{L_2(0,l)}$$

$$\forall t \in [0,T]\ .$$

Here and later on c is a constant not depending on estimated quantity and controls.

From this in particular follows

$$\int_0^l |\delta u(x,T)|^2\,dx \le c\,\|\delta v\|^2_{L_2(0,l)}\ . \tag{4.13}$$

Then from (4.11) and (4.13) we get

$$|R| \le c\,\|\delta v\|^2_{L_2(0,l)}\ . \tag{4.14}$$

Thus from (4.12) and estimate (4.14) it follows that the functional $J_\alpha(v)$ is differentiable in $L_2(0,l)$ and its differential and gradient are defined by the expressions

$$< J'_\alpha(v), \delta v >_{L_2(0,l)} = \int_0^l [\psi(x,0;v)+\alpha v(x)]\,\delta v(x)\,dx \tag{4.15}$$

and

$$J'_\alpha(v) = \psi(x,0;v) + \alpha v(x)\ . \tag{4.16}$$

Let's show that the mapping $v \to J'_\alpha(v)$, defined by the relation (4.15) acts continuously from V_d to $L_2(0,l)$. Let $\delta\psi(x,t) = \psi(x,t;v+\delta v) - \psi(x,t;v)$. As follows from (4.5)-(4.7) $\delta\psi(x,t)$ is a generalized solution from $W_2^1(Q)$ for the boundary problem

$$\begin{array}{ll} \frac{\partial^2\delta\psi}{\partial t^2} - \frac{\partial^2\delta\psi}{\partial x^2} = 0, & (x,t)\in Q \\ \delta\psi(x,T)=0, & \left.\frac{\partial\delta\psi}{\partial t}\right|_{t=T} = -\delta u(x,T),\ \ 0\le x\le l, \\ \delta\psi(0,t;v)=0, & \delta\psi(l,t;v)=0,\ \ 0\le t\le T. \end{array}$$

From this boundary problem similarly to [5, p. 213-215] one may obtain the estimate

$$\int_0^l \left[|\delta\psi(x,t)|^2 + \left|\frac{\partial\delta\psi(x,t)}{\partial x}\right|^2 + \left|\frac{\partial\delta\psi(x,t)}{\partial t}\right|^2 \right] dx \le C\,\|\delta u(x,T)\|^2_{L_2(0,l)}\,,\ \ \forall t\in[0,T]. \tag{4.17}$$

Then from (4.13) and(4.17) it follows that

$$\int_0^l \left[|\delta\psi(x,t)|^2 + \left|\frac{\partial\delta\psi(x,t)}{\partial x}\right|^2 + \left|\frac{\partial\delta\psi(x,t)}{\partial t}\right|^2 \right] dx \le C\,\|\delta v\|^2_{L_2(0,l)}\,,\ \ \forall t\in[0,T].$$

From this last in particular we get that

$$\int_0^l |\delta\psi(x,0)|^2\,dx \le C\,\|\delta v\|^2_{L_2(0,l)}\,. \tag{4.18}$$

Now using the formula (4.16) it is not difficult to obtain the inequality

$$\left\| J'_\alpha(v+\delta v - J'_\alpha(v)\right\|_{L_2(0,l)} \le C\left[\|\delta\psi(x,0)\|_{L_2(0,l)} + \|\delta v\|_{L_2(0,l)} \right].$$

Due to (4.18) the right hand side of this inequality tends to zero when $\|\delta v\|_{L_2(0,l)} \to 0$. It gives that $v \to J'_\alpha(v)$ is a continuous mapping from V_d to $L_2(0,l)$.
We prove the following theorem.
Theorem 4.1. *Let the condition set on the data of the problem* (2.1)-(2.3), (3.12) *be fulfilled. Then the functional* (3.12) *is continuous Frechet differentiable in* $L_2(0,l)$ *and its differential and gradient in the point* $v(x)\in V_d$ *at the increment* $\delta v(x)\in L_2(0,l)$ *are defined by the expressions* (4.15) *and* (4.16), *correspondingly.*
Theorem 4.2. *Let the condition set on the data of the problem* (2.1)-(2.3), (3.12) *be fulfilled. Then for the optimality of the control* $v_* = v_*(x)\in V_d$ *in the problem* (2.1)-(2.3), (3.12), *the fulfillment of the inequality*

$$\int_0^l \left[\psi(x,0;v_*) + \alpha v_*(x)\right]\left[v(x) - v_*(x)\right]dx \ge 0,\ \ \forall v\in V_d\,, \tag{4.19}$$

where $\psi(x,t;v_*)$ *is a solution of the adjoint problem* (4.5)-(4.7) *at* $v = v_*(x)$, *is necessary and sufficient.*
Proof. The set V_d is convex in $L_2(0,l)$, the functional $J_\alpha(v)$ is continuously Frechet differentiable on $L_2(0,l)$ and its differential and gradient in the point $v(x)$ are defined by the formulas (4.15), (4.16). Then due to known theorem $[8, p.\, 28]$ on the element $v_*\in V_d$ it is necessary and sufficient the fulfillment of the inequality $< J'_\alpha(v_*), v - v_* >_{L_2(0,l)} \ge 0$ or following to (4.15) fulfillment of the inequality (4.19). Thus theorem 4.2 is proved.

References

[1] A.R. Aliyeva, On a domain optimization problem for the Pauli operator, *Advanced Mathematical Models & Applications* **1(1)** (2016), 50-55.

[2] Y.S. Gasimov, On a shape design problem for one spectral functional, *Journal of Inverse and Ill-Posed Problems* **21(5)** (2012), 629–637.

[3] S.I. Kabanikhin, *Inverse and Ill-posed Problems, Second Edition*, Sib. Nauch. Izd., Novosibirsk, 2009, 457p.

[4] A.N. Kolmogorov and S.V. Fomin, *Elements of the Theory of Functions and Functional Analysis,* Moscow, Nauka, 1981, 543p.

[5] O.A. Ladyzhenskaya, *Boundary-Value Problems of Mathematical Physics,* Moscow, Nauka, 1973, 408 p.

[6] J.L. Lions, *Optimal Control of Systems Described by the Partial Differential Equations*, Moscow, Mir, 1972, 416 p.

[7] R.R. Safiullova, The inverse problem for the second-order hyperbolic equation with unknown coefficient, depending on the time, *Bulletin of South Ural State University, The series "Mathematical modeling and programming",* **6(4)** (2013), 73-86.

[8] F.P. Vasilyev, *Methods Solutions Extremal Problems,* Moscow, Nauka, 1981, 400p.

Gunay Ismayilova
Sumgait State University, Sumgait, Azerbaijan.
E-mail address: gunay_ismayilova_83@mail.ru

Received: February 13, 2017; Revised: December 22, 2017; Accepted: January 4, 2018