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# OPTIMIZATION OF FOURTH-ORDER DIFFERENTIAL INCLUSIONS

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**Abstract**. The present paper studies the sufficient conditions of optimality for Cauchy problem of fourth-order differential $(P_D)$ inclusions. Mainly our purpose is to derive sufficient optimality conditions for mentioned problems with fourth-order differential inclusions (DFIs) and transversality conditions. The basic idea of obtaining optimal conditions is the use of locally adjoint mappings (LAM), defined by Hamiltonian functions. Moreover, in the application of these results the fourth-order linear optimal control problems with linear differential inclusions are considered. We analyze the proposed method for a class of Lagrange problem with integrand of quadratic form involving symmetric nonnegative semidefinite matrix. An illustrative example is given. Theoretical analysis and practical results show that our method is simple and easy to implement and is efficient for computing optimal solution of the fourth order differential inclusions. The results reveal that the proposed method is very accurate and efficient.

## 1. Introduction

The investigated optimization problem is the logical continuation of the work done in previous paper of Mahmudov [19], where are mainly concerned with the necessary and sufficient conditions of optimality for Bolza problem with fourth-order discrete and discrete-approximate inclusions. In the last decade discrete and continuous time processes with lumped and distributed parameters found wide application in the field of mathematical economics and in problems of control dynamic system optimization and differential games (see [6]-[10], [13, 14, 16, 17], [20]-[22]); a great deal of studies on the optimal control problems with higher order ordinary and partial differential inclusions have been made by many authors [2]-[7], [11, 12, 15, 18, 19], [23]-[28]. As is pointed out in [26], boundary value problems (BVPs) for second and fourth-order differential equations play a very important role in both theory and applications. In recent years, BVPs for second and higher order differential equations have been extensively studied. In particular, fourth-order linear differential equations [26], subjected to some boundary conditions arise in the mathematical description of some physical systems (for

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example, the mathematical models of deflection of beams [25, 26]). These beams, which appear in many structures, deflect under their own weight or under the influence of some external forces. For example, if a load is applied to the beam in a vertical plane containing the axis of symmetry, the beam undergoes a distortion, and the curve connecting the centroids of all cross sections is called the deflection curve or elastic curve.

Along the way, the problems accompanied with the fourth-order discrete and differential inclusions are more complicated due to the higher-order derivatives and their discrete analogues. In fact, the difficulty is to construct adjoint the inclusions and the transversality conditions. A convenient procedure for eliminating this complication in optimal control theory involving higher order derivatives is a formal transformation of these problems to the system of first order differential inclusions or equations. It appears that in practice returning to the original higher order problem and expressing the obtained optimality conditions by original problem dataset, in general, is very difficult and sometimes impossible. Consequently, a lot of investigations on the second-order differential inclusions usually are devoted to existence and viability problems. The first viability result for second-order differential inclusions were given by Cernea [5]. The paper [4] gives necessary and sufficient conditions ensuring the existence of solutions to the second-order differential inclusions with state constraints. In the paper [24], a class of nonlinear BVPs for second-order differential inclusions with nonlinear perturbations is studied.

In this paper, we deal with the problem for fourth-order differential inclusions

$$\text{minimize } J[x(\cdot)] = \int_0^1 g\big(x(t), x'(t), x''(t), x'''(t), t\big)\,dt + \varphi\big(x(1), x'(1), x''(1), x'''(1)\big) \tag{1.1}$$

$$(P_{FDI}) \qquad \frac{d^4x(t)}{dt^4} \in F\big(x(t), x'(t), x''(t), x'''(t), t\big), \text{ a.e. } \quad t \in [0,1], \tag{1.2}$$

$$x(0) = \alpha_0,\ x'(0) = \alpha_1,\ x''(0) = \alpha_2,\ x'''(0) = \alpha_3, \tag{1.3}$$

Here $F(\cdot, t) : \mathbb{R}^{4n} \rightrightarrows \mathbb{R}^n$ is a set-valued mapping, $g(\cdot, t) : \mathbb{R}^n \to \mathbb{R}^1$ is continuous function with respect to $x$, $\varphi : \mathbb{R}^n \to \mathbb{R}^1$ -proper function and $\alpha_k,\ k = 0, 1, 2, 3$ are fixed vectors. The problem is to find a trajectory $\tilde{x}(t)$ of the Cauchy problem (1.1) (1.3) for the fourth-order differential inclusions satisfying (1.2) almost everywhere (a.e.) on $[0, 1]$ and the initial conditions (1.3) that minimizes the Bolza functional $J[x(\cdot)]$. We label this problem as $(P_{FDI})$. Here, a feasible trajectory $x(\cdot)$ is an absolutely continuous function having absolutely continuous derivatives up to order three on a time interval $[0, 1]$ for which $\frac{d^4x(\cdot)}{dt^4} \in L_1^n\big([0, 1]\big)$. We observe that such class of functions $W_{1,4}^n\big([0, 1]\big)$ is a Banach space, endowed with the different equivalent norms. For instance,
$||x(\cdot)|| = \sum_{k=0}^{3} |x^{(k)}(0)| + ||x^{IV}(\cdot)||_1$ or $||x(\cdot)|| = \sum_{k=0}^{4} ||x^{(k)}(\cdot)||_1$, where $||x^{(k)}(\cdot)||_1 = \int_0^1 |x^{(k)}(t)|dt$, and $|x|$ is an Euclidean norm in $\mathbb{R}^n$.

In what follows we give an overview of mathematical optimization with higher order differential inclusions, focusing on the special role of convex and nonconvex optimization.

In the paper [23], the nonlinear fourth-order differential equation $u^{(IV)} \pm F(u,x)u = 0$ is considered, where $F$ is a positive monotone function of $u$. In the paper [7], the existence of solutions of a class of four-point boundary value problems for a fourth-order ordinary differential equation is studied. In the paper [25], a fourth-order differential equation with nonlinear boundary condition is considered and the existence and uniqueness of a solution is proved. In [26], some sufficient conditions for (2,2)-disconjugacy are established and the distribution of zeros of nontrivial solutions of fourth-order differential equations are studied. The results are extended to cover some boundary value problems in the bending of beams. The main results are proved by making use of a generalization of Hardy's inequality and some Opial-type inequalities. In [2] an asymptotic theory for a class of fourth-order differential equations is developed. Under general conditions on the coefficients of the differential equation, the forms of the asymptotic solutions such that the solutions have different orders of magnitude for large $x$, are obtained. In paper [11] the existence of solutions of a class of four-point boundary-value problems for fourth-order ordinary differential equations are proved. This analysis relies on a fixed-point theorem due to Krasnoselskii and Zabreiko. In [28], by the method of variation, the existence, nonexistence, and multiplicity of solutions of an Ambrosetti-Prodi type problem for a system of second and fourth-order ordinary differential equations are studied. In [27], the sufficient conditions for the linear differential equations of fourth-order are established and a suitable Green's function and its estimates are used.

The present paper is devoted to one of the difficult and interesting field optimization of fourth-order ordinary discrete and differential inclusions. To the authors knowledge, this is the first paper, where optimization of fourth-order differential inclusions is discussed. The novelty of our approach is to use Euler-Lagrange and Hamiltonian type of adjoint differential inclusions to establish sufficient conditions of optimality for the fourth order differential inclusions. The stated problems and the corresponding results are new.

The paper is organized as follows.

Section 2 provides the needed facts from the book of Mahmudov [14]; Hamiltonian function $H$ and argmaximum sets of a set-valued mapping $F$, the LAM introduced and the basic idea-discretization method to establish the Euler-Lagrange inclusion for problem $(P_{FDI})$ is described.

In Section 3 via Euler-Lagrange and Hamiltonian type of adjoint inclusions the sufficient conditions of optimality for the problem $(P_{FDI})$ are proved.

In Section 4, some interesting applications of Theorems 3.1 and 3.2 are given. Namely, in the form of Weierstrass-Pontryagin maximum principle the sufficient condition of optimality for fourth-order linear discrete inclusions with a quadratic form involving symmetric nonnegative semidefinite matrix are derived. In particular, it is shown that this method can also play an important role in computational procedures for of the solution.

The conclusion is made in Section 5.

## 2. Notations and preliminaries

Let $\langle x, y\rangle$ be an inner product of elements $x, y \in \mathbb{R}^n$. Assume that $F : \mathbb{R}^{4n} \rightrightarrows \mathbb{R}^n$ is a set-valued mapping from $\mathbb{R}^{4n} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ into the set of subsets of $\mathbb{R}^n$. Then a set-valued mapping $F : \mathbb{R}^{4n} \rightrightarrows \mathbb{R}^n$ is convex if its $gphF = \{(x, v, v_4) : v_4 \in F(x, v)\},\ v = (v_1, v_2, v_3), x, v_i \in \mathbb{R}^n,\ i = 1, 2, 3$ is a convex subset of $\mathbb{R}^{5n}$. It is convex-valued if $F(x, v)$ is a convex set for each $(x, v) \in domF = \{(x, v) : F(x, v) \neq \varnothing\}$. Hamiltonian function and argmaximum set for set-valued mapping $F$ is defined as follows

$$H_F(x, v, v_4^*) = \sup_{v_4} \left\{ \left\langle v_4, v_4^* \right\rangle : v_4 \in F(x, v) \right\}, v_4^* \in \mathbb{R}^n,$$

$$F_{Arg}(x, v; v_4^*) \equiv F_A(x, v; v_4^*) = \left\{ v_4 \in F(x, v) : \langle v_4, v_4^* \rangle = H_F(x, v, v_4^*) \right\},$$

respectively. For convex $F$ we set $H_F(x, v, v_4^*) = -\infty$ if $F(x, v) = \varnothing$. For the readers convenience let us mention the following definition from the book of Mahmudov [14].

**Definition 2.1.** A convex cone $K_A(z_0)$ is called the cone of tangent directions at a point $z_0 = (x^0, v^0, v_4^0) \in A (A \subset \mathbb{R}^{5n})$ if from $\bar{z} = (\bar{x}, \bar{v}, \bar{v}_4) \in K_A(z_0)$ it follows that $\bar{z}$ is a tangent vector to the set $A$, i.e., there exists a function $\mu(\lambda) \in \mathbb{R}^{5n}$ satisfying $z_0 + \lambda\bar{z} + \mu(\lambda) \in A$ for sufficiently small $\lambda > 0$, where $\lambda^{-1}\mu(\lambda) \to 0$, as $\lambda \downarrow 0$.

Note that this cone may be regarded as a particular case of the corresponding tangent cone described in [3].

Clearly, for a convex set $A$ at a point $(x^0, v_1^0, v_2^0) \in A$ we have $\mu(\lambda) \equiv 0$.

In general, for a mapping $F$ a set-valued mapping $F^*(\cdot, z_0) : \mathbb{R}^n \rightrightarrows \mathbb{R}^{4n}$ defined by

$$F^*\left(v_4^*; (x^0, v^0, v_4^0)\right) := \left\{(x^*, v^*) : (x^*, v^*, -v_4^*) \in K_{gphF}^*(x^0, v^0, v_4^0)\right\}$$

is called a locally adjoint set-valued mapping (LAM) to $F$ at a point $(x^0, v^0, v_4^0) \in gphF$, where $K_{gphF}^*(x^0, v^0, v_4^0)$ is the dual to a cone of tangent vectors $K_{gphF}(x^0, v^0, v_4^0)$. In what follows another way to define LAMs in the "non-convex" case is the next one

$$F^*\left(v_4^*; (x^0, v^0, v_4^0)\right) := \left\{(x^*, v^*) : H_F(x, v, v_4^*) - H_F(x^0, v^0, v_4^*) \right.$$
$$\left. \leq \langle x^*, x - x^0\rangle + \langle v^*, v - v^0\rangle, \forall (x, v) \in \mathbb{R}^{4n}\right\},\ v_4 \in F_A(x, v; v_4^*),$$

which is called the LAM to non-convex mapping $F$ at a point $(x^0, v^0, v_4^0) \in gphF$. The main advantage of this definition is its simplicity. Clearly, for the convex mapping $F$ the Hamiltonian function $H_F(\cdot, v_4^*)$ is concave and the latter definition of LAM coincide with the previous definition of LAM.

We note that the method of discrete-approximation of $(P_{FDI})$ with fourth order differential inclusions has been very effective in the investigation of optimality conditions [19], where the basic idea was to study the fourth-order discrete-approximation problem:

$$\begin{aligned}
&\text{minimize } J[x(\cdot)] = \sum_{t=4\delta}^{1-4\delta} \delta g\big(x(t), \Delta x(t), \Delta^2 x(t), \Delta^3 x(t), t\big) \\
&\qquad + \varphi\big(x(1-3\delta), \Delta x(1-3\delta), \Delta^2 x(1-3\delta), \Delta^3 x(1-3\delta)\big), \\
&\qquad \Delta^4 x(t) \in F\big(x(t), \Delta x(t), \Delta^2 x(t), \Delta^3 x(t), t\big)\ ,\ t = 0, \delta, ..., 1-4\delta, \qquad (2.1) \\
&\qquad x(0) = \alpha_0,\ \Delta x(0) = \alpha_1,\ \Delta^2 x(0) = \alpha_2,\ \Delta^3 x(0) = \alpha_3.
\end{aligned}$$

Here $s$th-order $(s = 1, 2, 3, 4)$ difference operator is defined as follows:

$$\Delta^s x(t) = \frac{1}{\delta^s} \sum_{k=0}^{s} (-1)^k C_s^k x\big(t+(s-k)\delta\big),\ C_s^K = \frac{s!}{k!(s-k)!},\ t = 0, \delta, ..., 1-\delta. \quad (2.2)$$

In the paper [19] for the Cauchy problem (2.1) is applied a generalized discrete-approximate Euler-Lagrange transformation formula. It appears that by passing to the limit in necessary and sufficient conditions of optimality for problem (2.1), (2.2) as $\delta \to 0$ (at least formally), we can establish the sufficient conditions of optimality for continuous problem $(P_{FDI})$. But in the presented paper to avoid a long calculations connected with the discretization method, establishment of optimality and endpoint conditions at the endpoint $t = 1$ for the discrete-approximate problem (2.1), (2.2) are omitted. In the next section are studied sufficient conditions of optimality for problem $(P_{FDI})$.

## 3. Sufficient conditions of optimality for $(P_{FDI})$

In this section we introduce the basic notions and notation to be used in the rest of the paper. At first consider a convex optimization problem, where $F(\cdot, t) : \mathbb{R}^{4n} \rightrightarrows \mathbb{R}^n$ is a convex set-valued mapping, $g(\cdot, t), \varphi : \mathbb{R}^{4n} \to \mathbb{R}^1$ is continuous and convex with respect to the four components.

As a result of approximation method described at the end of Section 2 we establish so-called the fourth-order Euler-Lagrange differential inclusion for the convex optimization problem $(P_{FDI})$:

$$\begin{aligned}
\text{(i)}\ &\Big(\tfrac{d^4 x^*(t)}{dt^4} + \tfrac{d\eta_3^*(t)}{dt}, \eta_3^*(t) + \tfrac{d\eta_2^*(t)}{dt}, \eta_2^*(t) + \tfrac{d\eta_1^*(t)}{dt}, \eta_1^*(t)\Big) \\
&\in F^*\big(x^*(t); (\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), \tilde{x}'''(t)), t\big) - \partial_{(x,v)} g\big(\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), \tilde{x}'''(t), t\big), \\
&\text{a.e. } t \in [0, 1].
\end{aligned}$$

It is important to note a subtlety in our definition of endpoint conditions in the convex optimization problem at the endpoint $t = 1$,

$$\begin{aligned}
\text{(ii)}\ &\Big(\tfrac{d^3 x^*(1)}{dt^3} + \eta_3^*(1), -\tfrac{d^2 x^*(1)}{dt^2} + \eta_2^*(1), +\tfrac{dx^*(1)}{dt} + \eta_1^*(1), -x^*(1)\Big) \\
&\in \partial\varphi\big(\tilde{x}(1), \tilde{x}'(1), \tilde{x}''(1), \tilde{x}'''(1)\big).
\end{aligned}$$

We emphasize again that our notation and terminology are generally consistent with those in Mordukhovich [22], Mahmudov [14] for first order ordinary differential inclusions.

Later on we suppose that $x^*(t), t \in [0,1]$, is absolutely continuous function with the higher order derivatives until three and $\frac{d^4x^*(\cdot)}{dt^4} \in L_1^n\big([0,1]\big)$. In addition let $\eta_k^*(t), k=1,2,3, t\in[0,1]$ be absolutely continuous and $\frac{d\eta_k^*(\cdot)}{dt} \in L_1^n\big([0,1]\big), k = 1,2,3$.

Besides, in terms of argmaximum set we shall offer a condition providing that the LAM $F^*$ is nonempty at a given point:

(iii) $\frac{d^4\tilde{x}(t)}{dt^4} \in F_A\big(\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), \tilde{x}'''(t); x^*(t), t\big)$, a.e. $t \in [0,1]$.

It turns out that the following theorem is true.

**Theorem 3.1.** *Let $g(\cdot,t), \varphi : \mathbb{R}^4 n \to \mathbb{R}^1$ be continuous convex functions, $F$ be a convex set-valued mapping. Then for the optimality of the trajectory $\tilde{x}(t)$ in the convex optimization problem $(P_{FDI})$ it is sufficient that there exists a collection of absolutely continuous functions $\{x^*(t), \eta_k^*(t), k = 1,2,3\}, t\in[0,1]$ satisfying a.e. the fourt-order Euler-Lagrange differential inclusion (i), (iii) and endpoint condition (ii) at the endpoint $t = 1$.*

*Proof.* We remind that by Theorem 2.1 [16] the LAM $F^*\big(v_4^*; (x,v), t\big) = \partial_{(x,v)} H_F(x, v, v_4^*, t), v_4 \in F_A(x, v; v_4^*, t), v = (v_1, v_2, v_3)$. On the other hand by convention $-\partial_{(x,v)} g(\cdot,t) = \partial_{(x,v)}(-g(\cdot,t))$. Then taking into account the Moreau-Rockafellar theorem [14, 22] from the condition (i) in term of Hamiltonian function we obtain the fourth-order adjoint differential inclusion

$$\Big(\frac{d^4x^*(t)}{dt^4} + \frac{d\eta_3^*(t)}{dt}, \eta_3^*(t) + \frac{d\eta_2^*(t)}{dt}, \eta_2^*(t) + \frac{d\eta_1^*(t)}{dt}, \eta_1^*(t)\Big)$$
$$\in \partial_{(x,v)}\Big[H_F\big(\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), \tilde{x}'''(t), x^*(t), t\big) - g\big(\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), \tilde{x}'''(t), t\big)\Big].$$

By using of the classical subdifferential definition, we rewrite the last relation in the form:

$$\begin{aligned} H_F\big(x(t), x'(t), x''(t), x'''(t), x^*(t), t\big) - H_F\big(\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), \tilde{x}'''(t), x^*(t), t\big) \\ -g\big(x(t), x'(t), x''(t), x'''(t), t\big) + g\big(\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), \tilde{x}'''(t), t\big) \\ \le \Big\langle \frac{d^4x^*(t)}{dt^4} + \frac{d\eta_3^*(t)}{dt}, x(t) - \tilde{x}(t)\Big\rangle + \Big\langle \eta_3^*(t) + \frac{d\eta_2^*(t)}{dt}, x'(t) - \tilde{x}'(t)\Big\rangle \quad (3.1) \\ +\Big\langle \eta_2^*(t) + \frac{d\eta_1^*(t)}{dt}, x''(t) - \tilde{x}''(t)\Big\rangle + \langle \eta_1^*(t), x'''(t) - \tilde{x}'''(t)\rangle. \end{aligned}$$

It follows from the definition of Hamiltonian function and from (3.1) that

$$\begin{aligned} \Big\langle \frac{d^4x(t)}{dt^4}, x^*(t)\Big\rangle - \Big\langle \frac{d^4\tilde{x}(t)}{dt^4}, x^*(t)\Big\rangle - g\big(x(t), x'(t), x''(t), x'''(t), t\big) \\ g\big(\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), \tilde{x}'''(t), t\big) \le \Big\langle \frac{d^4x^*(t)}{dt^4}, x(t) - \tilde{x}(t)\Big\rangle + \frac{d}{dt}\langle \eta_3^*(t), x(t) - \tilde{x}(t)\rangle \\ +\frac{d}{dt}\langle \eta_2^*(t), x'(t) - \tilde{x}'(t)\rangle + \frac{d}{dt}\langle \eta_1^*(t), x''(t) - \tilde{x}''(t)\rangle. \end{aligned}$$

Now let us rewrite this inequality as follows

$$
\begin{aligned}
g\big(x(t), x'(t), x''(t), x'''(t), t\big) - g\big(\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), \tilde{x}'''(t), t\big) \\
\geq \Big\langle \frac{d^4(x(t)-\tilde{x}(t))}{dt^4}, x^*(t) \Big\rangle - \Big\langle \frac{d^4 x^*(t)}{dt^4}, x(t)-\tilde{x}(t) \Big\rangle \\
-\frac{d}{dt}\Big\langle \eta_1^*(t), \frac{d^2(x(t)-\tilde{x}(t))}{dt^2} \Big\rangle - \frac{d}{dt}\Big\langle \eta_2^*(t), \frac{d(x(t)-\tilde{x}(t))}{dt} \Big\rangle \\
-\frac{d}{dt}\big\langle \eta_3^*(t), x(t)-\tilde{x}(t) \big\rangle.
\end{aligned}
\tag{3.2}
$$

Integrating both sides of relation (3.2) we obtain

$$
\begin{aligned}
\int_0^1 \big[g\big(x(t), x'(t), x''(t), x'''(t), t\big) - g\big(\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), \tilde{x}'''(t), t\big)\big] dt \\
\geq \int_0^1 \Big[\Big\langle \frac{d^4(x(t)-\tilde{x}(t))}{dt^4}, x^*(t) \Big\rangle - \Big\langle \frac{d^4 x^*(t)}{dt^4}, x(t)-\tilde{x}(t) \Big\rangle\Big] \\
+\Big\langle \eta_1^*(0), \frac{d^2(x(0)-\tilde{x}(0)}{dt^2} \Big\rangle + \Big\langle \eta_2^*(0), \frac{d(x(0)-\tilde{x}(0)}{dt} \Big\rangle + \big\langle \eta_3^*(0), x(0)-\tilde{x}(0) \big\rangle. \\
-\Big\langle \eta_1^*(1), \frac{d^2(x(1)-\tilde{x}(1)}{dt^2} \Big\rangle - \Big\langle \eta_2^*(1), \frac{d(x(1)-\tilde{x}(1)}{dt} \Big\rangle - \big\langle \eta_3^*(1), x(1)-\tilde{x}(1) \big\rangle.
\end{aligned}
\tag{3.3}
$$

Let us denote

$$
A = \Big\langle \frac{d^4(x(t)-\tilde{x}(t))}{dt^4}, x^*(t) \Big\rangle - \Big\langle \frac{d^4 x^*(t)}{dt^4}, x(t)-\tilde{x}(t) \Big\rangle.
$$

Then this relation can be transformed to an equivalent form

$$
\begin{aligned}
A = \frac{d}{dt}\Big\langle \frac{d^3(x(t)-\tilde{x}(t))}{dt^3}, x^*(t) \Big\rangle - \frac{d}{dt}\Big\langle \frac{d^3 x^*(t)}{dt^3}, x(t)-\tilde{x}(t) \Big\rangle \\
-\frac{d}{dt}\Big\langle \frac{d^2(x(t)-\tilde{x}(t)}{dt^2}, \frac{dx^*(t)}{dt} \Big\rangle + \frac{d}{dt}\Big\langle \frac{d(x(t)-\tilde{x}(t)}{dt}, \frac{d^2 x^*(t)}{dt^2} \Big\rangle.
\end{aligned}
\tag{3.4}
$$

Consequently, in this way, repeating the techniques from [17] it can be shown the following remarkable integral representation of $A$ (see(3.4)):

$$
\begin{aligned}
\int_0^1 A dt = \Big\langle \frac{d^3(x(1)-\tilde{x}(1))}{dt^3}, x^*(1) \Big\rangle - \Big\langle \frac{d^3(x(0)-\tilde{x}(0))}{dt^3}, x^*(0) \Big\rangle \\
-\Big\langle \frac{d^3 x^*(1)}{dt^3}, x(1)-\tilde{x}(1) \Big\rangle + \Big\langle \frac{d^3 x^*(0)}{dt^3}, x(0)-\tilde{x}(0) \Big\rangle \\
-\Big\langle \frac{d^2(x(1)-\tilde{x}(1))}{dt^2}, \frac{dx^*(1)}{dt} \Big\rangle + \Big\langle \frac{d^2(x(0)-\tilde{x}(0))}{dt^2}, \frac{dx^*(0)}{dt} \Big\rangle \\
+\Big\langle \frac{d^2 x^*(1)}{dt^2}, \frac{d(x(1)-\tilde{x}(1))}{dt} \Big\rangle - \Big\langle \frac{d^2 x^*(0)}{dt^2}, \frac{d(x(0)-\tilde{x}(0))}{dt} \Big\rangle.
\end{aligned}
\tag{3.5}
$$

Therefore, substitution (3.5) into (3.3) and taking into account that $x(t), \tilde{x}(t)$ are feasible trajectories ($x(0) = \tilde{x}(0) = \alpha_0, x'(0) = \tilde{x}'(0) = \alpha_1, x''(0) = \tilde{x}''(0) =$

$\alpha_2, x'''(0) = \tilde{x}'''(0) = \alpha_3$) we have

$$\int_0^1 \big[g\big(x(t), x'(t), x''(t), x'''(t), t\big) - g\big(\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), \tilde{x}'''(t), t\big)\big]dt$$
$$\geq \Big\langle \frac{d^3(x(1)-\tilde{x}(1))}{dt^3}, x^*(1)\Big\rangle - \Big\langle \frac{d^3x^*(1)}{dt^3}, x(1)-\tilde{x}(1)\Big\rangle$$
$$+\Big\langle \frac{d^2x^*(1)}{dt^2}, \frac{d(x(1)-\tilde{x}(1))}{dt}\Big\rangle - \Big\langle \frac{d^2(x(1)-\tilde{x}(1))}{dt^2}, \frac{dx^*(1)}{dt}\Big\rangle$$
$$-\Big\langle \eta_1^*(1), \frac{d^2(x(1)-\tilde{x}(1))}{dt^2}\Big\rangle - \Big\langle \eta_2^*(1), \frac{d(x(1)-\tilde{x}(1))}{dt}\Big\rangle - \big\langle \eta_3^*(1), x(1)-\tilde{x}(1)\big\rangle.$$

Consequently, we have

$$\int_0^1 \big[g\big(x(t), x'(t), x''(t), x'''(t), t\big) - g\big(\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), \tilde{x}'''(t), t\big)\big]dt$$
$$\geq \Big\langle x^*(1), \frac{d^3(x(1)-\tilde{x}(1))}{dt^3}\Big\rangle - \Big\langle \eta_1^*(1) + \frac{dx^*(1)}{dt}, \frac{d^2(x(1)-\tilde{x}(1))}{dt^2}\Big\rangle$$
$$-\Big\langle \eta_2^*(1) - \frac{d^2x^*(1)}{dt^2}, \frac{d(x(1)-\tilde{x}(1))}{dt}\Big\rangle - \Big\langle \eta_3^*(1) + \frac{d^3x^*(1)}{dt^3}, x(1)-\tilde{x}(1)\Big\rangle. \tag{3.6}$$

But by the endpoint conditions (ii), at the endpoint $t = 1$

$$\varphi\big(x(1), x'(1), x''(1), x'''(1)\big) - \varphi\big(\tilde{x}(1), \tilde{x}'(1), \tilde{x}''(1), \tilde{x}'''(1)\big)$$
$$\geq \Big\langle \frac{d^3x^*(1)}{dt^3} + \eta_3^*(1), x(1)-\tilde{x}(1)\Big\rangle + \Big\langle \eta_2^*(1) - \frac{d^2x^*(1)}{dt^2}, x'(1)-\tilde{x}'(1)\Big\rangle$$
$$+\Big\langle \frac{dx^*(1)}{dt} + \eta_1^*(1), x''(1)-\tilde{x}''(1)\Big\rangle + \Big\langle -x^*(1), x'''(1)-\tilde{x}'''(1)\Big\rangle. \tag{3.7}$$

Now summing (3.6) and (3.7) we have

$$\int_0^1 \big[g\big(x(t), x'(t), x''(t), x'''(t), t\big) - g\big(\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), \tilde{x}'''(t), t\big)\big]dt$$
$$+\varphi\big(x(1), x'(1), x''(1), x'''(1)\big) - \varphi\big(\tilde{x}(1), \tilde{x}'(1), \tilde{x}''(1), \tilde{x}'''(1)\big) \geq 0$$

that is, $J[x(t)] \geq J[\tilde{x}(t)], \forall x(t), t \in [0,1]$ and $\tilde{x}(t), t \in [0,1]$ is optimal. □

**Corollary 3.1.** *For a closed set-valued mapping $F$ the conditions (i), (iii) of Theorem 3.1 can be rewritten in term of Hamiltonian function in much more convenient form:*

$$\Big(\frac{d^4x^*(t)}{dt^4} + \frac{d\eta_3^*(t)}{dt}, \eta_3^*(t) + \frac{d\eta_2^*(t)}{dt}, \eta_2^*(t) + \frac{d\eta_1^*(t)}{dt}, \eta_1^*(t)\Big)$$
$$\in \partial_{(x,v)}H_F\big(\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), \tilde{x}'''(t), x^*(t), t\big) - \partial_{(x,v)}g\big(\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), \tilde{x}'''(t), t\big),$$
$$\frac{d^4\tilde{x}(t)}{dt^4} \in \partial_{v_4^*}H_F\big(\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), \tilde{x}'''(t), x^*(t), t\big), \;\; a.e.\ t \in [0,1].$$

*Proof.* By Lemmas 2.1 and 2.2 [14, 16] we can write $F^*\big(v_4^*; (x, v_1, v_2, v_3, v_4), t\big) = \partial_{(x,v)}H_F\big(x, v_1, v_2, v_3, v_4^*), t\big), F_A\big(x, v_1, v_2, v_3, v_4^*), t\big) = \partial_{v_4^*}H_F\big(x, v_1, v_2, v_3, v_4^*), t\big)$. Then the assertions of corollary are equivalent with the conditions (i), (iii) of Theorem 3.1. □

**Corollary 3.2.** *Suppose that in the problem* $(P_{FDI})$ *the conditions* $F\big(x(t), x'(t), x''(t), x'''(t), t\big) \equiv F\big(x(t), t\big), g\big(x(t), x'(t), x''(t), x'''(t), t\big) \equiv g\big(x(t), t\big), \varphi\big(x(1), x'(1), x''(1), x'''(1)\big) \equiv \varphi\big(x(1)\big), t \in [0,1]$ *are satisfied, that is dependent variables* $x', x'', x'''$ *are missing. Then the conditions (i)-(ii) of Theorem 3.1 can be simplified as follows*

$$\frac{d^4x^*(t)}{dt^4} \in F^*(x^*(t); (\tilde{x}(t), \tilde{x}^{IV}(t)), t) - \partial_x g(\tilde{x}(t), t),\ t \in [0,1],\ \frac{d^3x^*(1)}{dt^3} \in \partial\varphi(\tilde{x}(1)).$$

*Proof.* Indeed, since in the presented case $F(x, v_1, v_2, v_3, t) \equiv F(x, t)$ we have $v^* = (v_1^*, v_2^*, v_3^*) = 0$, which implies that in the left hand side of the Euler-Lagrange inclusion (i) the last three components identically are equal to zero:

$$\eta_3^*(t) + \frac{d\eta_2^*(t)}{dt} \equiv 0,\ \eta_2^*(t) + \frac{d\eta_1^*(t)}{dt} \equiv 0,\ \eta_1^*(t) \equiv 0.$$

Consequently, by sequentially substitution, it follows that $\eta_3^*(t) \equiv 0$ and so the second term in the first component in (i) is equal to zero, that is $\frac{d\eta_3^*(t)}{dt} \equiv 0$ identically. Now taking into account that in the endpoint conditions (ii), (iii) $\eta_k^*(1) = 0 (k = 1,2,3)$, we have the desired result. The proof of corollary is completed. □

We can state the following theorem concerning optimization of $(P_{FDI})$ in the "non-convex" case.

**Theorem 3.2.** *Let (1.1)-(1.3) be nonconvex problem, that is* $g(\cdot, t), \varphi : \mathbb{R}^{4n} \to \mathbb{R}^1$ *be a non-convex function,* $F$ *be a non-convex set-valued mapping . Then for the optimality of the trajectory* $\tilde{x}(t), t \in [0,1]$ *in the problem (2.1)-(3.1) it is sufficient that there exists a collection of absolutely continuous functions* $\{x^*(t), \eta_k^*(t), k = 1,2,3\}, t \in [0,1]$ *satisfying the conditions:*

(a) $\Big(\frac{d^4x^*(t)}{dt^4} + \frac{d\eta_3^*(t)}{dt} + x^*(t), \eta_3^*(t) + \frac{d\eta_2^*(t)}{dt} + {x^*}'(t), \eta_2^*(t) + \frac{d\eta_1^*(t)}{dt} + {x^*}''(t), \eta_1^*(t) + {x^*}'''(t)\Big) \in F^*\big(x^*(t); (\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), \tilde{x}'''(t)), t\big)$ *a.e.* $t \in [0,1]$,

(b) $\varphi\big(x, v_1, v_2, v_3\big) - \varphi\big(\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), \tilde{x}'''(t)\big) \ge \Big\langle \frac{d^3x^*(1)}{dt^3} + \eta_3^*(1), x - \tilde{x}(1)\Big\rangle + \Big\langle -\frac{d^2x^*(1)}{dt^2} + \eta_2^*(1), v_1 - \tilde{x}'(1)\Big\rangle + \Big\langle -\frac{dx^*(1)}{dt} + \eta_1^*(1), v_2 - \tilde{x}''(1)\Big\rangle + \big\langle -x^*(1), v_3 - \tilde{x}'''(1)\big\rangle$,

(c) $g\big(x, v_1, v_2, v_3, t\big) - g\big(\tilde{x}_1, \tilde{x}'(1), \tilde{x}''(1), \tilde{x}'''(1), t\big) \ge \big\langle x^*(t), x - \tilde{x}(t)\big\rangle + \sum_{k=1}^{3} \Big\langle \frac{d^kx^*(t)}{dt^k}, v_k - \tilde{x}^{(k)}(t)\Big\rangle,\ \forall (x, v) \in \mathbb{R}^{4n},\ v = (v_1, v_2, v_3)$,

(d) $\Big\langle \frac{d^4\tilde{x}(t)}{dt^4}, x^*(t)\Big\rangle = H_F\big(\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), \tilde{x}'''(t), x^*(t), t\big)$, *a.e.* $t \in [0,1]$.

*Proof.* We proceed by analogy with the preceding derivation in the proof of Theorem 3.1:

$$H_F\big(x(t), x'(t), x''(t), x'''(t), x^*(t), t\big) - H_F\big(\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), \tilde{x}'''(t), x^*(t), t\big)$$
$$\le \Big\langle \frac{d^4x^*(t)}{dt^4} + \frac{d\eta_3^*(t)}{dt} + x^*(t), x(t) - \tilde{x}(t)\Big\rangle + \Big\langle \eta_3^*(t) + \frac{\eta_2^*(t)}{dt} + {x^*}'(t), x'(t) - \tilde{x}'(t)\Big\rangle$$
$$\Big\langle \eta_2^*(t) + \frac{\eta_1^*(t)}{dt} + {x^*}''(t), x''(t) - \tilde{x}''(t)\Big\rangle + \Big\langle \eta_1^*(t) + {x^*}'''(t), x'''(t) - \tilde{x}'''(t)\Big\rangle,$$

whereas

$$\Big\langle \frac{d^4x(t)}{dt^4}, x^*(t)\Big\rangle - \Big\langle \frac{d^4\tilde{x}(t)}{dt^4}, x^*(t)\Big\rangle \leq \Big\langle \frac{d^4x^*(t)}{dt^4} + \frac{d\eta_3^*(t)}{dt} + x^*(t), x(t) - \tilde{x}(t)\Big\rangle$$
$$\Big\langle \eta_3^*(t) + \frac{d\eta_2^*(t)}{dt} + x^{*'}(t), x'(t) - \tilde{x}'(t)\Big\rangle + \Big\langle \eta_2^*(t) + \frac{d\eta_1^*(t)}{dt} + x^{*''}(t), x''(t) - \tilde{x}''(t)\Big\rangle$$
$$+\Big\langle \eta_1^*(t) + x^{*'''}(t), x'''(t) - \tilde{x}'''(t)\Big\rangle.$$

Moreover, observe that for non-convex $g(\cdot, t)$ by the condition $(c)$ for all feasible trajectories $x(\cdot)$ the following inequality is satisfied

$$g\big(x(t), x'(t), x''(t), x'''(t), t\big) - g\big(\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), \tilde{x}'''(t), t\big)$$
$$\geq \Big\langle x^*(t), x(t) - \tilde{x}(t)\Big\rangle + \sum_{k=1}^{3} \Big\langle \frac{d^k x^*(t)}{dt^k}, x^{(k)}(t) - \tilde{x}^{(k)}(t)\Big\rangle.$$

Therefore, the relation (3.2) is justified. In what follows the proof of the second part runs similarly. □

Let us denote $\frac{\partial g}{\partial x^{(k)}} = \frac{\partial g\big(\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), \tilde{x}'''(t), t\big)}{\partial x^{(k)}}$, $k = 0, 1, 2, 3$ and require that

$$g\big(x, v_1, v_2, v_3, t\big) - g\big(\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), \tilde{x}'''(t), t\big) \geq \Big\langle \frac{\partial g}{\partial x}, x - \tilde{x}(t)\Big\rangle$$
$$+ \sum_{k=1}^{3} \Big\langle \frac{\partial g}{\partial x^{(k)}}, v_k - \tilde{x}^{(k)}(t)\Big\rangle, \ \forall (x, v) \in \mathbb{R}^{4n}, \ v = (v_1, v_2, v_3). \tag{3.8}$$

On the other hand, suppose that the following Euler-Lagrange inclusion is satisfied:

$$\Big(\frac{d^4x^*(t)}{dt^4} + \frac{d\eta_3^*(t)}{dt} + \frac{\partial g}{\partial x}, \eta_3^*(t) + \frac{d\eta_2^*(t)}{dt} + \frac{\partial g}{\partial x'}, \eta_2^*(t) + \frac{d\eta_1^*(t)}{dt} + \frac{\partial g}{\partial x''},$$
$$\eta_1^*(t) + \frac{\partial g}{\partial x'''}\Big) \in F^*\big(x^*(t); (\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), \tilde{x}'''(t)), t\big). \tag{3.9}$$

**Corollary 3.3.** *Under the conditions (3.8), (3.9) and conditions* $(b), (d)$ *of Theorem 3.2 the result of Theorem 3.2 remains true.*

*Proof.* The formulated corollary can be proved similarly to previous theorems, and so its proof is omitted. □

## 4. Some applications of optimization for fourth order differential inclusions $(P_{FDI})$

Now, let us consider optimization of the following higher order "linear" differential inclusion with initial value problem, labelled by $(P_{LCP})$:

$$\begin{aligned}
&\text{minimize} && J_0[x(\cdot)] = \textstyle\int_0^1 g_0\big(x(t), t\big)dt \\
&(P_{LCP}) && \tfrac{d^4x(t)}{dt^4} \in F\big(x(t), x'(t), x''(t), x'''(t)\big), \ \text{a.e. } t \in [0, 1], \\
& && x(0) = x_0^0, \ x'(0) = x_0^1, \ x_2^0 = x_0^2, \ x_2^0 = x_0^3, \\
& && F\big(x, v_1, v_2, v_3\big) \equiv A_0 x + A_1 v_1 + A_2 v_2 + A_3 v_3 + BU.
\end{aligned}$$

Here the integrand $g_0(\cdot,t)$ is equal to the quadratic form $g_0(x,t) = \frac{1}{2}\langle x, \Lambda x\rangle + \langle c, x\rangle$, where $\Lambda$ is a symmetric nonnegative semidefinite $n\times n$ matrix and $c$ and $x_0^k$ are fixed points; $x_0^k \in \mathbb{R}^n, k = 0,1,2,3$. Obviously, this function is convex and by Theorem 3.1 $\partial_{(x,v)} g\big(x, v, v_2, v_3, t\big) \equiv \{\partial_x g_0(x,t) \times (0,0,0)\}$, where $\partial_x g_0(x,t) = \{\Lambda x + c\}$. Moreover, $A_i, i = 0,1,2,3$, and $B$ are $n\times n$ and $n \times r$ matrices, respectively, $U$ is a convex closed subset of $\mathbb{R}^r$. In fact, the problem is to find a controlling parameter $\tilde{u}(t) \in U$ (say $\tilde{u}(\cdot)$ from the class of bounded measurable functions) for initial value problem with fourth-order "linear" differential inclusions and free endpoint constraints such that the arc $\tilde{x}(t)$ corresponding to it minimizes $J_0[x(\cdot)]$.

We will apply the Theorem 3.1. Since $\varphi(x, v_1, v_2, v_3) \equiv 0$ it follows that $\partial\varphi\big(\tilde{x}(1), \tilde{x}'(1), \tilde{x}''(1), \tilde{x}'''(1)\big) \equiv \{0\}\times\{0\}\times\{0\}\times\{0\}$. Consequently, the endpoint condition (ii) of Theorem 3.1 at the point $t = 1$ is transformed into relation

$$\Big(\frac{d^3x^*(1)}{dt^3} + \eta_3^*(1), -\frac{d^2x^*(1)}{dt^2} + \eta_2^*(1), \frac{dx^*(1)}{dt} + \eta_1^*(1), -x^*(1)\Big) \equiv \{0\}\times\{0\}\times\{0\}\times\{0\}$$

or in more detail into equations

$$\big(-1\big)^{k+1}\frac{d^kx^*(1)}{dt^k} + \eta_k^*(1) = 0\big(x^*(1) = 0\big),\ k = 1,2,3. \tag{4.1}$$

To formulate of fourth-order adjoint Euler-Lagrange differential inclusion for the convex optimization problem $(P_{LCP})$ we should compute $F^*\big(v_4^*; (x, v_1, v_2, v_3, v_4)\big)$. Taking into account that $F\big(x, v_1, v_2, v_3,\big) \equiv A_0x + \sum_{k=1}^{3} A_kv_k + BU$ it can be easily computed that

$$H_F(x, v, v_4^*) = \sup_{v_4}\Big\{\big\langle v_4, v_4^*\big\rangle : v_4 \in F(x,v)\Big\} = \sup_{v_4}\Big\{\Big\langle A_0x + \sum_{k=1}^{3} A_kv_k + Bu, v_4^*\Big\rangle :$$
$$v_4 \in F(x,v\Big\} = \big\langle x, A_0^*v_4^*\big\rangle \sum_{k=1}^{3}\big\langle v_k, A_k^*v_4^*\big\rangle + \sup_u\Big\{\big\langle Bu, v_4^*\big\rangle : u \in U\Big\},$$

where $A^*$ is adjoint (transposed) matrix of $A$. Then if $\tilde{v}_4 = A_0\tilde{x} + \sum_{k=1}^{3} A_k\tilde{v}_k + B\tilde{u}, \tilde{u} \in U$ as is shown in [19] one has

$$F^*\big(v_4^*; (\tilde{x}, \tilde{v}, \tilde{v}_4)\big) = \begin{cases} \big(A_0^*v_4^*, A_1^*v_4^*, A_2^*v_4^*, A_3^*v_4^*\big), & -B^*v_4^* \in K_U^*(\tilde{u}), \\ \varnothing\ , & -B^*v_4^* \notin K_U^*(\tilde{u}). \end{cases} \tag{4.2}$$

Thus, using (4.2) and the relation $\partial_x g_0(x,t) = \{\Lambda x + c\}$ by the condition (i) of Theorem 3.1 we have the following system of Euler-Lagranges type linear adjoint equations:

$$\frac{d^4x^*(t)}{dt^4} + \frac{d\eta_3^*(t)}{dt} = A_0^*x^*(t) - \Lambda\tilde{x}(t) - c,\ \eta_3^*(t) + \frac{d\eta_2^*(t)}{dt} = A_1^*x^*(t),$$
$$\eta_2^*(t) + \frac{d\eta_1^*(t)}{dt} = A_2^*x^*(t),\ \eta_1^*(t) = A_3^*x^*(t). \tag{4.3}$$

By sequentially substitution, we find that

$$\eta_1^*(t) = A_3^* x^*(t),\ \eta_2^*(t) = A_2^* x^*(t) - A_3^* \frac{dx^*(t)}{dt},$$
$$\eta_3^*(t) = A_1^* x^*(t) - A_2^* \frac{dx^*(t)}{dt} + A_3^* \frac{d^2x^*(t)}{dt^2}. \tag{4.4}$$

Then taking into account the relations (4.4) in (4.1) we can easily see that

$$-x^*(1) = 0,\ \frac{dx^*(1)}{dt} + A_3^* x^*(1) = 0,\ -\frac{d^2x^*(1)}{dt^2} + A_2^* x^*(1) - A_3^* \frac{dx^*(1)}{dt} = 0,$$
$$\frac{d^3x^*(1)}{dt^3} + A_1^* x^*(1) - A_2^* \frac{dx^*(1)}{dt} + A_3^* \frac{d^2x^*(1)}{dt^2} = 0.$$

Obviously, by sequentially substitution here we have

$$x^*(1) = 0,\ \frac{dx^*(1)}{dt} = 0,\ \frac{d^2x^*(1)}{dt^2} = 0,\ \frac{d^3x^*(1)}{dt^3} = 0. \tag{4.5}$$

In turn by substituting the expression for $\eta_k^*(t), k = 1, 2, 3$ into the first equation in (4.3) we can define the following Euler-Lagrange type adjoint differential inclusion (equation);

$$\frac{d^4x^*(t)}{dt^4} = A_0^* x^*(t) - A_1^* \frac{dx^*(t)}{dt} + A_2^* \frac{d^2x^*(t)}{dt^2} - A_3^* \frac{d^3x^*(t)}{dt^3} - \Lambda \tilde{x}(t) - c. \tag{4.6}$$

On the other hand, the Weierstrass-Pontryagin maximum principle [14, 22] of theorem is an immediate consequence of the conditions (iii) of Theorem 3.1 and formula (4.2):

$$\Big\langle B\tilde{u}(t), x^*(t) \Big\rangle = \sup_{u \in U} \Big\langle Bu, x^*(t) \Big\rangle. \tag{4.7}$$

Finally, we can formulate the obtained result as follows.

**Theorem 4.1.** *The arc $\tilde{x}(t)$ corresponding to the controlling parameter $\tilde{u}(t)$ minimizes the quadratic cost functional in the fourth-order linear optimal control problem $(P_{LCP})$ with initial value problem and free endpoint constraints, if there exists an absolutely continuous function $x^*(t)$ satisfying the fourth-order adjoint differential equation (4.6), the endpoint condition (4.5) and Weierstrass-Pontryagin maximum principle (4.7).*

It should be noted that in concrete problems, according to Theorem 4.1 and Weierstrass-Pontryagin maximum principle, an optimal solution of fourth-order linear differential optimal control problem can be successfully computed. Let us consider the following example.

**Example 4.1.** Suppose we have the following problem:

$$\text{minimize} \int_0^1 g_0\big(x(t), t\big)dt,\ \text{subject to}\ \frac{d^4x(t)}{dt^4} = -2x(t) + 3x''(t) + u,\ t \in [0, 1],$$
$$x(0) = 0,\ x'(0) = 1,\ x''(0) = 2,\ x'''(0) = 1,\ u \in U = [-1, +1], \tag{4.8}$$

where $g_0(x, t) = x$. It is required to find a control function $\tilde{u}(t) \in U$ such that the corresponding trajectory $\tilde{x}(t)$ minimizes the indicated functional in problem (4.8). Obviously, in this case $F(x, v_1, v_2, v_3) = -2x + 3v_2 + U$.

By Theorem 4.1 (see (4.6)) the Euler-Lagrange adjoint DFIs and transversality condition for this problem consist of the following

$$\frac{d^4x^*(t)}{dt^4} = -2x^*(t) + 3x^{*''}(t) - 1,\ t \in [0,1],\ x^*(1) = x^{*'}(1) = x^{*''}(1) = x^{*'''}(1) = 0. \tag{4.9}$$

Besides, Weierstrass-Pontryagin maximum condition (4.7) in our example has the form $\tilde{u}(t) \cdot x^*(t) = \max_{-1\le u\le 1} u \cdot x^*(t), t \in [0,1]$ whence $\tilde{u}(t) = sgnx^*(t)$ that is $\tilde{u}(t) = 1$ if $x^*(t) > 0$ and $\tilde{u}(t) = -1$ if $x^*(t) < 0$. We show that the values of adjoint variables $x^*(t)$, optimal control $\tilde{u}(t)$ can be easily computed. Let us solve the equation (4.9); using the classical theory of linear non-homogeneous differential equations we can find the corresponding characteristic equation $r^4 - 3r^2 + 2 = 0$ of homogeneous fourth-order differential equation. The four roots of this equation are real numbers $\pm 1$ and $\pm\sqrt{2}$ . According to these roots the general solution of corresponding fourth-order homogeneous differential equation is $C_1e^t + C_2e^{-t} + C_3e^{\sqrt{2}t} + C_4e^{-\sqrt{2}t}$ On the other it can be easily verified that the particular solution of non-homogeneous fourth-order differential equation is $-1/2$. As a result, the general solution of the adjoint Euler-Lagrange type equation (4.9) has the form

$$x^*(t) = C_1e^t + C_2e^{-t} + C_3e^{\sqrt{2}t} + C_4e^{-\sqrt{2}t} - \frac{1}{2}, \tag{4.10}$$

where $C_i, i = 1, 2, 3, 4$ are arbitrary constants to be determined.

For this purpose, from (4.10) we can derive that

$$\begin{aligned} \frac{dx^*(t)}{dt} &= C_1e^t - C_2e^{-t} + \sqrt{2}C_3e^{\sqrt{2}t} - \sqrt{2}C_4e^{-\sqrt{2}t}, \\ \frac{d^2x^*(t)}{dt^2} &= C_1e^t + C_2e^{-t} + 2C_3e^{\sqrt{2}t} + 2C_4e^{-\sqrt{2}t}, \\ \frac{d^3x^*(t)}{dt^3} &= C_1e^t - C_2e^{-t} + 2\sqrt{2}C_3e^{\sqrt{2}t} - 2\sqrt{2}C_4e^{-\sqrt{2}t}. \end{aligned} \tag{4.11}$$

Now taking into account in (4.9) the condition at the point $t = 1$ we deduce from (4.10) and (4.11) the following equations

$$\begin{aligned} C_1e + C_2e^{-1} + C_3e^{\sqrt{2}} + C_4e^{-\sqrt{2}} &= \frac{1}{2}, \\ C_1e - C_2e^{-1} + \sqrt{2}C_3e^{\sqrt{2}} - \sqrt{2}C_4e^{-\sqrt{2}} &= 0, \\ C_1e + C_2e^{-1} + 2C_3e^{\sqrt{2}} + 2C_4e^{-\sqrt{2}} &= 0, \\ C_1e - C_2e^{-1} + 2\sqrt{2}C_3e^{\sqrt{2}} - 2\sqrt{2}C_4e^{-\sqrt{2}} &= 0. \end{aligned} \tag{4.12}$$

By elementary way it can be checked that the solution to the system of algebraic equations (4.12) consist of the following

$$C_1 = \frac{1}{2e},\ C_2 = \frac{e}{2},\ C_3 = -\frac{e^{-\sqrt{2}}}{4},\ C_4 = -\frac{e^{\sqrt{2}}}{4}. \tag{4.13}$$

Then by substituting (4.13) into (4.10) we have the unique solution to the problem (4.9)

$$x^*(t) = \frac{1}{2}\Big(e^{t-1} + e^{1-t}\Big) - \frac{1}{4}\Big(e^{\sqrt{2}(t-1)} + e^{\sqrt{2}(1-t)}\Big) - \frac{1}{2}.$$

We recall that, $\tilde{u}(t) = -1$ if $x^*(t) > 0$ and $\tilde{u}(t) = 1$ if $x^*(t) > 0$. Thus, in the case, where $\left(e^{t-1} + e^{1-t}\right) - \frac{1}{2}\left(e^{\sqrt{2}(t-1)} + e^{\sqrt{2}(1-t)}\right) < 1$, we should solve the problem

$$\frac{d^4x(t)}{dt^4} = -2x(t) + 3x''(t) - 1, \ t \in [0,1],$$
$$x(0) = 0, \ x'(0) = 1, \ x''(0) = 2, \ x'''(0) = 1. \tag{4.14}$$

By analogy with the adjoint system (4.9) the general solution to the differential equation (4.14) has the form

$$x(t) = K_1 e^t + K_2 e^{-t} + K_3 e^{\sqrt{2}t} + K_4 e^{-\sqrt{2}t} - \frac{1}{2}, \tag{4.15}$$

where $K_i, i = 1,2,3,4$ are arbitrary constants to be defined. By using the initial condition in (4.14) we have the following system of linear algebraic equations

$$K_1 + K_2 + K_3 + K_4 = \frac{1}{2}, \quad K_1 + K_2 + 2K_3 + 2K_4 = 2,$$
$$K_1 - K_2 + \sqrt{2}K_3 - \sqrt{2}K_4 = 1, \quad K_1 - K_2 + \sqrt{2}K_3 - 2\sqrt{2}K_4 = 1.$$

The solution to this system with respect to $K_i, i = 1,2,3,4$ is given below:

$$K_1 = 0, \quad K_2 = -1, \quad K_3 = \frac{3}{4}, \quad K_4 = \frac{3}{4}.$$

Substituting this solution in (4.15) we have an optimal solution to the problem (4.14) corresponding to optimal control $\tilde{u}(t) = -1$:

$$\tilde{x}(t) = -e^{-t} + \frac{3}{4}\left(e^{\sqrt{2}t} + e^{-\sqrt{2}t}\right) - \frac{1}{2}.$$

By similar way it can be shown that if $\left(e^{t-1} + e^{1-t}\right) - \frac{1}{2}\left(e^{\sqrt{2}(t-1)} + e^{\sqrt{2}(1-t)}\right) > 1$ that is, if $\tilde{u}(t) = 1$ instead of (4.14) we have

$$\frac{d^4x(t)}{dt^4} = -2x(t) + 3x''(t) + 1, \ t \in [0,1],$$
$$x(0) = 0, \ x'(0) = 1, \ x''(0) = 2, \ x'''(0) = 1. \tag{4.16}$$

Since the homogeneous equations corresponding to the adjoint equation (4.9) and controlling fourth-order equation (4.16) coincides, its general solution is

$$\bar{K}_1 e^t + \bar{K}_2 e^{-t} + \bar{K}_3 e^{\sqrt{2}t} + \bar{K}_4 e^{-\sqrt{2}t},$$

where $\bar{K}_i, i = 1,2,3,4$ are arbitrary constants. It is easy to see that in this case the particular solution of (4.16) is $1/2$. Consequently, the solution to (4.16) has the form

$$x(t) = \bar{K}_1 e^t + \bar{K}_2 e^{-t} + \bar{K}_3 e^{\sqrt{2}t} + \bar{K}_4 e^{-\sqrt{2}t} + \frac{1}{2}.$$

To find $\bar{K}_i, i = 1,2,3,4$ by using initial condition in (4.16), we have the following system

$$\bar{K}_1 + \bar{K}_2 + \bar{K}_3 + \bar{K}_4 = -\frac{1}{2}; \ \bar{K}_1 - \bar{K}_2 + \sqrt{2}\bar{K}_3 - \sqrt{2}\bar{K}_4 = 1,$$
$$\bar{K}_1 + \bar{K}_2 + 2\bar{K}_3 + 2\bar{K}_4 = 2; \ \bar{K}_1 - \bar{K}_2 + 2\sqrt{2}\bar{K}_3 - 2\sqrt{2}\bar{K}_4 = 1.$$

The solution of this system consist of $\bar{K}_1 = -1, \bar{K}_2 = -2, \bar{K}_3 = \bar{K}_4 = 5/4$. Thus, the solution to initial-value problem (4.16) is

$$\tilde{x}(t) = -e^t - 2e^{-t} + \frac{5}{4}\left(e^{\sqrt{2}t} + e^{-\sqrt{2}t}\right) + \frac{1}{2}.$$

*Remark* 4.1. We remind [19] that according to problem $(P_{LCP})$ we have the following discrete-approximate equation

$$\Delta^4 x(t) = A_0 x(t) + A_1 \Delta x(t) + A_2 \Delta^2 x(t) + A_3 \Delta^3 x(t) + Bu(t),\ u(t) \in U,$$

$t = 0, \delta, ..., 1 - 4\delta$, where $\Delta^k, k = 1, 2, 3, 4$ are $k$-th order difference operators as is shown in (2.2):

$$\Delta x(t) = \frac{1}{\delta}\Big[x(t+\delta) - x(t)\Big],\ \Delta^2 x(t) = \frac{1}{\delta^2}\big[x(t+2\delta) - 2x(t+\delta) + x(t)\big],$$

$$\Delta^3 x(t) = \frac{1}{\delta^3}\Big[x(t+3\delta) - 3x(t+2\delta) + 3x(t+\delta) - x(t)\Big],$$

$$\Delta^4 x(t) = \frac{1}{\delta^4}\Big[x(t+4\delta) - 4x(t+3\delta) + 6x(t+2\delta) - 4x(t+\delta) + x(t)\Big].$$

By Theorem 4.2 [19] for optimality of the trajectory $\tilde{x}(t)$ in the "linear" discrete-approximate problem, it is necessary and sufficient that there exists $x^*(t)$ satisfying the adjoint Euler-Lagrange DFIs (equations)

$$\Delta^4 x^*(t) = A_0^* x^*(t+4\delta) - A_1^* \Delta x^*(t+3\delta) + A_2^* \Delta^2 x^*(t+2\delta)$$
$$-A_3^* \Delta^3 x^*(t+\delta) - \lambda g_0'(\tilde{x}(t), t),\ t = 0, ..., 1 - 4\delta,\ \lambda \in \{0, 1\}$$

with "nitial" (final) conditions $x^*(1) = 0, \Delta x^*(1) = 0, \Delta^2 x^*(1) = 0, \Delta^3 x^*(1) = 0$. Using this equation, we can formulate the discrete-approximate problem for above considered problem:

Minimize $\sum_{t=4\delta}^{1-4\delta} \delta x(t)$, subject to $\Delta^4 x(t) = -2x(t) + 3\Delta^2 x(t) + u(t), u(t) \in U, t = 0, ..., 1 - 4\delta, x(0) = 0, \Delta x(0) = 1, \Delta^2 x(0) = 2, \Delta^3 x(0) = 1$. Obviously, $x(0) = 0, x(\delta) = \delta, x(2\delta) = 2\delta + 2\delta^2, x(3\delta) = 3\delta + 6\delta^2 + \delta^3$. Then for our problem we have the adjoint equation and transversality condition of the form

$$\Delta^4 x^*(t) = -2x^*(t+4\delta) + 3\Delta^2 x^*(t+2\delta) - 1, t = 0, ..., 1 - 4\delta,$$
$$x^*(1) = 0, \Delta x^*(1) = 0, \Delta^2 x^*(1) = 0, \Delta^3 x^*(1) = 0.$$

## 5. Conclusion

In this paper a new method for solving a Bolza problem with fourth-order differential inclusions which are often used to describe various processes in science and engineering is presented. This approach plays a much more important role in derivation of fourth-order adjoint DFIs. Thus, a sufficient conditions of optimality for such problems are deduced. There has been a significant development in the study of optimization for differential and difference equations and inclusions in recent years [9, 14, 22]. Finally, it is concluded that the proposed method is reliable for solving the various optimization problems with fourth-order discrete and differential inclusions. Theoretical analysis and practical results show that our method is simple and easy to implement and is efficient for computing optimal solution of the fourth order differential inclusions. At last, an example is given for illustrating our results.

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